Extremal Properties of Balanced Tri-Diagonal Matrices

By Peter A. Businger

Abstract. If A is a square matrix with distinct eigenvalues and D a nonsingular matrix, then the angles between row- and column-eigenvectors of $D^{-1}AD$ differ from the corresponding quantities of A . Perturbation analysis of the eigenvalue problem motivates the minimization of functions of these angles over the set of diagonal similarity transforms; two such functions which are of particular interest are the spectral and the Euclidean condition numbers of the eigenvector matrix X of $D^{-1}AD$. It is shown that for a tri-diagonal real matrix A both these condition numbers are minimized when D is chosen such that the magnitudes of corresponding sub- and super-diagonal elements are equal. ■

If a tri-diagonal matrix A is such that corresponding sub- and super-diagonal elements have equal magnitude then A is said to be *balanced* or *equilibrated*. Wilkinson [5, p. 424] uses norms of balanced tri-diagonal matrices for error analysis of the eigenvalue problem. He observes that, given a tri-diagonal matrix $A = [a_{ij}]$ all of whose sub- and super-diagonal elements are nonzero, a diagonal matrix $D = \text{diag}(d_1, d_2, \dots, d_n)$ can be found such that $D^{-1}AD$ is balanced. In fact, such a D is defined by

$$d_{i+1}/d_i = (|a_{i+1,i}|/|a_{i,i+1}|)^{1/2}, \quad i = 1, 2, \dots, n-1.$$

If some sub- or super-diagonal element of A is zero then finding its eigenvalues can be reduced to finding the eigenvalues of submatrices, each of which can be balanced separately.

It is an immediate consequence of Osborne's Lemma 2 [3] that a balanced tri-diagonal matrix A has the extremal property

$$\|A\|_E = \inf_D \|D^{-1}AD\|_E,$$

where $\|\cdot\|_E$ denotes the *Euclidean matrix norm* (*Schur norm*, *Frobenius norm*). Our Theorem 1 states the analogous result for the spectral norm; Theorems 2 and 3 show that the eigenvalue problem of a balanced tri-diagonal matrix is optimally conditioned in the sense that no matrix of the form $D^{-1}AD$ has smaller angles between corresponding row- and column-eigenvectors.

We use $\|\cdot\|$ to denote the *Euclidean vector norm*, $\|\cdot\|_2$ for the subordinate matrix bound (the *spectral matrix norm*), $k_2(\cdot)$ for the *spectral condition number* of a nonsingular matrix, and $k_E(\cdot)$ for the *Euclidean condition number* (defined by $k_E(X) = \|X\|_E \|X^{-1}\|_E$). Absolute value signs applied to vectors are understood component-wise. D , D_1 , and D_2 denote diagonal matrices with positive diagonal elements.

THEOREM 1. *If A is a balanced tri-diagonal real matrix then*

$$\|A\|_2 = \inf_D \|D^{-1}AD\|_2.$$

Proof. There exists a real diagonal matrix E with $|E| = I$ such that $B = EA$ is symmetric. Since, for all D ,

$$\|D^{-1}AD\|_2 = \|ED^{-1}AD\|_2 = \|D^{-1}EAD\|_2 = \|D^{-1}BD\|_2,$$

the conclusion follows from the observation that for a symmetric matrix B , $\|B\|_2 \leq \|D^{-1}BD\|_2$ for all D .

The following theorem deals with the secants $\|y^H\| \|x\|/|y^Hx|$ of the angles between corresponding row- and column-eigenvectors of a matrix.

THEOREM 2. *If A is a balanced tri-diagonal real matrix with distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$, corresponding column-eigenvectors $x_1, x_2, \dots, x_n$, and corresponding row-eigenvectors $y_1^H, y_2^H, \dots, y_n^H$ then*

$$\frac{\|y_i^H\| \|x_i\|}{|y_i^Hx_i|} = \inf_D \frac{\|y_i^H D\| \|D^{-1}x_i\|}{|y_i^Hx_i|}$$

for $i = 1, 2, \dots, n$.

Proof. Basing his argument on a theorem due to Stoer and Witzgall [4], Bauer [1] showed that for any vector pair y^H and x ,

$$\inf_D \frac{\|y^H D\| \|D^{-1}x\|}{|y^Hx|} = \frac{|y^Hx|}{|y^Hx|}.$$

Since $A = EA^T E$ for some real diagonal matrix E with $|E| = I$, $y_i = c_i E x_i$ for some scalars c_i . Hence

$$\frac{\|y_i^H\| \|x_i\|}{|y_i^Hx_i|} = \frac{|y_i^H| \|x_i\|}{|y_i^Hx_i|}$$

for $i = 1, 2, \dots, n$, which completes the proof.

COROLLARY. *A has an eigenvector matrix $X = [x_1, x_2, \dots, x_n]$ such that $k_E(X) = \inf_{D_1, D_2} k_E(D_1^{-1}XD_2)$.*

Proof. By Theorem 2, each term in the sum on the right of the relationship

$$\inf_{D_1, D_2} k_E(D_1^{-1}XD_2) = \inf_{D_1} \sum_{i=1}^n \frac{\|y_i^H D_1\| \|D_1^{-1}x_i\|}{|y_i^Hx_i|}$$

is minimized when $D_1 = I$. This implies the corollary.

THEOREM 3. *If A is a balanced tri-diagonal real matrix with distinct eigenvalues then A has an eigenvector matrix $X = [x_1, x_2, \dots, x_n]$ such that $k_2(X) = \inf_{D_1, D_2} k_2(D_1^{-1}XD_2)$.*

Proof. Bauer [2] showed that

$$\inf_{D_1, D_2} k_2(D_1^{-1}XD_2) \geq \rho(E_1 X^{-1} E_2 X)$$

for all diagonal matrices E_1 and E_2 for which $|E_1| = |E_2| = I$ (ρ denotes the spectral radius). Hence it suffices for us to obtain equality for some eigenvector matrix X of A and for some such E_1 and E_2 .

Let Q be a unitary matrix such that if $Z = XQ$ then $J = Z^{-1}AZ$ is the direct sum of 1 by 1 and 2 by 2 matrices. (The latter are of the form

$$\begin{bmatrix} \lambda & \mu \\ -\mu & \lambda \end{bmatrix}$$

and correspond to conjugate complex pairs of eigenvalues $\lambda \pm i\mu$.) If the permutation matrix P is chosen such that $\bar{X} = XP$, invariance of k_2 implies that for all D_1 and D_2

$$\begin{aligned} k(D_1^{-1}XD_2) &= k(\overline{D_1^{-1}XD_2}) = k(D_1^{-1}\bar{X}D_2) \\ &= k(D_1^{-1}XPD_2) = k(D_1^{-1}X(PD_2P^T)) . \end{aligned}$$

Hence no generality is lost if we assume that those pairs of diagonal elements of D_2 are equal which correspond to a complex conjugate pair of eigenvectors. Under this assumption

$$k(D_1^{-1}XD_2) = k(D_1^{-1}XD_2Q) = k(D_1^{-1}XQD_2) ,$$

which allows us to replace the problem of minimizing $k(D_1^{-1}XD_2)$ by that of finding $\inf_{D_1, D_2} k(D_1^{-1}ZD_2)$. Now $Z^{-1}AZ = J$ implies

$$Z^T A^T Z^{-T} = J^T = E_1 J E_1$$

for some real diagonal matrix E_1 such that $|E_1| = I$. Hence, if $A^T = E_2 A E_2$, it follows that $E_2 Z^{-T} E_1 = Z D_2$ for some diagonal matrix D_2 . Thus there exists a matrix Z_0 such that $Z_0^{-1} A Z_0 = J$ as well as $Z_0^{-1} = E_1 Z_0^T E_2$. Hence

$$k(Z_0) = \|Z_0\|_2 \|E_1 Z_0^T E_2\|_2 = \|Z_0\|_2^2 = \rho(Z_0^T Z_0) = \rho(E_1 Z_0^{-1} E_2 Z_0) .$$

The result of Bauer stated at the beginning of this proof now establishes the theorem.

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