On the Computation of the Eigenvalues of a Tridiagonal Matrix

By I. Gargantini*

Abstract. A recent algorithm for the simultaneous approximation of all zeros of a polynomial is applied to the computation of the eigenvalues of a tridiagonal matrix. The method works in the presence of multiplicity and degeneracy and has been tested in a multitude of cases; its practical limitation on a computer is the large number of locations required for matrices of high order. ■

The efficiency of the second algorithm described in [1] for determining the zeros of a polynomial has been investigated in connection with the computation of the eigenvalues of a tridiagonal matrix, all the eigenvalues being computed at the same time together with their range of approximation.

Let A be the given matrix of order N whose coefficients (real or complex) are arranged in the following way:

$$A = \begin{bmatrix} \alpha_1 & \beta_2 & & & & \\ \gamma_2 & \alpha_2 & \beta_3 & & & \\ & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot \\ & & & & \gamma_N & \alpha_N \end{bmatrix}.$$

The evaluation of the characteristic polynomial $P(z) = |A - zI|$ and of its successive derivatives $P^{(k)}(z)$, $k = 1, 2, \dots, N$, can be carried out by means of the formulas listed below:

$$P_0^{(0)}(z) = 1,$$

$$P_1^{(0)}(z) = \alpha_1 - z,$$

$$P_{r+1}^{(0)}(z) = (\alpha_{r+1} - z)P_r^{(0)}(z) - \gamma_{r+1}\beta_{r+1}P_{r-1}^{(0)}(z), \quad r = 1, 2, \dots, N-1,$$

where

$$P_N^{(0)} = P(z),$$

$$P_k^{(k)} = -kP_{k-1}^{(k-1)}(z),$$

$$P_{k+1}^{(k)} = (\alpha_{k+1} - z)P_k^{(k)}(z) - kP_k^{(k-1)}(z),$$

$$P_{r+1}^{(k)}(z) = (\alpha_{r+1} - z)P_r^{(k)}(z) - kP_r^{(k-1)}(z) - \gamma_{r+1}\beta_{r+1}P_{r-1}^{(k)}(z),$$

$$r = k+1, \dots, N-1; k = 1, 2, \dots, N,$$

where

$$P_N^{(k)}(z) = P^{(k)}(z), \quad k = 1, 2, \dots, N.$$

Received May 13, 1968, revised September 23, 1968.

* Formerly at the IBM Zurich Laboratory, now at the University of Western Ontario, London, Canada.

Since the zero-searching routine operates in the square Q_0 centered at the origin and with side 2, we first transform the matrix A into a new matrix having all the eigenvalues inside Q_0 . The algorithm consists in the construction of rectangles R_i ($i = 1, 2, \dots, I, I \leq N$) with the properties: (1) at least one zero of $P(z)$ belongs to R_i ; (2) the exact number of zeros inside R_i can be determined by applying a discrete form of the argument principle.

The given matrix can be real or complex, but in the real case, we can restrict our search to the superior or inferior half of the square Q_0 . As a rectangle having a side belonging to the x -axis can have a real zero on the boundary, it is necessary to make this region symmetric with respect to the axis of abscissas before applying the argument principle.

A program was written for the IBM 360/40 in Fortran and the routine tested in some examples in which the results were known. The procedure was found to give correct results in the cases tested; however, some shortcomings have to be mentioned. First, a certain amount of calculation is required, the number of multiplications necessary to evaluate $P(z)$ and $P^{(k)}(z)$ being of the order of $5N^2$ in the real case. Secondly, the number of locations necessary to store the components of the rectangles R_i is not known a priori and increases with the order of the matrix: for $N = 30$ the average number of components to be stored is of the order of 10^3 .

TABLE I

0.55335693256745740 01	0.0	0.22152527E-12
-0.25889514550243420 01	0.0	0.22152527E-12
-0.11640327929446930 01	0.0	0.22152527E-12
0.54013675314595310-01	0.0	0.22152527E-12
0.46157208991438380 00	-0.76229593153588840 00	0.18087463E-12
0.46157208991438380 00	0.76229593153588840 00	0.18087463E-12
0.13338102225606140 01	-0.13397201297733090 01	0.18087463E-12
0.13338102225606140 01	0.13397201297733090 01	0.18087463E-12
0.20307074541242390 01	-0.18195936489965580 01	0.28998792E-12
0.20307074541242390 01	0.18195936489965580 01	0.28998792E-12
0.12683218413541360 01	0.0	0.22152527E-12
0.20212614808906860 01	0.0	0.22152527E-12
0.21041753135262260 01	-0.35399301136210530 01	0.18087463E-12
0.21041753135262260 01	0.35399301136210530 01	0.18087463E-12
0.25282503109991250 01	-0.45995025638457320 00	0.18087463E-12
0.25282503109991250 01	0.45995025638457320 00	0.18087463E-12
0.32931444731942820 01	0.0	0.22152527E-12
0.46656404692911460 01	0.0	0.22152527E-12

In the following, we denote by α, β, γ the vectors of components $\{\alpha_1, \alpha_2, \dots, \alpha_N\}$, $\{\beta_2, \beta_3, \dots, \beta_N\}$, $\{\gamma_2, \gamma_3, \dots, \gamma_N\}$ respectively. A typical result of the routine is in Table I for the matrix with

$$\begin{aligned}\alpha &= \{1, 2, 1, 2, 1, 1, 2, 3, 1, 2, 4, 1, 3, 2, 1, 4, 1, -2\}, \\ \beta &= \{-3, 2, 1, -3, 2, -4, -5, -5, -3, -2, 1, 2, 3, 3, 4, 1, 2\}, \\ \gamma &= \{1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1\}.\end{aligned}$$

Experimentally, regions containing a multiple zero were found to be larger than regions containing a simple root. It so happens that for the same degree of refinement of the initial square Q_0 , the error bound for a multiple zero is larger than for a simple one, and increases with increasing multiplicity. As an example, we give in Table II the results for the matrix with

$$\alpha = \{1, 1, 1, -2, -2, 3\}, \beta = \{0, 1, 0, -1, 0\}, \gamma = \{-2, 0, 1, 0, 1\}.$$

Degeneracy is ignored by the algorithm. Numerical confirmation of this fact is illustrated by the computation of the eigenvalues of the matrix with $\alpha = \{0, 0, 0, 0, 0, 0\}$, $\beta = \{1, 1, 0, 1, 0\}$, $\gamma = \{0, 0, 0, 0, 0\}$, which results are in Table III.

TABLE II

0.2999999999999999 01	0.0	0.23360153E-05
-0.1999999999999999 01	0.0	0.46720306E-05
-0.1999999999999999 01	0.0	0.46720306E-05
0.9999999999999999 00	0.0	0.70080459E-05
0.9999999999999999 00	0.0	0.70080459E-05
0.9999999999999999 00	0.0	0.70080459E-05

TABLE III

0.0	0.0	0.40880268E-05
0.0	0.0	0.40880268E-05
0.0	0.0	0.40880268E-05
0.0	0.0	0.40880268E-05
0.0	0.0	0.40880268E-05
0.0	0.0	0.40880268E-05

Acknowledgment. The author wishes to thank Dr. J. H. Wilkinson for the most valuable material supplied for testing.

University of Western Ontario
London, Ontario, Canada

1. P. HENRICI & I. GARGANTINI, *Uniformly Convergent Algorithms for the Simultaneous Approximation of all Zeros of a Polynomial*, Proc. Sympos. on Constructive Aspects of the Fundamental Theorem of Algebra, Wiley, New York. (To appear.)