On Amicable and Sociable Numbers*

By Henri Cohen

Abstract. An exhaustive search has yielded 236 amicable pairs of which the lesser number is smaller than 10^8 , 57 pairs being new.

It has also yielded 9 new sociable groups of order 10 or less, of which the lesser number is smaller than $6 \cdot 10^7$; the 9 sociable groups are all of order 4.

The sequence of iterates of the function $s(n) = \sigma(n) - n$ starting with 276 has also been extended to 119 terms.

Introduction. Let $n \geq 2$ be an integer, and

$$\sigma(n) = \sum_{d|n} d, \quad s(n) = \sigma(n) - n = \sum_{d|n, d \neq n} d.$$

We wish to study the behavior of the sequence:

$$\begin{aligned} a_0(n) &= n \\ a_{k+1}(n) &= s(a_k(n)) \end{aligned}$$

which will be called the aliquot series of n .

It is clear that if this sequence is bounded as $k \rightarrow \infty$ it is periodic, since $a_k(n)$ can take only a finite number of values.

Consequently the sequence can have essentially three distinct behaviors:

- (a) The sequence converges, i.e. there exists a k for which $a_k = 1$ (or equivalently a_{k-1} prime).
- (b) The sequence is periodic of period t : there exists k_0 such that

$$a_{k+t} = a_k \quad \text{for all } k \geq k_0.$$

If one can take $k_0 = 0$, the sequence is purely periodic; in this case: if $t = 1$, n is a perfect number, if $t = 2$, $(n, s(n))$ is a pair of amicable numbers, and in general the t -uplet $(a_0, \dots, a_{t-1})$ is a sociable group of order t .

- (c) The sequence is unbounded.

Results on Amicable Numbers. Two recent papers [6], [7], listed all pairs of amicable numbers up to 10^6 and 10^7 respectively.

Table 1 extends these lists and contains all amicable pairs with the lesser number between 10^7 and 10^8 . The 57 pairs marked with an asterisk are not found in the lists given by Escott [1], Poulet [2], Garcia [3], Lee [4], Lee [5], and seem to be new.

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TABLE 1
AMICABLE NUMBERS

10254970 = 2. 5. 11. 53. 1759	10273670 = 2. 5. 11. 59. 1583
10533296 = 2 ⁴ . 19. 34649	10949704 = 2 ³ . 29. 109. 433
10572550 = 2. 5 ² . 19. 31. 359	10854650 = 2. 5 ² . 31. 47. 149
10596368 = 2 ⁴ . 29. 41. 557	11199112 = 2 ³ . 53. 61. 433
* 10634085 = 3 ⁴ . 5. 7. 11 ² . 31	14084763 = 3. 7. 11 ² . 23. 241
* 10992735 = 3 ² . 5. 13. 19. 23. 43	12070305 = 3 ² . 5. 13. 47. 439
* 11173460 = 2 ² . 5. 53. 83. 127	13212076 = 2 ² . 31. 47. 2267
* 11252648 = 2 ³ . 11. 71. 1801	12101272 = 2 ³ . 67. 107. 211
11498355 = 3 ⁴ . 5. 11. 29. 89	12024045 = 3 ⁴ . 5. 11. 2699
11545616 = 2 ⁴ . 19. 163. 233	12247504 = 2 ⁴ . 491. 1559
11693290 = 2. 5. 7. 167047	12361622 = 2. 7 ² . 13. 31. 313
11905504 = 2 ⁵ . 13. 28619	13337336 = 2 ³ . 107. 15581
12397552 = 2 ⁴ . 23. 59. 571	13136528 = 2 ⁴ . 359. 2287
* 12707704 = 2 ³ . 17. 41. 43. 53	14236136 = 2 ³ . 107. 16631
* 13671735 = 3. 5. 7 ² . 11. 19. 89	15877065 = 3. 5. 17. 19. 29. 113
* 13813150 = 2. 5 ² . 13. 79. 269	14310050 = 2. 5 ² . 29. 71. 139
13921528 = 2 ³ . 19. 67. 1367	13985672 = 2 ³ . 19. 101. 911
14311688 = 2 ³ . 17. 47. 2239	14718712 = 2 ³ . 23. 167. 479
* 14426230 = 2. 5. 7. 13. 83. 191	18087818 = 2. 7. 31. 71. 587
14443730 = 2. 5. 7 ³ . 4211	15882670 = 2. 5. 19. 179. 467
14654150 = 2. 5 ² . 7. 149. 281	16817050 = 2. 5 ² . 179. 1879
15002464 = 2 ⁵ . 37. 12671	15334304 = 2 ⁵ . 227. 2111
* 15363832 = 2 ³ . 11. 71. 2459	16517768 = 2 ³ . 53. 163. 239
15938055 = 3 ² . 5. 7. 19. 2663	17308665 = 3 ² . 5. 11. 73. 479
* 16137628 = 2 ² . 13. 23. 103. 131	16150628 = 2 ² . 13. 31. 43. 233
16871582 = 2. 7 ² . 13. 17. 19. 41	19325698 = 2. 7 ² . 19. 97. 107
* 17041010 = 2. 5. 7. 31. 7853	19150222 = 2. 7. 13. 43. 2447
* 17257695 = 3. 5. 7. 13. 47. 269	17578785 = 3. 5. 7. 23. 29. 251
17754165 = 3 ² . 5. 11. 13. 31. 89	19985355 = 3 ² . 5. 13. 127. 269
17844255 = 3 ² . 5. 11. 13. 47. 59	19895265 = 3 ² . 5. 13. 71. 479
17908064 = 2 ⁵ . 53. 10559	18017056 = 2 ⁵ . 79. 7127
* 18056312 = 2 ³ . 17. 103. 1289	18166888 = 2 ³ . 19. 107. 1117

TABLE 1 (Continued)

* 18194715 = $3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 59 \cdot 89$	22240485 = $3^2 \cdot 5 \cdot 31 \cdot 107 \cdot 149$
18655744 = $2^9 \cdot 83 \cdot 439$	19154336 = $2^5 \cdot 619 \cdot 967$
* 20014808 = $2^3 \cdot 11 \cdot 79 \cdot 2879$	21457192 = $2^3 \cdot 47 \cdot 149 \cdot 383$
20022328 = $2^3 \cdot 17 \cdot 23 \cdot 37 \cdot 173$	22823432 = $2^3 \cdot 1367 \cdot 2087$
20308995 = $3^3 \cdot 5 \cdot 7 \cdot 21491$	20955645 = $3^3 \cdot 5 \cdot 17 \cdot 23 \cdot 397$
21448630 = $2 \cdot 5 \cdot 7 \cdot 131 \cdot 2339$	23030090 = $2 \cdot 5 \cdot 19 \cdot 53 \cdot 2287$
* 22227075 = $3^3 \cdot 5^2 \cdot 13 \cdot 17 \cdot 149$	24644925 = $3^3 \cdot 5^2 \cdot 29 \cdot 1259$
22249552 = $2^4 \cdot 13 \cdot 41 \cdot 2609$	25325528 = $2^3 \cdot 587 \cdot 5393$
22508145 = $3^3 \cdot 5 \cdot 11 \cdot 23 \cdot 659$	23111055 = $3^3 \cdot 5 \cdot 11 \cdot 79 \cdot 197$
22608632 = $2^3 \cdot 19 \cdot 23 \cdot 29 \cdot 223$	25775368 = $2^3 \cdot 1439 \cdot 2239$
23358248 = $2^3 \cdot 23 \cdot 37 \cdot 47 \cdot 73$	25233112 = $2^3 \cdot 37 \cdot 85247$
23389695 = $3^3 \cdot 5 \cdot 7 \cdot 53 \cdot 467$	25132545 = $3^3 \cdot 5 \cdot 17 \cdot 47 \cdot 233$
* 23628940 = $2^2 \cdot 5 \cdot 37^2 \cdot 863$	27428276 = $2^2 \cdot 17 \cdot 251 \cdot 1607$
* 24472180 = $2^2 \cdot 5 \cdot 17 \cdot 167 \cdot 431$	30395276 = $2^2 \cdot 47 \cdot 107 \cdot 1511$
25596544 = $2^7 \cdot 311 \cdot 643$	25640096 = $2^5 \cdot 67 \cdot 11959$
25966832 = $2^4 \cdot 29 \cdot 191 \cdot 293$	26529808 = $2^4 \cdot 47 \cdot 35279$
26090325 = $3^2 \cdot 5^2 \cdot 17 \cdot 19 \cdot 359$	26138475 = $3^2 \cdot 5^2 \cdot 11 \cdot 59 \cdot 179$
* 28118032 = $2^4 \cdot 47 \cdot 139 \cdot 269$	28128368 = $2^4 \cdot 59 \cdot 83 \cdot 359$
* 28608424 = $2^3 \cdot 13 \cdot 139 \cdot 1979$	29603576 = $2^3 \cdot 23 \cdot 349 \cdot 461$
30724694 = $2 \cdot 7 \cdot 11 \cdot 13 \cdot 103 \cdot 149$	32174506 = $2 \cdot 7 \cdot 13 \cdot 17 \cdot 10399$
30830696 = $2^3 \cdot 13 \cdot 521 \cdot 569$	31652704 = $2^5 \cdot 449 \cdot 2203$
31536855 = $3^2 \cdot 5 \cdot 7 \cdot 53 \cdot 1889$	32148585 = $3^2 \cdot 5 \cdot 7 \cdot 102059$
* 31818952 = $2^3 \cdot 11 \cdot 41 \cdot 8819$	34860248 = $2^3 \cdot 97 \cdot 167 \cdot 269$
32205616 = $2^4 \cdot 17 \cdot 167 \cdot 709$	34352624 = $2^4 \cdot 2147039$
* 32642324 = $2^2 \cdot 11 \cdot 13 \cdot 149 \cdot 383$	35095276 = $2^2 \cdot 17 \cdot 47 \cdot 79 \cdot 139$
32685250 = $2 \cdot 5^3 \cdot 13 \cdot 89 \cdot 113$	34538270 = $2 \cdot 5 \cdot 13 \cdot 379 \cdot 701$
* 33501825 = $3^2 \cdot 5^2 \cdot 7 \cdot 89 \cdot 239$	36136575 = $3^2 \cdot 5^2 \cdot 19 \cdot 79 \cdot 107$
34256222 = $2 \cdot 7 \cdot 11 \cdot 13 \cdot 71 \cdot 241$	35997346 = $2 \cdot 7 \cdot 11 \cdot 23 \cdot 10163$
* 34364912 = $2^4 \cdot 43 \cdot 199 \cdot 251$	34380688 = $2^4 \cdot 47 \cdot 131 \cdot 349$
34765731 = $3^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 227$	36939357 = $3^2 \cdot 7 \cdot 13 \cdot 23 \cdot 37 \cdot 53$
* 35115795 = $3^3 \cdot 5 \cdot 11 \cdot 13 \cdot 17 \cdot 107$	43266285 = $3^3 \cdot 5 \cdot 53 \cdot 6047$
35361326 = $2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 1039$	40117714 = $2 \cdot 7 \cdot 13 \cdot 53 \cdot 4159$

TABLE 1 (Continued)

$35373195 = 3^2 \cdot 5 \cdot 11 \cdot 13 \cdot 23 \cdot 239$	$40105845 = 3^2 \cdot 5 \cdot 13 \cdot 179 \cdot 383$
$35390008 = 2^3 \cdot 19 \cdot 23 \cdot 53 \cdot 191$	$39259592 = 2^3 \cdot 71 \cdot 69119$
$35472592 = 2^4 \cdot 43 \cdot 47 \cdot 1097$	$36415664 = 2^4 \cdot 53 \cdot 42943$
* $37363095 = 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 41 \cdot 263$	$45663849 = 3^2 \cdot 7 \cdot 11 \cdot 131 \cdot 503$
* $37784810 = 2 \cdot 5 \cdot 7 \cdot 539783$	$39944086 = 2 \cdot 7 \cdot 13 \cdot 41 \cdot 53 \cdot 101$
$37848915 = 3^2 \cdot 5 \cdot 13 \cdot 23 \cdot 29 \cdot 97$	$39202605 = 3^2 \cdot 5 \cdot 13 \cdot 19 \cdot 3527$
* $38400512 = 2^9 \cdot 179 \cdot 419$	$38938288 = 2^4 \cdot 83 \cdot 109 \cdot 269$
* $38637016 = 2^3 \cdot 11 \cdot 359 \cdot 1223$	$40678184 = 2^3 \cdot 29 \cdot 271 \cdot 647$
$38663950 = 2 \cdot 5^2 \cdot 13 \cdot 17 \cdot 3499$	$43362050 = 2 \cdot 5^2 \cdot 59 \cdot 14699$
* $38783992 = 2^3 \cdot 13 \cdot 37 \cdot 10079$	$41654408 = 2^3 \cdot 47 \cdot 139 \cdot 797$
$38807968 = 2^5 \cdot 37 \cdot 73 \cdot 449$	$40912232 = 2^3 \cdot 37 \cdot 89 \cdot 1553$
* $43096904 = 2^3 \cdot 17 \cdot 41 \cdot 59 \cdot 131$	$46715896 = 2^3 \cdot 53 \cdot 239 \cdot 461$
* $44139856 = 2^4 \cdot 29 \cdot 251 \cdot 379$	$44916944 = 2^4 \cdot 83 \cdot 149 \cdot 227$
$45263384 = 2^3 \cdot 17 \cdot 59 \cdot 5641$	$46137016 = 2^3 \cdot 19 \cdot 433 \cdot 701$
* $46237730 = 2 \cdot 5 \cdot 7 \cdot 11^2 \cdot 53 \cdot 103$	$61319902 = 2 \cdot 7 \cdot 83 \cdot 113 \cdot 467$
* $46271745 = 3^2 \cdot 5 \cdot 13 \cdot 19 \cdot 23 \cdot 181$	$49125375 = 3^2 \cdot 5^3 \cdot 13 \cdot 3359$
* $46521405 = 3^3 \cdot 5 \cdot 7 \cdot 19 \cdot 2591$	$53011395 = 3^3 \cdot 5 \cdot 31 \cdot 53 \cdot 239$
* $46555250 = 2 \cdot 5^3 \cdot 7 \cdot 37 \cdot 719$	$55880590 = 2 \cdot 5 \cdot 103 \cdot 227 \cdot 239$
* $46991890 = 2 \cdot 5 \cdot 11 \cdot 29 \cdot 14731$	$48471470 = 2 \cdot 5 \cdot 19^2 \cdot 29 \cdot 463$
* $48639032 = 2^3 \cdot 13 \cdot 29 \cdot 16127$	$52967368 = 2^3 \cdot 59 \cdot 293 \cdot 383$
$48641584 = 2^4 \cdot 29 \cdot 104831$	$48852176 = 2^4 \cdot 47 \cdot 167 \cdot 389$
* $49215166 = 2 \cdot 7 \cdot 11 \cdot 13^2 \cdot 31 \cdot 61$	$55349570 = 2 \cdot 5 \cdot 31 \cdot 61 \cdot 2927$
* $50997596 = 2^2 \cdot 13 \cdot 19 \cdot 71 \cdot 727$	$51737764 = 2^2 \cdot 13 \cdot 23 \cdot 181 \cdot 239$
$52695376 = 2^4 \cdot 17 \cdot 151 \cdot 1283$	$56208368 = 2^4 \cdot 3513023$
$56055872 = 2^6 \cdot 79 \cdot 11087$	$56598208 = 2^6 \cdot 383 \cdot 2309$
* $56512610 = 2 \cdot 5 \cdot 7 \cdot 11 \cdot 23 \cdot 3191$	$75866014 = 2 \cdot 7^2 \cdot 774143$
$56924192 = 2^5 \cdot 13 \cdot 193 \cdot 709$	$64562488 = 2^3 \cdot 283 \cdot 28517$
* $58580540 = 2^2 \cdot 5 \cdot 23 \cdot 347 \cdot 367$	$70507972 = 2^2 \cdot 23 \cdot 521 \cdot 1471$
* $59497888 = 2^5 \cdot 41 \cdot 101 \cdot 449$	$61953512 = 2^3 \cdot 29 \cdot 97 \cdot 2753$
* $63560025 = 3^3 \cdot 5^2 \cdot 17 \cdot 29 \cdot 191$	$65003175 = 3^3 \cdot 5^2 \cdot 23 \cdot 53 \cdot 79$
$63717615 = 3^2 \cdot 5 \cdot 13 \cdot 17 \cdot 43 \cdot 149$	$66011985 = 3^2 \cdot 5 \cdot 13 \cdot 19 \cdot 5939$
$66595130 = 2 \cdot 5 \cdot 7 \cdot 31 \cdot 30689$	$74824390 = 2 \cdot 5 \cdot 31 \cdot 59 \cdot 4091$

TABLE 1 (Concluded)

* 66854710 = 2. 5. 13 ³ . 17. 179	71946890 = 2. 5. 17. 83. 5099
67729064 = 2 ³ . 13. 431. 1511	69439576 = 2 ³ . 23. 107. 3527
67738268 = 2 ² . 13. 17. 19. 37. 109	79732132 = 2 ² . 19. 263. 3989
68891992 = 2 ³ . 13. 23. 83. 347	78437288 = 2 ³ . 587. 16703
71015260 = 2 ² . 5. 23. 263. 587	85458596 = 2 ² . 41. 47. 11087
71241830 = 2. 5. 11. 19. 89. 383	78057370 = 2. 5. 17. 359. 1279
* 72958556 = 2 ² . 11. 19. 197. 443	74733604 = 2 ² . 11. 53. 73. 439
73032872 = 2 ³ . 11. 71. 11689	78469528 = 2 ³ . 59. 83. 2003
74055952 = 2 ⁴ . 23. 61. 3299	78166448 = 2 ⁴ . 197. 24799
* 74386305 = 3 ² . 5. 7. 17. 29. 479	87354495 = 3 ² . 5. 19. 71. 1439
* 74769345 = 3 ³ . 5. 7 ² . 89. 127	82824255 = 3 ³ . 5. 17. 151. 239
75171808 = 2 ⁵ . 53. 127. 349	77237792 = 2 ⁵ . 479. 5039
75226888 = 2 ³ . 11. 59. 14489	81265112 = 2 ³ . 53. 137. 1399
* 78088504 = 2 ³ . 13. 31. 53. 457	88110536 = 2 ³ . 167. 65951
* 78447010 = 2. 5. 17. 19. 149. 163	80960990 = 2. 5. 11. 491. 1499
* 79324875 = 3 ² . 5 ³ . 7 ² . 1439	87133365 = 3 ² . 5. 11. 103. 1709
80422335 = 3 ³ . 5. 7. 85103	82977345 = 3 ³ . 5. 11. 71. 787
82633005 = 3 ² . 5. 7. 13. 17. 1187	104177619 = 3 ² . 7. 13. 131. 971
83135650 = 2. 5 ² . 13. 79. 1619 [*]	85603550 = 2. 5 ² . 19. 251. 359
* 84521745 = 3 ³ . 5. 7. 11. 47. 173	107908335 = 3 ³ . 5. 383. 2087
* 84591405 = 3 ³ . 5. 17. 29. 31. 41	89590995 = 3 ³ . 5. 13. 71. 719
86158220 = 2 ² . 5. 41. 105071	99188788 = 2 ² . 23. 43. 25073
* 87998470 = 2. 5. 7. 29. 67. 647	102358010 = 2. 5. 47. 89. 2447
* 88144630 = 2. 5. 7 ² . 29. 6203	102814490 = 2. 5. 37. 269. 1033
89477984 = 2 ⁵ . 59. 83. 571	92143456 = 2 ⁵ . 1637. 1759
90437150 = 2. 5 ² . 19. 23. 4139	94372450 = 2. 5 ² . 23. 137. 599
* 91996816 = 2 ⁴ . 29. 331. 599	93259184 = 2 ⁴ . 79. 89. 829
93837808 = 2 ⁴ . 19. 83. 3719	99899792 = 2 ⁴ . 1399. 4463
95629904 = 2 ⁴ . 37. 67. 2411	97580944 = 2 ⁴ . 67. 227. 401
95791430 = 2. 5. 7. 17. 101. 797	115187002 = 2. 7. 17. 113. 4283
* 96304845 = 3. 5. 7 ² . 13. 10079	96747315 = 3. 5. 7 ² . 23. 59. 97
97041735 = 3 ² . 5. 7. 71. 4339	97945785 = 3 ² . 5. 7. 239. 1301

explain, at least partly, why only one group has been found of odd order > 1 . It does not explain why no groups of order 6, 8 or 10 have been found.

Results on Unbounded Sequences. It has been conjectured by Catalan (see revision by Dickson [10]) that the aliquot series of n is never unbounded. It is known to be bounded for $2 \leq n \leq 275$. The smallest n for which the behavior is not known is 276. G. A. Paxson [9] has calculated 67 terms of this sequence. I have extended this to 119 terms and found:

$$a_{118}(276) = 2133148752623068133100.$$

Conclusion. From these results a number of conjectures can be made.

Let $A(x)$ be the number of amicable pairs of which the smaller number is less than x ; then empirically one can conjecture:

Conjecture 1. There exists $\beta > 0$ such that

$$\text{Log } A(x) \sim \beta \cdot \text{Log}(x).$$

This conjecture of course implies the as yet unknown fact that there exists an infinity of amicable pairs.

From Table 1 and preceding tables a least square method gives

$$\beta = 0.29 \dots$$

A heuristic computation of β would be welcome.

Conjecture 2. There exists an infinity of sociable groups of order 4.

This is a particular case of a general conjecture of Erdős [11]. Furthermore in the same paper Erdős states that the density of sociable groups of any order is 0. Combining this with Catalan's conjecture as revised by Dickson one obtains:

Conjecture 3. For almost all n (i.e. with density 1) the associated sequence converges. These conjectures seem very difficult to prove.

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