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Whittaker's Cardinal Function in Retrospect*

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Introductory Note. Eoin L. Whitney died of a heart attack on November 21, 1966. His colleagues knew him to be a very good mathematician. He was an inspiration to many, in particular to the co-authors of the present paper.

Abstract. This paper exposes properties of the Whittaker cardinal function and illustrates the use of this function as a mathematical tool. The cardinal function is derived using the Paley-Wiener theorem. The cardinal function and the central-difference expansions are linked through their similarities. A bound is obtained on the difference between the cardinal function and the function which it interpolates. Several cardinal functions of a number of special functions are examined. It is shown how the cardinal function provides a link between Fourier series and Fourier transforms, and how the cardinal function may be used to solve integral equations.

1. Introduction and Summary. The Whittaker cardinal function was discovered by E. T. Whittaker [1], who wanted to know whether there exists in the class of all functions which take on the same values at the set of points $A = \{kh\}_{k=-\infty}^{\infty}$, $h > 0$, "a function of royal blood whose distinguished properties set it apart from its bourgeois brethren". This function then played a fundamental role in the development of the theory of central difference processes, a theory which was also originated by E. T. Whittaker [1]. Somewhat later J. M. Whittaker and his co-workers [2], [3] produced a considerable enrichment of this theory.

A cognate but independent theory has developed more recently in engineering literature on the communication of information; this theory stems mainly from papers of Hartley [4], Nyquist [5] and Shannon [6], and is usually termed as sampling theory.

Presently, Schoenberg and his students [7] are extending the Whittaker theory to splines.

Our purpose in the present paper is to expose the properties of the Whittaker cardinal function $C(g, h, x)$ of a function g and to illustrate by example the use of the cardinal function $C(g, h, x)$ as a mathematical tool for the study of some numerical processes.

In Section 2, we define the cardinal function, and we use the Paley-Wiener theorem to describe a class of functions $B(h)$ for which $C(g, h, x) = g(x)$. We also show that the cardinal function $C(g, h, x)$ is actually an orthogonal expansion of every g in $B(h)$.

In Section 3, we expose some connections between central-difference series and the cardinal function.

In Section 4, we obtain a bound on the difference between $g(x)$ and $C(g, x, h)$

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in the case when $g(z)$ is analytic in the region $\{z = x + iy : |y| \leqq d\}$ for some $d > 0$, and $\int_{-\infty}^{\infty} |g(x + iy)|^2 \, dx < \infty$ for all $|y| \leqq d$.

In Section 5, we give a number of examples illustrating the use of the cardinal function. More specifically, we examine the cardinal functions $C(1/(x - a), h, x)$, $C(1/(x^2 + a^2), h, x)$, $C(e^{i\lambda x}, h, x)$, $C(1/\Gamma(x), h, x)$ and $C(|x|, h, x)$. We also illustrate a connection between the trapezoidal formula and the cardinal function, and we illustrate the use of the cardinal function for solving integral equations. Finally, we show how the cardinal function provides a link between Fourier transforms and Fourier series.

2. Representation of a Class of Functions by Means of Ordinates.

2.1. *Whittaker's cardinal function.* Let us begin by defining the cardinal function of a function g defined on the real line R.

Definition 2.1. *Let the function* sinc x *be defined by*

$$\operatorname{sinc} x \equiv \frac{\sin \pi x}{\pi x}. \tag{2.1}$$

Let g be a function defined on R and let $h > 0$. The formal series

$$\sum_{k=-\infty}^{\infty} g(kh) \operatorname{sinc} \frac{x - kh}{h} \tag{2.2}$$

will be called the cardinal series of the function g with respect to the tabular interval h. If the series (2.2) *converges, we denote its sum by $C(g, h, x)$, and the function $C(g, h, x)$ will be called the cardinal function* (*or Whittaker cardinal function*) *of the function g.*

We next state some known properties of Fourier transforms which can be found in Titchmarsh [15] and which will enable us to determine a class $B(h)$ of functions $g(x)$ for which $C(g, h, x) \equiv g(x)$.

Let $L^p(a, b)$ denote the set of all complex Lebesgue measurable functions $f(x)$ such that $\int_a^b |f(x)|^p \, dx < \infty$. Every function $f \in L^2(R)$ has the Fourier transform $F \in L^2(R)$ given by

$$F(x) = \int_R e^{ixt} f(t) \, dt. \tag{2.3}$$

Given F, we can recover f by use of

$$f(t) = \frac{1}{2\pi} \int_R e^{-ixt} F(x) \, dx. \tag{2.4}$$

If F is the Fourier transform of $f \in L^2(R)$ and G is the Fourier transform of $g \in L^2(R)$, then the product FG is again the Fourier transform of a function $\theta \in L^2(R)$, provided that $FG \in L^2(R)$. If $FG \in L^2(R)$ the function θ is given in terms of f and g by

$$\theta(t) = \int_R f(\tau) g(t - \tau) \, d\tau. \tag{2.5}$$

If F and G are the respective Fourier transforms of f and $g \in L^2(R)$, then Parseval's theorem states that

$$\int_R f(t)\overline{g(t)} \, dt = \frac{1}{2\pi} \int_R F(x)\overline{G(x)} \, dx, \tag{2.6}$$

where $\bar{g}$ and $\bar{G}$ denote the complex conjugates of g and G respectively.

Definition 2.2. Let $B(h)$ denote the set of all functions g such that $g(z) = g(x + iy)$ is an entire function, such that $g(x) \in L^2(R)$, and such that

$$|g(z)| \leqq C \exp [\pi |y|/h] \tag{2.7}$$

for some constant C.

The following theorem of Paley and Wiener proven in [8, pp. 197–200] appeared after many properties of the Whittaker cardinal function were known.

THEOREM 2.3. *Every function $g \in B(h)$ can be represented in the form*

$$g(z) = \frac{1}{2\pi} \int_{-\pi/h}^{\pi/h} e^{-iwz} G(w)\, dw \tag{2.8}$$

where $G \in L^2(-\pi/h, \pi/h)$.

If we combine the Eqs. (2.4) and (2.8) and interchange the order of integration, we obtain

THEOREM 2.4. *If $g \in B(h)$, then*

$$g(x) = \frac{1}{h} \int_R g(t) \operatorname{sinc} \frac{x - t}{h}\, dt. \tag{2.9}$$

By applying Schwarz's inequality to the integral

$$g(z) = \frac{1}{h} \int_R f(t) \operatorname{sinc} \frac{z - t}{h}\, dt \tag{2.10}$$

where $f \in L^2(R)$ we see that the function $g(z)$ is an entire function which satisfies (2.7). Furthermore, using the identities,

$$\begin{aligned} \frac{1}{h} \int_R e^{ixt} \operatorname{sinc} \frac{t}{h}\, dt &= 1 \quad \text{if } |x| < \pi/h, \\ &= \tfrac{1}{2} \quad \text{if } |x| = \pi/h, \\ &= 0 \quad \text{if } |x| > \pi/h, \end{aligned} \tag{2.11}$$

derived below in (2.13) and (2.14), together with (2.5), we see that the Fourier transform of the function $g(x)$ in (2.10) is

$$\begin{aligned} G(w) &= F(w) \quad \text{if } |w| < \pi/h, \\ &= 0 \qquad\ \ \text{if } |w| > \pi/h, \end{aligned} \tag{2.12}$$

where $w \in R$, and where F denotes the Fourier transform of the function f. By an application of Theorem 2.3, it now follows that the function $g(z)$ in (2.10) is in $B(h)$. We have thus proved

THEOREM 2.5. *If $f \in L^2(R)$, then the function $g(z)$ given in* (2.10) *is in $B(h)$.*

If we apply the trapezoidal sum formula to the integral on the right of (2.10) we obtain the cardinal series of the function $f(t)$

$$\sum_{k=-\infty}^{\infty} f(kh) \operatorname{sinc} \frac{x - kh}{h}.$$

We shall see in the following section that if $g \in B(h)$, then $C(g, h, x) \equiv g(x)$.

Let us illustrate by an example that (2.9) is not generally valid if $g \notin B(h)$. If in (2.10) we take $f(t) = \cos(\pi t/bh)$ and replace z by x and t by $x + t$, we obtain

$$\begin{aligned} g(x) &= \frac{1}{\pi}\int_R \frac{\sin(\pi t/h)}{t} \cos[\pi(t+x)/bh]\, dt \\ &= \frac{1}{\pi}\cos(\pi x/bh)\int_R \frac{\sin(\pi t/h)}{t}\cos(\pi t/bh)\, dt, \end{aligned} \tag{2.13}$$

i.e.

$$\begin{aligned} g(x) &= \cos(\pi x/bh) && \text{if } b > 1, \\ &= (1/2)\cos(\pi x/bh) && \text{if } b = 1, \\ &= 0 && \text{if } b < 1. \end{aligned} \tag{2.14}$$

2.2. *The Cardinal Series as an Orthogonal Expansion.* We shall show that if $g \in B(h)$ then $C(g, h, x)$ is an orthogonal expansion of g.

Using (2.3), (2.4) and (2.11), we see immediately that

$$\frac{1}{2\pi}\int_{-\pi/h}^{\pi/h} e^{-iwx}\, dw = \frac{1}{h}\operatorname{sinc}\frac{x}{h}, \tag{2.15}$$

and Eq. (2.6) therefore enables us to deduce that

$$\int_R \operatorname{sinc}\frac{s-x}{h}\operatorname{sinc}\frac{x-t}{h}\, dx = \frac{h^2}{2\pi}\int_{-\pi/h}^{\pi/h} e^{iw(s-t)}\, dw = h\operatorname{sinc}\frac{s-t}{h}. \tag{2.16}$$

Putting $s = mh$, $t = nh$ in this equation where m and n are integers, we have

LEMMA 2.6. *If m and n are integers, then*

$$\begin{aligned} \frac{1}{h}\int_R \operatorname{sinc}\frac{x-mh}{h}\operatorname{sinc}\frac{x-nh}{h}\, dx &= 0 \quad \text{if } m \neq n, \\ &= 1 \quad \text{if } m = n. \end{aligned} \tag{2.17}$$

Next, we prove

THEOREM 2.7. *If $g \in B(h)$, then*

$$g(z) = \sum_{k=-\infty}^{\infty} a_k \operatorname{sinc}\frac{z-kh}{h} \tag{2.18}$$

where

$$a_k = \frac{1}{h}\int_R g(x)\operatorname{sinc}\frac{x-kh}{h}\, dx = g(kh). \tag{2.19}$$

Proof. Clearly, $e^{-izw} \in L^2(-\pi/h, \pi/h)$ with respect to w for all $z \in R$. Hence, the Fourier series for e^{-izw} converges in $L^2(-\pi/h, \pi/h)$ to e^{-izw}. If we multiply e^{-izw} by e^{-inwh} and integrate over $(-\pi/h, \pi/h)$, we see that

$$e^{-izw} = \frac{h}{\pi}\sin\frac{\pi z}{h}\sum_{k=-\infty}^{\infty}\frac{(-1)^k e^{ikwh}}{z+kh} \tag{2.20}$$

a.e. on $(-\pi/h, \pi/h)$. Using (2.20), we replace e^{-izw} by its series expansion in (2.8). We then interchange the order of integration and summation, as we may, since the

mth partial sum of the series on the right of (2.20) is bounded by a constant independent of m, to obtain (2.18), where

$$a_k = \frac{1}{2\pi} \int_{-\pi/h}^{\pi/h} e^{-ikwh} G(w)\, dw. \tag{2.21}$$

By Theorem 2.3, $a_k = g(kh)$. This completes the proof.

Note that, since $G(w) \in L^2(-\pi/h, \pi/h)$, we have $\sum_{(k)} |a_k|^2 < \infty$. By Schwarz's inequality, it follows that the series on the right of (2.18) is absolutely convergent. Hence, if we use (2.18) to form the product of $g(x)$ and its complex conjugate, integrate over R, and use Lemma 2.6, we obtain

THEOREM 2.8. *If $g \in B(h)$, then*

$$\int_R |g(x)|^2\, dx = h \sum_{n=-\infty}^{\infty} |g(nh)|^2. \tag{2.22}$$

The set

$$\left\{ \frac{1}{h^{1/2}} \operatorname{sinc} \frac{x - nh}{h} \right\}_{n=-\infty}^{\infty}$$

is therefore a complete orthonormal set in $B(h)$.

In engineering terminology, the functions which satisfy (2.8) are termed band-limited. It is unusual that the coefficients in an orthogonal expansion should depend only on a single ordinate of the function and this property exhibits the nature of band-limited functions. It is, however, natural to anticipate that an orthogonal expansion of the conventional kind should exist for band-limited functions, i.e., an expansion in which each coefficient a_n depends on all the values of a function in a line segment; as a typical expansion of the conventional kind we may cite the representation of a function in terms of Legendre polynomials in which the general coefficient depends on all the function values on $(-1, 1)$. A remarkable expansion of this kind for band-limited functions has recently been obtained and it has led to important new results in the theory of band-limited and nearly band-limited functions (Pollak [11]).

3. Summability and Connection with Central Difference Series. In this section, we shall link the central-difference formulas with the cardinal function. We begin with a theorem of Ferrar on the summability of the cardinal series (see Whittaker [3, p. 69]).

THEOREM 3.1. *Let $\{a_n\}_{n=-\infty}^{\infty}$ be such that*

$$\sum_{n=2}^{\infty} (|a_n| + |a_{-n}|) \frac{\log n}{n} < \infty. \tag{3.1}$$

Then the series

$$C(x) = a_0 \operatorname{sinc} \frac{x}{h} + \sum_{n=1}^{\infty} \left(a_n \operatorname{sinc} \frac{x - nh}{h} + a_{-n} \operatorname{sinc} \frac{x + nh}{h} \right) \tag{3.2}$$

is absolutely convergent, and for all $a \in R$,

$$C(x) = \sum_{n=-\infty}^{\infty} C(a + nh) \operatorname{sinc} \frac{x - a - nh}{h}. \tag{3.3}$$

Let us now examine the formal Everett series

$$E(g, h, x) = \sum_{k=0}^{\infty}\left[\begin{bmatrix} s+k \\ 2k+1 \end{bmatrix}\delta^{2k}g_0 + \begin{bmatrix} r+k \\ 2k+1 \end{bmatrix}\delta^{2k}g_1\right] \tag{3.4}$$

where r and s are defined by $r = x/h$, $s = 1 - r$, $g_0 = g(0)$, $g_1 = g(h)$, and

$$\begin{bmatrix} r \\ m \end{bmatrix} = \frac{r(r-1)(r-2)\cdots(r-m+1)}{m!}. \tag{3.5}$$

If we express the Everett expansion $E(g, h, x)$ of $g(x)$ in terms of ordinates rather than differences, we obtain

$$\begin{aligned} E(g, h, x) = {} & \sum_{n=0}^{\infty}\begin{bmatrix} s+n \\ 2n+1 \end{bmatrix}\sum_{k=-n}^{n}(-1)^{n-k}\begin{bmatrix} 2n \\ n-k \end{bmatrix}g_k \\ & + \sum_{n=0}^{\infty}\begin{bmatrix} r+n \\ 2n+1 \end{bmatrix}\sum_{k=-n}^{n}(-1)^{n-k}\begin{bmatrix} 2n \\ n-k \end{bmatrix}g_{k+1}. \end{aligned} \tag{3.6}$$

We now formally interchange the order of summation to get

$$\begin{aligned} E(g, h, x) = \sum_{k=-\infty}^{\infty} g_k\Bigg[& \sum_{n=k}^{\infty}\begin{bmatrix} s+n \\ 2n+1 \end{bmatrix}(-1)^{n-k}\begin{bmatrix} 2n \\ n-k \end{bmatrix} \\ & + \sum_{n=k-1}^{\infty}\begin{bmatrix} r+n \\ 2n+1 \end{bmatrix}(-1)^{n-k-1}\begin{bmatrix} 2n \\ n-k+1 \end{bmatrix}\Bigg]. \end{aligned} \tag{3.7}$$

The quantity in square brackets on the right of (3.7) is easily recognized (see [9, p. 211]) to be the convergent expansion of sinc $[(x - kh)/h]$, i.e.,

$$\begin{aligned} \operatorname{sinc}\frac{x-kh}{h} = {} & \sum_{n=k}^{\infty}\begin{bmatrix} s+n \\ 2n+1 \end{bmatrix}(-1)^{n-k}\begin{bmatrix} 2n \\ n-k \end{bmatrix} \\ & + \sum_{n=k-1}^{\infty}\begin{bmatrix} r+n \\ 2n+1 \end{bmatrix}(-1)^{n-k+1}\begin{bmatrix} 2n \\ n-k+1 \end{bmatrix}. \end{aligned} \tag{3.8}$$

Hence, we formally have $E(g, h, x) = \sum_{k=-\infty}^{\infty} g(kh)$ sinc $[(x - kh)/h]$. Let us state a theorem of Whittaker [3, p. 64] which relates the convergence of these two series.

THEOREM 3.2. *If the series on the right of* (3.4) *converges, then the right of* (2.2) *is summable* (V.P. $1/n^2$)** *to the same sum. If the right of* (2.2) *converges, then the right of* (3.4) *converges to the same sum.*

If we replace the right of (3.4) by either the Gauss series, the Stirling series, or the Bessel series [9, Chapter 8], the statement of Theorem 3.2 remains valid.

We emphasize the dependence of the right of (2.2) and (3.4) on h, since even if these series converge, their sum will not in general be identical with $g(x)$. For example, if we sample the function $f = \sin \pi x/h$ at spacing h, the cardinal series (2.2) yields

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** De La Vallée Poussin summability.

$$C(f, h, x) = \frac{h}{\pi} \sin \frac{\pi}{h} (x - \tfrac{1}{2}h) \lim_{N\to\infty} \sum_{n=-N}^{N} \frac{1}{x - (2n+1)h/2}$$

$$= \sin \frac{\pi}{h} (x - \tfrac{1}{2}h) \cot \frac{\pi}{h} (x - \tfrac{1}{2}h) = \sin \frac{\pi}{h} x.$$

If we again sample $\sin \pi x/h$ at spacing $2h$, using the ordinates $(4n + 1)h/2$, (2.2) yields

$$C(f, 2h, x) = \frac{2h}{\pi} \sin \frac{\pi}{2h} (x - \tfrac{1}{2}h) \lim_{N\to\infty} \sum_{n=-N}^{N} \frac{(-1)^n}{x - (4n+1)h/2}$$

$$+ \sin \frac{\pi}{2h} (x - \tfrac{1}{2}h) \operatorname{cosec} \frac{\pi}{2h} (x - \tfrac{1}{2}h) = 1.$$

The following theorem, due to Nörlund [9, pp. 209, 218], gives the conditions under which $E(g, x, h) = g(x)$. In it we assume that x in (3.4) is replaced by a complex number z.

THEOREM 3.3. *If the Everett series on the right of* (3.4) *converges for some* z_0 *not an integer, then it converges for all complex* z *to an entire function* $E(g, h, z)$ *which satisfies* (2.7). *If* $g \in L^2(R)$ *and if* $g(z)$ *is an entire function which satisfies* (2.7), *then the right of* (3.4) *converges to* $g(z)$.

Finally, we illustrate a method of increasing the rate of convergence of the cardinal series, in the case when a derivative of $g(x)$ is known at $x = a$.

THEOREM 3.4. *Let* $g(x) = C(g, h, x)$. *Then*

$$g(x) = g(a) \operatorname{sinc} \left[\frac{x-a}{h}\right] + \frac{h}{\pi} g'(a) \sin \pi[(x-a)/h] + \frac{x-a}{h} \sum_{n=-\infty, n\neq 0}^{\infty} \frac{1}{n} g(a+nh) \operatorname{sinc} \left[\frac{x-a-nh}{h}\right]. \tag{3.9}$$

Proof. Since $g(x) = C(g, h, x)$, we have for all $a \in R$,

$$g(x) = \sum_{n=-\infty}^{\infty} g(a+nh) \operatorname{sinc} \frac{x-a-nh}{h}. \tag{3.10}$$

Differentiating this series, we have

$$g'(a) = - \sum_{n=-\infty, n\neq 0}^{\infty} \frac{(-1)^n}{nh} g(a+nh). \tag{3.11}$$

If we multiply both sides of (3.11) by $(h/\pi) \sin \pi(x - a)/h$ and substitute into (3.10), we obtain (3.9).

4. The Difference Between $g(x)$ **and** $C(g, h, x)$. We shall now obtain a bound on the difference between an analytic function $g(z)$ and $C(g, h, z)$, in the case when z is real.

THEOREM 4.1. *Let* $g(z) = g(x + iy)$ *be analytic within the strip* $|y| \leqq d$, *where* $d > 0$. *For all* $|y| \leqq d$, *let* $g(x + iy) \to 0$ *as* $x \to \pm \infty$, *and let* $g(x + iy) \in L^2(R)$ *with respect to* x. *Then, for all* $x \in R$,

$$g(x) - C(g, h, x) = \epsilon \tag{4.1}$$

where

$$|\epsilon| \leqq \frac{\left|\sin \frac{\pi x}{h}\right|}{\sinh \frac{\pi d}{h}} \frac{M(d)}{d} \tag{4.2}$$

and where

$$M(d) = \max_{v=\pm d} \left\{\int_R |g(x + iv)|^2 \, dx\right\}^{1/2}. \tag{4.3}$$

Proof. Let r and s be real, and let a contour L_n be defined by $L_n = \{r + is : |r| \leqq n + \frac{1}{2}$ and $s = \pm d; r = \pm (n + \frac{1}{2})$ and $|s| \leqq d\}$. Then L_n encloses the points $x = kh$, $k = 0, \pm 1, \cdots, \pm n$. Hence, we deduce from Cauchy's theorem that

$$g(x) = \sum_{k=-n}^{n} g(kh) \operatorname{sinc} \frac{x - kh}{h} + \frac{\sin (\pi x/h)}{2\pi i} \int_{L_n} \frac{g(z) \, dz}{(z - x) \sin (\pi z/h)}. \tag{4.4}$$

Upon letting $n \to \infty$, we obtain the equation

$$\epsilon = \lim_{n\to\infty} \frac{\sin (\pi x/h)}{2\pi i} \int_{-(n+1/2)}^{(n+1/2)} \left[\frac{g(t + id)}{t - x + id} - \frac{g(-t - id)}{t + x + d}\right] \frac{dt}{\sinh \pi(d + it)/h} \tag{4.5}$$

since $g(x \pm iy) \to 0$ as $x \to \pm \infty$, for all $|y| \leqq d$. Upon noting that

$$|\sinh [\pi(d + it)/h]| \geqq e^{\pi d/h}(1 - e^{-2\pi d/h})/2,$$

replacing each term on the right by its absolute value, and using Schwarz's inequality for integrals, we obtain (4.2).

Letting the bound in (4.2) depend upon d enables us to get better bounds by varying d. For example, if

$$M(d) \leqq (a - d)^{-m} \tag{4.6}$$

where a, $m > 0$, then

$$|\epsilon| \leqq \frac{(\pi/(mh))^m}{a - mh/\pi} \cdot \frac{\left|\sin \frac{\pi}{h} x\right|}{\sinh \left[\frac{\pi}{h}\left(a - \frac{mh}{\pi}\right)\right]}, \tag{4.7}$$

while if

$$M(d) \leqq \exp (ad^m) \tag{4.8}$$

where a, $m > 1$, then

$$|\epsilon| \leqq \left(\frac{amh}{\pi}\right) 1/(m - 1) \cdot \frac{\left|\sin \frac{\pi}{h} x\right|}{\sinh \left[\frac{\pi}{h}\left(1 - \frac{1}{m}\right)\left(\frac{\pi}{mh}\right)^{1/(m-1)}\right]}. \tag{4.9}$$

5. Examples.

5.1. *The Approximation of* $1/(x - a)$, $a \neq nh$. If we take $g(x) = 1/(x - a)$ and choose the contour L_n as in (4.4) so that the point $z = a \neq nh$ is in L_n, we obtain an explicit error term since the contour integral tends to zero as $n \to \infty$. Thus

$$\frac{1}{z-a} - C(g, h, z) = \frac{\sin \pi z/h}{\sin \pi a/h} \cdot \frac{1}{z-a}, \tag{5.1}$$

and we note the following:

1. $C(g, h, z)$ is an entire function (and hence the singularity at $z = a$ is annihilated).
2. The difference $1/(z-a) - C(g, h, z)$ vanishes at $z = nh$.

If in (5.1) we first replace a by ia, then replace a by $-ia$ and subtract the two results thus obtained, we get

$$C\left(\frac{1}{x^2 + a^2}, h, z\right) = \frac{1}{z^2 + a^2}\left[1 + \frac{z \sin (\pi z/h)}{a \sin (\pi a/h)}\right]. \tag{5.2}$$

This example has been much discussed—sometimes inadequately—in the literature. The singularities of the function $1/(z^2 + a^2)$ are irrelevant in discussing the convergence of the central difference expansion of this function. It can be seen that if a is real the discrepancy between $1/(z^2 + a^2)$ and $C(1/(x^2 + a^2), h, z)$ can be made as small as we please by choosing h sufficiently small, provided $|y| < a$.

To illustrate, let us apply the Newton-Gauss series

$$\begin{aligned} \mathrm{NG}(g, h, z) = \mathrm{NG}(g, 1, z) &= g(0) + \left[z\delta_{1/2} + \frac{z(z-1)}{2!}\delta_0^2\right]g \\ &+ \left[\frac{z(z^2-1)}{3!}\delta_{1/2}^3 + \frac{z(z^2-1^2)(z^2-2^2)}{4!}\delta_0^4\right]g + \cdots, \end{aligned} \tag{5.3}$$

where, e.g., $\delta_{1/2}g = \delta g(\frac{1}{2})$, to $g(z) = 1/(1 + z^2)$ at $z = i$. We then get the sum

$$\sum_{k=0}^{\infty} \frac{1}{1 + k^2} = \tfrac{1}{2}(1 + \pi \coth \pi); \tag{5.4}$$

this can also be obtained by setting $a = 1$ and taking the limit as $z \to i$ in (5.2).

5.2. *The Approximation of* $\exp(i\lambda x)$: *Aliasing*. When $g(z) = \exp(i\lambda z)$, the conditions of Theorem 4.1 are not satisfied and it is not convenient to use the contour integral method of Section 2.4 to discuss the error in approximating $g(z)$ by its Whittaker cardinal function

$$C(g, h, z) = \sin \frac{\pi z}{h} \sum_{p=-\infty}^{\infty} \frac{(-1)^p \exp (i\lambda ph)}{\pi(z - ph)/h}. \tag{5.5}$$

This series can be evaluated if we make use of the following known result (see Bromwich [12, p. 393]).

THEOREM 5.1. *Let θ be the difference between v and the integer nearest to v, and let $|\theta| \neq \frac{1}{2}$. Then*

$$\frac{\sin \pi t}{\pi} \sum_{p=-\infty}^{\infty} (-1)^p \frac{\exp (2p\pi i v)}{t - p} = \exp (2t\pi i\theta). \tag{5.6}$$

Hence if we write

$$\lambda = 2\pi N/h + w; \qquad |w| < \pi/h \tag{5.7}$$

and make use of (5.6), we get

$$C(g, h, z) = \exp [iwz], \tag{5.8}$$

i.e., we approximate the short-period ($<2h$) function $\exp(i\lambda x)$ by a function whose period in x is greater than $2h$. As we might expect, if a rapidly oscillating function is sampled at spacing h, the rapidity of the oscillation cannot in general be discovered. Since N in (5.7) is *any* integer, $C(g, h, z)$ is alias for the infinite set of functions

$$\exp(i\lambda x) = \exp[i(2\pi N/h + w)], \qquad N = 0, \pm 1, \pm 2, \cdots. \tag{5.9}$$

Thus, while the higher frequencies are distorted in approximation of g by its cardinal function, more faithful representation can be obtained by decreasing h. Note, however, that $C(\sin 1/x, h, z) \not\equiv \sin 1/z$ for any $h > 0$ and we must in general always expect some distortion.

5.3. *An Expansion of* $1/\Gamma(x)$. Let us take $h = 1$ and let us note that $1/\Gamma(x)$ vanishes for $x = 0, -1, -2, \cdots$. We have

$$C(1/\Gamma, 1, z) = \frac{\sin \pi z}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{(z-n)(n-1)!}. \tag{5.10}$$

We know, however, from analysis that we can write $1/\Gamma(z)$ in the form

$$1/\Gamma(z) = \frac{\sin \pi z \Gamma(1-z)}{\pi} = \frac{\sin \pi z}{\pi} \int_0^{\infty} e^{-t} t^{-z}\, dt, \qquad (\mathrm{Rl}\, z < 1). \tag{5.11}$$

If we split the range of integration into two ranges $(0, 1)$ and $(1, \infty)$, expand e^{-t} in powers of t and integrate each term over $(0, 1)$, we obtain

$$1/\Gamma(z) = \frac{\sin \pi z}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{(z-n)(n-1)!} + \frac{\sin \pi z}{\pi} \int_1^{\infty} e^{-t} t^{-z}\, dt \tag{5.12}$$

without any restriction on z.

Upon comparing (5.10) and (5.12), we see that *** the approximation $C(1/\Gamma, 1, z)$ is very good when Rl z is large and positive, but very poor when Rl z is large and negative.

5.4 *Harmonic Analysis.*† In the interval $-1 \leqq x \leqq 1$, the function $|x|$ has the Fourier expansion

$$|x| = \frac{1}{2} - \frac{4}{\pi^2} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \cos[(2n+1)\pi x]. \tag{5.13}$$

An alternate periodic representation can be obtained by harmonic analysis, using only the ordinates at the points $0, \pm\frac{1}{4}, \pm\frac{1}{2}, \pm\frac{3}{4}, \pm 1$ ($h = \frac{1}{4}$):

$$|x| \cong \frac{1}{2} - \frac{2+\sqrt{2}}{8} \cos \pi x - \frac{2-\sqrt{2}}{8} \cos 3\pi x. \tag{5.14}$$

The series (5.13) can be converted into the right of (5.14) by application of the procedure of Section 5.2 as follows. By taking $h = \frac{1}{4}$, the term $\cos 5\pi x$ is converted into $\cos 3\pi x$ (note that $\cos 5\pi x = \cos 3\pi x$ at $x = nh$). Similarly the term $\cos 7\pi x$ is converted into $\cos \pi x$. Carrying out the conversion term by term, the Fourier series (5.13) converts to

*** We cannot obtain a better approximation in this instance by decreasing h since the cardinal series for $1/\Gamma(x)$ diverges when $h < 1$.

† The name is unfortunate but the process is frequently used in the periodic analysis of empirical data, such as hourly temperature records.

$$\tfrac{1}{2} - 4/\pi^2(1/1^2 + 1/7^2 + 1/9^2 + 1/15^2 + 1/17^2 + \cdots) \cos \pi x$$
$$- 4/\pi^2(1/3^2 + 1/5^2 + 1/11^2 + 1/13^2 + \cdots) \cos 3\pi x;$$

this is identical with the representation on the right of (5.14).

Note the considerable distortion introduced in converting short period harmonics into long period harmonics. A similar phenomenon occurs when $1/\Gamma(z)$ is represented by its cardinal series, with $h = 1$. The function $1/\Gamma(z)$ oscillates with large amplitude for negative real z and conversion (i.e. the cardinal representation) introduces considerable distortion.

5.5. *Quadrature by Trapezoidal Rule.* If we integrate the cardinal series for g termwise, we obtain the trapezoidal rule approximation to $\int_R g(x)\,dx$, i.e.

$$\int_R g(x)\,dx = h \sum_{n=-\infty}^{\infty} g(nh) + \eta. \tag{5.15}$$

We use a contour integral method to obtain a bound on η. Letting L_n be defined as in the proof of Theorem 4.1, we see if $g(z)$ is analytic in the interior of L_n, and continuous on L_n, then

$$\tfrac{1}{2}\pi i \int_{L_n} \pi g(z) \cot \pi z/h\,dz = h \sum_{k=-n}^{n} g(kh). \tag{5.16}$$

Setting $(\tfrac{1}{2}i)\cot(\pi z/h) = -\tfrac{1}{2} - e^{2\pi i z/h}/(1 - e^{2\pi i z/h})$ on the upper, and $(\tfrac{1}{2}i)\cot(\pi z/h) = \tfrac{1}{2} - e^{-2\pi i z/h}/(1 - e^{-2\pi i z/h})$ on the lower segment of L_n that is parallel to the x-axis, we obtain

THEOREM 5.2. *Let $g(z) = g(x + iy)$ be analytic in the strip $|y| \leqq d$ and let $g(x + iv) \to 0$ as $x \to \pm\infty$ for all $|v| \leqq d$. If $g(x + iy) \in L^1(R)$ with respect to x for all $|y| \leqq d$, then η in* (5.15) *is bounded as*

$$|\eta| \leqq \frac{2e^{-2\pi d/h}}{1 - e^{-2\pi d/h}} N(d), \tag{5.17}$$

where

$$N(d) = \max_{v=\pm d} \int_R |g(x + iv)|\,dx. \tag{5.18}$$

Proof. If the conditions of the theorem are satisfied, then

$$\eta = -\int_R \frac{e^{-2\pi(d-it)/h}}{1 - e^{-2\pi(d-it)/h}} g(t + id)\,dt - \int_R \frac{e^{-2\pi(d+it)/h}}{1 - e^{-2\pi(d+it)/h}} g(t - id)\,dt \tag{5.19}$$

from which (5.17) follows.

For example, if

$$N(d) \leqq (a - d)^{-m} \qquad (a,\ m > 0), \tag{5.20}$$

then

$$|\eta| \leqq (2/mh)^m e^{-2\pi(a-mh/\pi)/h}/[1 - e^{-2\pi(a-mh/\pi)/h}]. \tag{5.21}$$

If

$$N(d) \leqq e^{ad^m} \qquad (m > 1), \tag{5.22}$$

then

$$(5.23)\quad |\eta| \leqq \exp\left[a(1-m)\left(\frac{2}{amh}\right)^{m/(m-1)}\right] \Big/ \left\{1 - \exp\left[a(1-m)\left(\frac{2}{amh}\right)^{m/(m-1)}\right]\right\}.$$

It is interesting to compare numerical values of these bounds and those in Section 4. Upon taking $g(x) = e^{-x^2}$ we tabulate bounds given by (4.9) and (5.23) in Table 1 for several values of h.

TABLE I

h	*bound on* ϵ	*bound on* η
1	1.9×10^{-1}	1.0×10^{-4}
$\frac{1}{2}$	6.0×10^{-5}	5.3×10^{-9}
$\frac{1}{4}$	1.6×10^{-9}	1.4×10^{-17}

5.6. *The Approximate Solution of Integral Equations.* Let us consider the cardinal representation applied to the approximate solution of the integral equation

$$(5.24)\qquad g(x) = \int_R K(x, y)g(y)\, dy + f(x).$$

Instead of (5.24), we consider the equation

$$(5.25)\qquad \sum_{k=-m}^{m} g_k C_k(x) = \int_R \sum_{j=-m}^{m} \sum_{k=-m}^{m} K_{k,j} g_j C_k(x) C_k(y)\, dy + \sum_{k=-m}^{m} f_k C_k(x)$$

where $C_k(x) = \operatorname{sinc}[(x - kh)/h]$, $g_k = g(kh)$, $K_{k,j} = K(kh, jh)$ and $f_k = f(kh)$. Using the orthogonality property of the C_k's, we thus obtain

$$(5.26)\qquad \sum_{k=-m}^{m} g_k C_k(x) = \sum_{k=-m}^{m} \left[f_k + h \sum_{j=-m}^{m} K_{k,j} g_j\right] C_k(x).$$

Multiplying through by $C_k(x)$ and integrating over R leads us to the equations

$$(5.27)\qquad \sum_{j=-m}^{m} [\delta_{kj} - hK_{k,j}]g_j = f_k, \qquad k = -m, -m+1, \cdots, m,$$

where $\delta_{kj} = 1$ if $k = j$, 0 if $k \neq j$.

Solving this linear system for the g_j's, we obtain an approximation to the solution of (5.24):

$$(5.28)\qquad g(x) \cong \sum_{j=-m}^{m} g_j C_j(x).$$

5.7. *Fourier Transforms and Fourier Series.* Let $g \in L^2(R)$. Let us multiply the cardinal series $C(g, h, t)$ of g by e^{ixt} (x, t real) and let us formally integrate each term of the resulting series. Using (2.11), we obtain

$$(5.29)\qquad \begin{aligned} \int_R e^{ixt} C(g, h, t)\, dt &= h \sum_{k=-\infty}^{\infty} g(kh) e^{ikhx} && \text{if } |x| < \pi/h, \\ &= 0 && \text{if } |x| > \pi/h. \end{aligned}$$

Now, suppose for example that $g(t)e^{itx}$ satisfies the conditions of Theorem 5.2. Then, clearly, the right-hand side of (5.29) approaches the Fourier transform of g as $h \to 0$.

Let $f \in L^2(-\pi/h, \pi/h)$. Then f has a Fourier series representation

$$f(x) = \sum_{k=-\infty}^{\infty} a_k e^{ikhx} \tag{5.30}$$

on $(-\pi/h, \pi/h)$, and furthermore

$$\int_{-\pi/h}^{\pi/h} |f(x)|^2 \, dx = \sum_{k=-\infty}^{\infty} |a_k|^2. \tag{5.31}$$

Now, suppose that

$$h \sum_{k=-\infty}^{\infty} |g(kh)|^2 \to \int_R |g(t)|^2 \, dt \tag{5.32}$$

as $h \to 0$. Using Theorem 2.8 we obtain the following theorem:

THEOREM 5.3. *There exists an isomorphism between* $B(h)$ *and* $L^2(-\pi/h, \pi/h)$. *If the function g satisfies* (5.32), *then the right of* (5.29) *converges in* $L^2(R)$ *to the Fourier transform of g as* $h \to 0$.

Of course, we could also have deduced the existence of an isomorphism between $B(h)$ and $L^2(-\pi/h, \pi/h)$ from Theorem 2.3. However, our motivation for the above approach stems from textbooks in mathematics (see [13, p. 88]) which deal with Fourier series and Fourier integrals, and which state that it would be natural to obtain the Fourier transform of a function from its Fourier series over a finite interval $[-T, T]$ by letting $T \to \infty$. We have not seen a natural development of this type in any textbook; see however Warmbrod [14] where a procedure of this type is carried out.

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