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Miniaturized Tables of Bessel Functions. II*

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Abstract. In a previous study, we discussed the expansion of two-parameter functions in a double series of Chebyshev polynomials, and, in particular, we presented coefficients for the evaluation of the modified Bessel function $(2z/\pi)^{1/2}e^zK_\nu(z)$ to 20 decimals for all $z \geqq 5$ and all ν, $0 \leqq \nu \leqq 1$. In the present study, we give similar coefficients for the evaluation of $ge^{-z}z^{-\mu}I_\nu(z)$ to at least 20 decimals where $I_\nu(z)$ is the modified Bessel function of the first kind and g and μ are certain constants which depend on the range of the parameter and variable for four different situations. The ranges are (1) $0 < z \leqq 8$, $0 \leqq \nu \leqq 4$; (2) $0 < z \leqq 8$, $4 \leqq \nu \leqq 8$; (3) $z \geqq 8$, $-1 \leqq \nu \leqq 0$; (4) $z \geqq 8$, $0 \leqq \nu \leqq 1$.

1. Introduction. In a previous study [1], we discussed the expansion of two-parameter functions in a double series of Chebyshev polynomials, and, in particular, we presented coefficients for the evaluation of the modified Bessel function $(2z/\pi)^{1/2} \times e^zK_\nu(z)$ to 20 decimals for all $z \geqq 5$ and all ν, $0 \leqq \nu \leqq 1$. Since $K_\nu(z) = K_{-\nu}(z)$ and $K_\nu(z)$ satisfies a three-term recurrence formula which is stable in the forward direction, we have in essence coefficients for the evaluation of $K_\nu(z)$ for all $z \geqq 5$ and all $\nu \geqq 0$.

In the present study, we give similar coefficients for the evaluation of $ge^{-z}z^{-\mu}I_\nu(z)$ to at least 20 decimals where $I_\nu(z)$ is the modified Bessel function of the first kind and g and μ are certain constants which depend on the range of the parameter and variable for four different situations as follows.

	z range	*ν range*	μ	g
(1)	$0 < z \leqq 8$	$0 \leqq \nu \leqq 4$	ν	1
(2)	$0 < z \leqq 8$	$4 \leqq \nu \leqq 8$	ν	1
(3)	$z \geqq 8$	$-1 \leqq \nu \leqq 0$	$-\frac{1}{2}$	$(2\pi)^{-1/2}$
(4)	$z \geqq 8$	$0 \leqq \nu \leqq 1$	$-\frac{1}{2}$	$(2\pi)^{-1/2}$

The recursion formula for $I_\nu(z)$ is always stable in the backward direction but only conditionally stable in the forward direction. Thus, even with the coefficients given here, we still lack coefficients to compute $e^{-z}I_\nu(z)$ for all real z and for ν sufficiently large. A study to correct this deficiency is under way and will be reported at a later date.

2. Chebyshev Expansions for $I_\nu(z)$. In [2, Vol. 2, pp. 338–340, 359–367], we gave coefficients for the expansion of $z^{-\nu}I_\nu(z)$ in series of Chebyshev polynomials for

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$0 < z \leqq 8$, $\nu = 0, \pm\frac{1}{4}, \pm\frac{1}{3}, \pm\frac{1}{2}, \pm\frac{2}{3}, \pm\frac{3}{4}, 1$, and, similarly, for the expansion of $(2\pi z)^{-1/2}e^{-z}I_\nu(z)$ for $z \geqq 8$, $\nu = 0, \frac{1}{4}, \frac{1}{3}, \frac{2}{3}, \frac{3}{4}, 1$. The coefficients for the range $0 < z \leqq 8$ are based on the ${}_0F_1$ representation for $I_\nu(z)$ which does not directly reflect the fact that for fixed ν, $I_\nu(z)$ grows exponentially with z as z increases in the sector $|\arg z| < \pi/2$. Now, $I_\nu(z)$ has a representation in terms of a ${}_1F_1$ which does reflect this exponential behavior and, in this present paper, development of the desired coefficients is based on this representation for $0 < z \leqq 8$. The representation used in the cited reference for $z \geqq 8$ is also used in our present study to derive the desired coefficients already noted.

From [2, Vol. 1, p. 213], we have

$$z^{-\nu}e^{-z}I_\nu(z) = [2^\nu\Gamma(\nu+1)]^{-1}\, {}_1F_1(a; c; -2z), \quad a = c/2 = \tfrac{1}{2} + \nu. \tag{1}$$

In general, from [2, Vol. 2, p. 35],

$$ {}_1F_1(a; c; z) = \sum_{k=0}^{\infty} G_k(a, c, \lambda)T_k^*(z/\lambda), \tag{2}$$

$$G_k(a, c, \lambda) = \frac{\epsilon_k(a)_k\lambda^k}{2^{2k}(c)_k k!}\, {}_2F_2\left(\begin{matrix} a+k, & \frac{1}{2}+k \\ c+k, & 1+2k \end{matrix}\,\middle|\, \lambda\right), \tag{3}$$

$$\begin{aligned} \frac{2G_k(a, c, \lambda)}{\epsilon_k} = {} & \frac{(k+1)}{(k+a)}\left\{-\frac{(k+3-a)}{(k+2)} + \frac{4(k+c)}{\lambda}\right\}G_{k+1}(a, c, \lambda) \\ & + \frac{2}{(k+a)}\left\{\frac{1}{2}(k+a) + \frac{2(k+1)(k+3-c)}{\lambda}\right\}G_{k+2}(a, c, \lambda) \\ & + \frac{(k+1)(k+3-a)}{(k+2)(k+a)}G_{k+3}(a, c, \lambda). \end{aligned} \tag{4}$$

In the above,

$$\epsilon_k = 1 \quad \text{if } k = 0, \qquad \epsilon_k = 2 \quad \text{if } k > 0. \tag{5}$$

Using [2, Vol. 1, p. 244], we find that for a, c and λ fixed,

$$\begin{aligned} G_k(a, c, \lambda) = {} & \frac{\Gamma(c)(\lambda/4)^k k^{a-c}}{\Gamma(a)k!} \\ & \cdot\left[1 + \frac{\lambda^2 - 8\lambda(c-a) - 8(c-a)(c+a-1)}{16k} + O(k^{-2})\right]. \end{aligned} \tag{6}$$

Thus, the expansion formula (2) converges and since the ${}_1F_1$ in (2) is one when $z = 0$, it follows that

$$\sum_{k=0}^{\infty} (-)^k G_k(a, c, \lambda) = 1. \tag{7}$$

Further, after the manner of the discussion given in [2, Vol. 2, pp. 159–166], we can show that use of the recursion formula (4) in the backward direction is convergent. Thus, for a fixed λ, we can generate the coefficients $G_k(a, c, \lambda)$ for given values of a and c. Suppose for example that c is fixed and we permit a to vary. Then, we can find coefficients $D_{r,k}(c, \lambda)$ such that

$$(8) \qquad G_k(a, c, \lambda) = \sum_{r=0}^{\infty} D_{r,k}(c, \lambda) T_r^*(a/\omega), \qquad 0 \leqq a \leqq \omega,$$

and so achieve a double series of Chebyshev polynomials for the evaluation of ${}_1F_1(a; c; z)$ for c fixed, valid for $0 \leqq z \leqq \lambda$ and $0 \leqq a \leqq \omega$. The manner of getting $D_{r,k}(c, \lambda)$ has been given in [1] and we omit further details.

Next, we seek a descending-type expansion in series of Chebyshev polynomials for the evaluation of $I_\nu(z)$ in the neighborhood of $z = +\infty$. To this end, we can write [2, Vol. 1, p. 226, Eq. (9)], [2, Vol. 2, p. 22, Eq. (10)],

$$(9) \qquad I_\nu(z) = (2\pi z)^{-1/2} e^z F_\nu(z),$$

$$(10) \qquad F_\nu(z) = G_{1,2}^{1,1}\left(2z \,\middle|\, \begin{matrix} 1 \\ \frac{1}{2}+\nu, \quad \frac{1}{2}-\nu \end{matrix}\right),$$

$$(11) \qquad F_\nu(z) = \sum_{k=0}^{\infty} M_k(\nu, \lambda) T_k^*(\lambda/z), \qquad \lambda \text{ fixed}, \lambda/z \leqq 1, z > 0,$$

$$(12) \qquad \begin{aligned} M_k(\nu, \lambda) &= \pi^{-1/2} \epsilon_k (-)^k G_{2,3}^{2,1}\left(2\lambda \,\middle|\, \begin{matrix} 1-k, & k+1 & \\ \frac{1}{2}, & \frac{1}{2}+\nu, & \frac{1}{2}-\nu \end{matrix}\right), \\ M_k(\nu, \lambda) &= \pi^{-1/2} \epsilon_k (-)^k G_{3,2}^{1,2}\left(\frac{1}{2\lambda} \,\middle|\, \begin{matrix} \frac{1}{2}, & \frac{1}{2}-\nu, & \frac{1}{2}+\nu \\ k, & -k & \end{matrix}\right), \end{aligned}$$

and from [2, Vol. 2, pp. 153, 154 and Remark 1, p. 155], we have the recursion formula

$$(13) \qquad \begin{aligned} \frac{2M_k(\nu, \lambda)}{\epsilon_k} = {} & 2(k+1)\left\{1 - \frac{(2k+3)(k+3/2+\nu)(k+3/2-\nu)}{2(k+2)(k+\frac{1}{2}+\nu)(k+\frac{1}{2}-\nu)} \right. \\ & \left. + \frac{8\lambda}{(k+\frac{1}{2}+\nu)(k+\frac{1}{2}-\nu)}\right\} M_{k+1}(\nu, \lambda) \\ & + \left\{1 - \frac{2(k+1)(2k+3+4\lambda)}{(k+\frac{1}{2}+\nu)(k+\frac{1}{2}-\nu)}\right\} M_{k+2}(\nu, \lambda) \\ & - \frac{(k+1)(k+5/2+\nu)(k+5/2-\nu)}{(k+2)(k+\frac{1}{2}+\nu)(k+\frac{1}{2}-\nu)} M_{k+3}(\nu, \lambda), \qquad k \geqq 0. \end{aligned}$$

Actually, (13) is not valid if ν is half an odd integer unless $k + \frac{1}{2} - \nu > 0$. If, for example, $\nu = \frac{1}{2} + n$, (13) is only valid for $k > n$. However, we can get a further relation if first we multiply through by $k + \frac{1}{2} - \nu$ and then set $k + \frac{1}{2} - \nu = 0$ for $k = n$. In particular, if $\nu = \frac{1}{2}$, we have

$$(14) \qquad \begin{aligned} \frac{2M_k(\frac{1}{2}, \lambda)}{\epsilon_k} = {} & \left[\frac{8\lambda - 3(k+1)}{k}\right] M_{k+1}(\tfrac{1}{2}, \lambda) - \left[\frac{8\lambda + 3(k+2)}{k}\right] M_{k+2}(\tfrac{1}{2}, \lambda) \\ & - \frac{(k+3)}{k} M_{k+3}(\tfrac{1}{2}, \lambda), \qquad k > 0, \end{aligned}$$

$$(15) \qquad (8\lambda - 3) M_1(\tfrac{1}{2}, \lambda) = (8\lambda + 6) M_2(\tfrac{1}{2}, \lambda) + 3M_3(\tfrac{1}{2}, \lambda).$$

It can be shown that

$$(16) \qquad M_0(\tfrac{1}{2}, \lambda) = 2\pi^{-1/2} \operatorname{Erf}(x), \qquad x^2 = 2\lambda, \qquad \operatorname{Erf}(x) = \int_0^x e^{-t^2}\, dt,$$

(17) $$M_1(\tfrac{1}{2}, \lambda) = 8\lambda[1 - M_0(\tfrac{1}{2}, \lambda)] - 4(2\lambda/\pi)^{1/2}e^{-2\lambda}.$$

With $z \to +\infty$, (11) yields the useful normalization equation

(18) $$\sum_{k=0}^{\infty} (-)^k M_k(\nu, \lambda) = 1.$$

From [2, Vol. 2, pp. 23, 24],

(19) $$M_k(\nu, \lambda) \sim k^{-1}[u \exp\{-3(2\lambda k^2 e^{i\pi})^{1/3}\} + v \exp\{-3(2\lambda k^2 e^{-i\pi})^{1/3}\}]$$

where u and v are constants. The two other linearly independent solutions of (14) can be taken in a form such that they are

(20) $$O(k^{-1} \exp\{3(2\lambda k^2)^{1/3}\}) \quad \text{and} \quad O(k^{-1} \exp\{-3(2\lambda k^2 e^{i\pi})^{1/3}\}).$$

It follows that the desired solution of (14) is not minimal in the sense of Gautschi [3] or not antidominant in the sense of Wimp [4], and consequently the backward recursion process for the evaluation of $M_k(\nu, \lambda)$ will fail unless modified. The necessary modification is discussed in [2, Vol. 2, pp. 163–164] and studied further in Wimp [4, Theorem 3]. We now describe this procedure.

Let N be a large positive integer. Put

(21) $$g_{N+k}^{(N)} = 0, \qquad k = 2, 3, \cdots, \qquad g_{N+1}^{(N)} = 1$$

and compute $g_n^{(N)}$, $n = N, N-1, \cdots, 0$ from (13) with $M_k(\nu, \lambda)$ replaced by $g_k^{(N)}$. (Here we assume that ν is not half an odd integer. The case when $\nu = \frac{1}{2}$ is treated later.) Put

(22) $$M_n^{(N)}(\nu, \lambda) = \rho^{(N)} g_n^{(N)}, \qquad n = 0, 1, \cdots, N+1, \quad \rho^{(N)} = \left(\sum_{n=0}^{N+1} (-)^n g_n^{(N)}\right)^{-1}.$$

Let N_1, N_2 be two different N values. We can find a number μ depending on N_1 and N_2 such that

(23) $$\mu \sum_{k=0}^{N_1+1} M_k^{(N_1)}(\nu, \lambda) + (1-\mu) \sum_{k=0}^{N_2+1} M_k^{(N_2)}(\nu, \lambda) = (2\pi\lambda)^{1/2} e^{-\lambda} I_\nu(\lambda).$$

Then

(24) $$\lim_{N_1\to\infty; N_2\to\infty; N_1\neq N_2} [\mu M_k^{(N_1)}(\nu, \lambda) + (1-\mu) M_k^{(N_2)}(\nu, \lambda)] = M_k(\nu, \lambda), \qquad k = 0, 1, \cdots.$$

If ν is half an odd integer, another technique must be used as the process just described breaks down due to the presence of the product $(k + \frac{1}{2} + \nu)(k + \frac{1}{2} - \nu)$. To illustrate, consider the case $\nu = \frac{1}{2}$. In this event,

(25) $$F_{1/2}(z) = 1 - e^{-2z}.$$

We have need for the three normalization relations

(26) $$1 - e^{-2\lambda} = \sum_{k=0}^{\infty} M_k(\tfrac{1}{2}, \lambda),$$

(27) $$1 = \sum_{k=0}^{\infty} (-)^k M_k(\tfrac{1}{2}, \lambda),$$

$$(28) \qquad 1 - e^{-4\lambda} = \sum_{k=0}^{\infty} (-)^k M_{2k}(\tfrac{1}{2}, \lambda),$$

which come from (9)–(11) when $\nu = \frac{1}{2}$ and $z = 2\lambda, +\infty$ and 4λ, respectively.

Again, let N be a large positive integer, set

$$(29) \qquad g_{N+k}^{(N)} = 0, \qquad k = 2, 3, \cdots, \qquad g_{N+1}^{(N)} = 1,$$

and compute

$$g_n^{(N)}, \qquad n = N, N-1, \cdots, 1,$$

from (13) with $M_k(\frac{1}{2}, \nu)$ replaced by $g_k^{(N)}$. Let

$$(30) \qquad M_k^{(N)} = \rho^{(N)} g_k^{(N)}, \qquad k = 1, 2, \cdots, \qquad M_0^{(N)} = g_0^{(N)}.$$

Then from (26) and (27), respectively, we have

$$(31) \qquad g_0^{(N)} + \rho^{(N)} \sum_{k=1}^{\infty} g_k^{(N)} = 1 - e^{-2\lambda},$$

$$(32) \qquad g_0^{(N)} + \rho^{(N)} \sum_{k=1}^{\infty} (-)^k g_k^{(N)} = 1.$$

Thus

$$(33) \qquad \rho^{(N)} = \frac{-e^{-2\lambda}}{2 \sum_{k=0}^{\infty} g_{2k+1}^{(N)}}$$

and $g_0^{(N)}$ can be recovered from either (31) or (32). Let N_1, N_2 be two different N numbers. We can find a number μ depending on N_1 and N_2 such that

$$(34) \qquad \mu \sum_{k=0}^{N_1+1} (-)^k M_{2k}^{(N_1)}(\tfrac{1}{2}, \lambda) + (1-\mu) \sum_{k=0}^{N_2+1} (-)^k M_{2k}^{(N_2)} (\tfrac{1}{2}\, \lambda) = 1 - e^{-4\lambda}.$$

Then

$$(35) \qquad \lim_{N_1 \to \infty; N_2 \to \infty; N_1 \neq N_2} [\mu M_k^{(N_1)}(\tfrac{1}{2}, \nu) + (1-\mu) M_k^{(N_2)}(\tfrac{1}{2}, \nu)] = M_k(\tfrac{1}{2}, \lambda),$$
$$k = 0, 1, \cdots.$$

The coefficients can be checked using (16) and (17). Alternatively, we can make use of (17) to find a number μ^* such that

$$(36) \qquad \begin{aligned} &8\lambda[\mu^* M_0^{(N_1)}(\tfrac{1}{2}, \lambda) + (1-\mu^*) M_0^{(N_2)}(\tfrac{1}{2}, \lambda)] \\ &\quad + \mu^* M_1^{(N_1)}(\tfrac{1}{2}, \lambda) + (1-\mu^*) M_1^{(N_2)}(\tfrac{1}{2}, \lambda) = 8\lambda - 4(2\lambda/\pi)^{1/2} e^{-2\lambda}. \end{aligned}$$

Then $M_k(\frac{1}{2}, \lambda)$ follows as in (35) with μ replaced by μ^* and (28) can be used as a check.

Another scheme to compute $M_k(\nu, \lambda)$ for ν half an odd integer is to use the procedure described by (21)–(24) to get $M_k(\nu, \lambda)$ for ν in the neighborhood of half an odd integer and then employ the Lagrangian interpolation formula.

3. Numerical Results. From (1), (2) and (8), with a slight change of notation, we have

$$I_\nu(z) = z^\nu e^z \sum_{k=0}^{\infty} H_k(\nu) T_k^*(z/8), \qquad 0 < z \leqq 8,$$
(37)
$$H_k(\nu) = \sum_{r=0}^{\infty} D_{r,k} T_r^*\left(\frac{\nu - s}{t}\right), \qquad s \leqq \nu \leqq s + t.$$

In Tables 1 and 2 of the microfiche section we present values of $D_{r,k}$ which were evaluated by the technique described in [1] for $s = 0$, $t = 4$ and $s = t = 4$, respectively. To develop the numerics, values of $\Gamma(\nu + 1)$ were required. These were obtained by use of the schema of my previous paper [5]. Numerous checks were made on the coefficients. In addition to those of the kind discussed in [1], checks were also made using the recurrence formula for $I_\nu(z)$, namely

$$I_{\nu+1}(z) + \frac{2\nu}{z} I_\nu(z) - I_{\nu-1}(z) = 0. \tag{38}$$

Further checks were accomplished by comparing values deduced from (37) with those computed from power series, especially when ν is half an odd integer, for in this instance

$$e^{-z} I_{n+1/2}(z) = (2\pi z)^{-1/2}[A_n(z) + (-)^{n+1} e^{-2z} A_n(-z)],$$
(39)
$$A_n(z) = {}_2F_0\left(-n, n + 1; \frac{1}{2z}\right),$$

and $A_n(z)$ is a polynomial in z^{-1} of degree n. Wronskian relations were also used to get checks. The computations were designed so that the coefficients for $0 \leqq \nu \leqq 4$ are accurate to about 25D while those for $4 \leqq \nu \leqq 8$ are accurate to about 27D. To evaluate $e^{-z}I_\nu(z)$, we must incorporate the value of z^ν. As $0 < z \leqq 8$, we see that the coefficients are sufficiently accurate to produce $e^{-z}I_\nu(z)$ to about 20 decimals at least.

From (9)–(11) with a slight change of notation we write

$$I_\nu(z) = (2\pi z)^{-1/2} e^z \sum_{k=0}^{\infty} M_k(\nu) T_k^*(8/z), \qquad z \geqq 8, \tag{40}$$

$$M_k(\nu) = \sum_{r=0}^{\infty} E_{r,k} T_r^*(\nu), \qquad 0 \leqq \nu \leqq 1, \tag{41}$$

$$M_k(\nu) = \sum_{r=0}^{\infty} F_{r,k} T_r^*(-\nu), \qquad -1 \leqq \nu \leqq 0. \tag{42}$$

In Tables 3 and 4 of the microfiche section we give values of $E_{r,k}$ and $F_{r,k}$, respectively. In the development of these coefficients, the appropriate values of $(2\pi\lambda)^{1/2}e^{-\lambda}I_\nu(\lambda)$, as required by (23), were obtained from (37) for $\nu > 0$ and (38) was used to get the values needed for $\nu < 0$. Again, the coefficients were subjected to numerous checks. For example, for $z = 8$, we compared values of $I_\nu(z)$, as obtained from (40)–(42), with those obtained from (37) and (38) when appropriate. We also used the defining relation for $K_\nu(z)$ in terms of $I_\nu(z)$ and $I_{-\nu}(z)$ to compare values obtained using the coefficients in [1] and the coefficients in the present tables. Further checks were gotten by use of a Wronskian relation. The coefficients are sufficiently accurate to enable the computation of $e^{-z}(2\pi z)^{1/2}I_\nu(z)$ to about 22 decimals.

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