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Minimal Quadratures for Functions of Low-Order Continuity

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Abstract. An analog of Wilf's quadrature is developed for functions of low-order continuity. This analog is used to demonstrate that the order of convergence of Wilf's quadrature is at least $1/n$.

1. Introduction. From the work done in minimal norm quadratures for Hilbert spaces of analytic functions by Wilf [7], Barnhill [1], Eckhardt [2], Richter [6], and others, it is natural to consider an extension of this concept for functions of low-order continuity. In this paper, we consider functions with a uniformly convergent Fourier-Chebyshev expansion on the interval $[-1, 1]$

$$f(x) = \sum_{i=0}^{\infty}{}' a_i T_i(x),$$

$$a_i = \frac{2}{\pi} \int_{-1}^{1} (1 - x^2)^{-1/2} f(x) T_i(x)\, dx,$$

where $T_i(x)$ is the ith degree Chebyshev polynomial of the first kind and the prime on the sum indicates the first term is to be halved. We also restrict $f(x)$ to have the property that $\sum_{i=0}^{\prime\infty} |a_i|$ converges, e.g. when $f'(x)$ is of bounded variation on $[-1, 1]$. For error bounds of Gaussian quadrature for functions of this type, see Rabinowitz [5].

2. Minimal Quadratures. Let $\sum_{s=0}^{n} H_s f(x_s)$ be an $(n + 1)$-point quadrature formula. We define $R_n(f) = \int_{-1}^{1} f(x)\, dx - \sum_{s=0}^{n} H_s f(x_s)$ and note from the expansion of $f(x)$ that $R_n(f) = \sum_{i=0}^{\prime\infty} a_k R_n(T_i)$. Using both the triangle and Schwarz inequalities we obtain the error estimate

$$|R_n(f)| \leqq \left(\sum_{i=0}^{k}{}'' a_i^2 \right)^{1/2} \left(\sum_{i=0}^{k}{}'' R_n(T_i)^2 \right)^{1/2} + \sum_{i=k}^{\infty}{}' |a_i R_n(T_i)| \tag{1}$$

where the double prime indicates both first and last terms are to be halved.

If $f(x)$ satisfies mild smoothness restrictions (cf. Elliott, [3]), then the coefficients a_i satisfy $|a_i| \leqq C/i^2$. In this case, since $R_n(T_i)$ is bounded for $i \geqq k$, the last term of the inequality is of order $1/k$. Thus, it appears worthwhile to consider, as in Wilf [7], minimizing $W(n, k)$ where

$$W(n, k) = \sum_{i=0}^{k}{}'' R_n(T_i)^2. \tag{2}$$

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We note $W(n, k) = 0$ for $k < 2n + 2$, since the problem is solved by Gauss-Legendre quadrature.

To minimize $W(n, k)$, of course, we must solve the $2n + 2$ simultaneous equations $\partial W(n, k)/\partial H_s = 0$ and $\partial W(n, k)/\partial x_s = 0$, $0 \leqq s \leqq n$. An analytic solution does not seem feasible, so we consider the less restrictive problem of choosing weights to minimize $W(n, k)$ with a given fixed set of nodes. In doing so we are able to answer a question posed by Wilf (see Section 4).

Solving $\partial W(n, k)/\partial H_s$, $0 \leqq s \leqq n$, leads to the system

$$\text{(3)} \qquad \sum_{i=0}^{k}{}'' R_n(T_i)T_i(x_s) = 0; \qquad s = 0, \cdots, n.$$

Setting

$$\alpha_i = \int_{-1}^{1} T_i(x)\,dx = -2/(i^2 - 1), \qquad i \text{ even},$$
$$= 0, \qquad i \text{ odd},$$

thus (3) becomes

$$\text{(4)} \qquad \sum_{i=0}^{k}{}'' \alpha_i T_i(x_s) = \sum_{i=0}^{k}{}'' \sum_{j=0}^{n} H_j T_i(x_j) T_i(x_s).$$

If $H_0^*, \cdots, H_n^*$ satisfy (4), then $\sum_{s=0}^{n} H_s^* f(x_s)$ is called a minimal quadrature.

Let $g_k(x) = \sum_{i=0}^{\prime\prime k} \alpha_i T_i(x)$ and $f_k(x) = \sum_{i=0}^{\prime\prime k} R_n(T_i)T_i(x)$. Then,

$$R_n(g_k) = \sum_{i=0}^{k}{}'' \alpha_i R_n(T_i) = \int_{-1}^{1} f_k(x)\,dx = R_n(f_k) - \sum_{s=0}^{n} H_s f_k(x_s).$$

If $H_0, \cdots, H_n$ is a solution of (4), then (3) the quadrature sum is zero. Further, as $R_n(f_k) = W(n, k)$, we have $R_n(g_k) = W(n, k)$ so $W(n, k)$, for any minimal quadrature, is the error made in approximating the integral of $g_k(x)$. We note here that $\sum_{i=0}^{\prime\infty} \alpha_i T_i(x)$ is the Fourier-Chebyshev expansion for $F(x) = \frac{1}{2}\pi(1 - x^2)^{1/2}$ on $[-1, 1]$, and since $F(x)$ is continuous and of bounded variation the series is uniformly convergent.

Let H denote the $(n + 1)$-dimensional vector $H = (H_0, \cdots, H_n)$ and define $\varphi : E^{n+1} \to E^{k+1}$ by $\varphi(H) = (R_n(T_0), \cdots, R_n(T_k))$, where $R_n(T_i) = \alpha_i - \sum_{s=0}^{n} H_s T_i(x_s)$. It is immediate from Hilbert space properties that there is a unique point H^* in E^{n+1} such that $\|\varphi(H^*)\|_2$ is minimal. Thus, the existence of a unique minimal quadrature is guaranteed.

3. Special Case. When $k = n$, the minimal quadrature is of course the interpolatory quadrature on $x_0, \cdots, x_n$. In the case $x_i = \cos(i\pi/n)$, the interpolatory quadrature is Clenshaw-Curtis quadrature. If we use the well-known orthogonality properties for $T_i(x)$ in (4) with $x_i = \cos(i\pi/n)$, we obtain immediately $\frac{1}{2}nH_i = g_n(x_i)$, $i = 1, \cdots, n - 1$, and $nH_i = g_n(x_i)$ for $i = 0$ or n. These are the same expressions found by Imhof [4], which he used to show the Clenshaw-Curtis weights were positive.

4. Improvement of a Result of Wilf. In [7] Wilf minimizes $W_n = \sum_{k=0}^{\infty} R_n(x^k)^2$. Let R_n^* denote the remainder for optimal quadrature in the set of functions analytic in $|z| < 1$ and $\mathfrak{L}^2$ on the unit circle, and let

$$||f||^2 = \frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta})|^2\, d\theta.$$

Thus $|R_n^*(f)| \leqq W_n^{1/2}\, ||f||$. Wilf was unable to give explicit solutions for the weights and nodes, but was able to show that W_n is the magnitude of the error in integrating $x^{-1}\log(1-x)^{-1}$ by the minimal formula. He derives the result $W_n \leqq O(\ln(n)/n)$ and leaves as an open question whether this result can be improved. On [0, 1] the Clenshaw-Curtis weights and nodes are, respectively,

$$w_i = \frac{4}{n+1} g_{n+1}(x_i),\ i = 1, \cdots, n, \qquad x_i = \left[\cos\frac{i\pi}{2(n+1)}\right]^2,\ i = 0, \cdots, n+1.$$
$$= \frac{2}{n+1} g_{n+1}(x_i),\ i = 0 \text{ or } n+1;$$

For ease of computation, and since $g_{n+1}(x)$ is uniformly convergent to $\frac{1}{2}\pi(1-x^2)^{1/2}$ on [0, 1], we shall use instead the weights

$$H_i = \frac{2\pi}{n+1}(1-(x_i)^2)^{1/2}, \qquad i = 1, \cdots, n,$$
$$= \frac{\pi}{n+1}(1-(x_i)^2)^{1/2}, \qquad i = 0 \text{ or } n+1,$$

and we note $H_0 = x_{n+1} = 0$.

Since Clenshaw-Curtis quadrature is exact for polynomials of degree less than $n+2$,

$$\sum_{s=0}^{n+1} w_s(x_s)^k = \frac{1}{k+1} \quad \text{for } 0 \leqq k \leqq n+1.$$

Then

$$\sum_{k=0}^{n} R_n(x_k)^2 \equiv \sum_{k=0}^{n}\left(1/(k+1) - \sum_{s=1}^{n} H_s x_s^k\right)^2$$
$$= \sum_{k=0}^{n}\left(\sum_{s=0}^{n+1}(w_s - H_s)x_s^k\right)^2$$
$$\leqq \sum_{k=0}^{n}\left(w_0 + \sum_{s=1}^{n}|w_s - H_s|\right)^2.$$

For $1 \leqq s \leqq n$,

$$|H_s - w_s| = (4/(n+1))\left|\sum_{i=n+1}^{\infty}\alpha_i T_i(x_s)\right|$$
$$\leqq (4/(n+1))\sum_{i=n+1}^{\infty}|\alpha_i| \leqq 4/(n^2-1).$$

Since $w_0 = 1/((n+1)^2 - 1)$, then by inserting these bounds we get

$$(5) \qquad \sum_{k=0}^{n} R_n(x^k)^2 \leqq (n+1)(1+4n)^2/(n^2-1)^2 \leqq C_1/n.$$

Now, for $k > n$, define

$$Q_n(x^k) \equiv \sum_{s=1}^{n} H_s x_s^k \leqq \frac{2\sqrt{2\pi}}{n+1} \sum_{s=1}^{n} \sin\left(\frac{s\pi}{2n+2}\right)\left(\cos\left(\frac{s\pi}{2n+2}\right)\right)^{2k}. \tag{6}$$

In what follows we will show that $Q_n(x^k) = O(1/k)$ for $k > n$. We first show that if n is sufficiently large and $k \geqq (n+1)^2$, then $kQ_n(x^k)$ is bounded independently of k. We then show by an integral bound that $Q_n(x^k) = O(1/k)$ for the remaining k in $(n(n+1)^2)$.

First let us assume that $k \geqq (n+1)^2$, then

$$y(k) \equiv kQ_n(x^k) \leqq \sum_{s=1}^{n} \sin\left(\frac{s\pi}{2n+2}\right) V_s(k), \tag{7}$$

where $V_s(k) = k(\cos(s\pi/(2n+2)))^{2k}$. Then it is easily verified that for n sufficiently large (say $n \geqq M$) and $k \geqq (n+1)^2$, $V_s'(k) < 0$, and thus $y(k)$ is bounded for all such k.

Now we consider a fixed $n \geqq M$ and any $k < (n+1)^2$. We define

$$z(s) = \left(\frac{2\sqrt{2\pi}}{n+1}\right) \sin\left(\frac{s\pi}{2n+2}\right)\left(\cos\left(\frac{s\pi}{2n+2}\right)\right)^{2k}. \tag{8}$$

Then $z'(s^*) = 0$ for $\tan(s^*\pi/(2n+2)) = (1/2k)^{1/2}$, and s^* is unique in $(0, n+1)$. Thus if m is the greatest integer in s^*, and since $z(s) \geqq 0$ in $(0, n+1)$ and is maximal at s^*

$$\sum_{s=1}^{m-1} z(s) + \sum_{s=m+2}^{n} z(s) \leqq \int_0^{n+1} z(t)\, dt < \frac{2\sqrt{2}}{k}. \tag{9}$$

Since $z(s^*) \leqq (2\sqrt{2\pi}/(n+1))(2k+1)^{-1/2}$, then

$$Q_n(x^k) \leqq \sum_{s=1}^{n} z(s) \leqq \frac{2\sqrt{2}}{k} + 2z(s^*) < \frac{2(\sqrt{2}+2\pi)}{k}. \tag{10}$$

Thus, combining these two cases with $C_2 = 2(\sqrt{2}+2\pi)$,

$$\begin{aligned}\sum_{k=n+1}^{\infty} R_n(x^k)^2 &\leqq \sum_{k=n+1}^{\infty} (1/(k+1)^2 + 2C_2/k(k+1) + C_2^2/k^2) \\ &= O(1/n).\end{aligned} \tag{11}$$

Combining this with (5) we get the desired result,

$$W_n \leqq \sum_{k=0}^{\infty} R_n(x^k)^2 = O(1/n). \tag{12}$$

Reflection on the magnitude of $R_n^*(x^k)^2$, i.e. $(1/(k+1) - Q_n^*(x^k))^2$, and the number of free parameters available leads us to conjecture that $O(1/n)$ is the best possible bound for W_n.

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