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Some Polynomials for Complex Quadrature

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Abstract. Equal-weight Chebyshev quadrature is not generally used because the nodes become complex for large n. However, interest in these schemes remains because of recent work on minimal norm quadrature as well as schemes for doing real integrals of analytic functions by complex methods. This note presents some properties of these Chebyshev quadratures that may be of interest to other researchers in this area. Proofs are sketched to save space.

Equal-weight Chebyshev quadrature is not generally used because the nodes $\{x_i^{(n)}\}_{i=1}^n$ become complex for $n \geqq 10$. However, interest in these schemes remains because of recent work on minimal norm quadrature [1], [2], and [3] as well as schemes for doing real integrals of analytic functions by complex methods [5]. This note presents some properties of these Chebyshev quadratures that may be of interest to other researchers in this area. Proofs are sketched to save space.

The nodes for Chebyshev quadrature are defined as the unique solution set of the system

$$\frac{2}{n} \sum_{i=1}^{n} [x_i^{(n)}]^j = \int_{-1}^{1} x^j \, dx, \qquad j = 1, \cdots, n.$$

Let $P_n(x) = \prod_{i=1}^n (x - x_i^{(n)})$.

THEOREM 1. *If $n = 2m$, $P_n(x)$ has at least two real zeros in $(-\xi_n, \xi_n)$ where ξ_n is the zero of largest magnitude of the nth Legendre polynomial.*

The proof is immediate by using a Gauss quadrature formula on $P_n(x)$. A little known result of Kuzmin [6] is

THEOREM 2. *$P_n(x)$ has $O(\log n)$ real zeros.*

Using this, we can prove

COROLLARY. *Again, with $n = 2m$, Theorem 1 is true with the smaller interval $(-\xi_m, \xi_m)$.*

For $n = 2m \leqq 100$, computation gives exactly two real zeros of $P_{2m}(x)$. Hence, using the known symmetry of P_{2m}, we get

COROLLARY. *The positive real zero of $P_{2m}(x)$ lies in the interval (ξ_m, ξ_{m+1}), $2m \leqq 100$.*

The zeros of $P_n(z)$ are given for $n \leqq 47$ in the microfiche section of this issue.

THEOREM 3. *Let $f(z)$ be analytic in a closed domain including the curve Γ (defined below) in its interior. Let I_n be given by*

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$$I_n = \frac{2}{n} \sum_{i=1}^{n} f(x_i^{(n)}) \approx \int_{-1}^{1} f(x)\, dx = I.$$

Then $I_n \to I$.

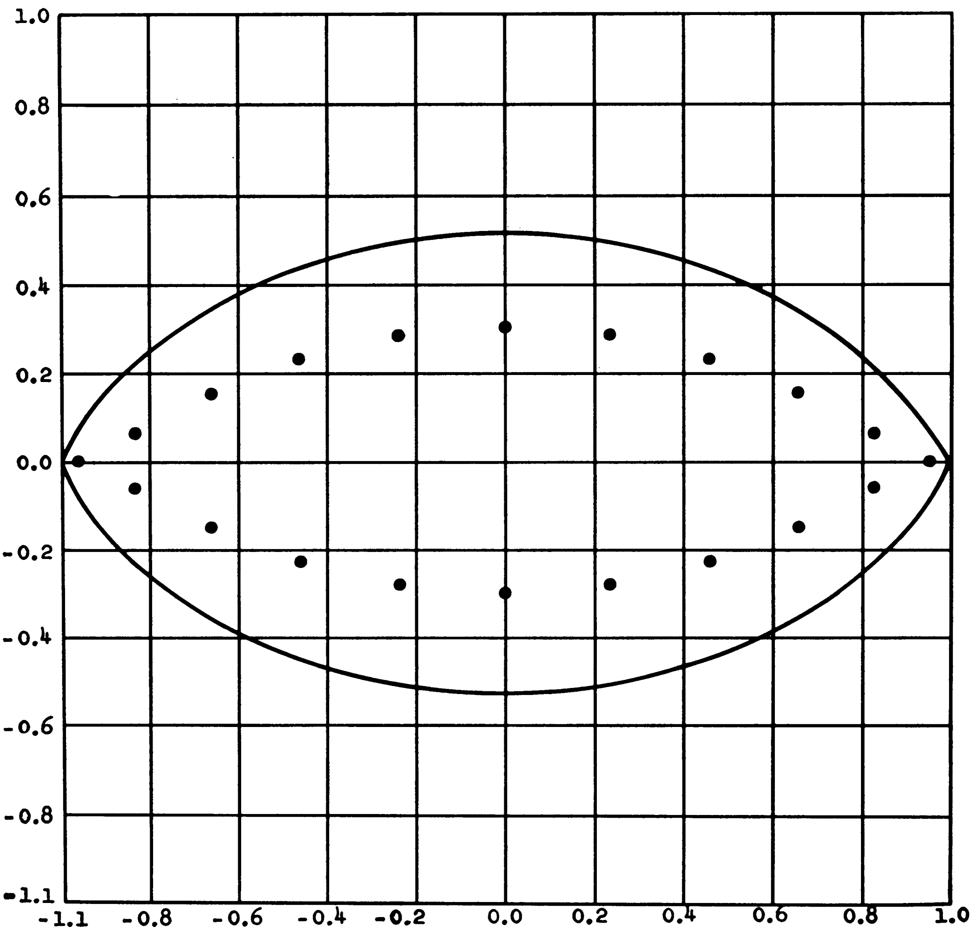
Proof. The curve Γ is the logarithmic potential curve

$$\Gamma = \left\{ z : \int_{-1}^{1} \log |z - t|\, dt = \int_{-1}^{1} \log |1 - t|\, dt \right\}.$$

Kuzmin [6] has shown that the zeros of $P_n(z)$ have an asymptotic distribution about Γ, if the zero at the origin for odd n is excluded. This eye-shaped curve has a maximum height of .52 at $x = 0$. Numerically, the zeros approach Γ quite slowly from the inside.

FIGURE 1

Logarithmic Potential Curve



The curve is given by $\Gamma = \{z: \int_{-1}^{1} \log |z - t|\, dt = \int_{-1}^{1} \log |1 - t|\, dt\}$. *The interior tic-marks are the zeros of* $P_{20}(z)$.

By Runge's Theorem [4], we may approximate $f(x)$ uniformly in Γ by a complex polynomial. Hence the quadrature sum $(2/n) \sum_{i=1}^{n} f(x_i^{(n)})$ may be replaced by an expression of the form

$$\sum_{j=0}^{N} a_j\left(\frac{2}{n}\sum_{i=1}^{n}(x_i^{(n)})^j\right)$$

with an error ϵ, independent of n. Since the quadrature is exact for polynomials, the theorem follows.

COROLLARY. *If* $|u| > 1 + \epsilon$,

$$\text{(a)} \qquad \lim_{n\to\infty} \log(P_n(u))^{1/n} = \frac{1}{2}\int_{-1}^{1} \log(u - t)\, dt + k,$$

$$\text{(b)} \qquad \lim_{n\to\infty}(P_n(u))^{1/n} = C \exp\left(\frac{1}{2}\int_{-1}^{1} \log(u - t)\, dt\right),$$

where the principal branch of both the log *and nth root functions are used.*

Proof. We can show [8] that if $|u| > 1 + \epsilon$,

$$\int_{-1}^{1} \frac{dx}{u - x} = \frac{2}{n}\frac{d}{du} \log P_n(u) + E\left[\frac{1}{u - x}\right],$$

where $E[1/(u - x)]$ is the error in the estimate of the integral of $f(x) = 1/(u - x)$. If we integrate from u_0 to U with respect to u (u_0, U and the path of integration remain outside the circle $|z| = 1 + \epsilon$ and on the principal branch of the logarithm),

$$\int_{-1}^{1} \log(U - x)\, dx = \frac{2}{n}\log[P_n(U)] + \int_{u_0}^{U} E\left[\frac{1}{u - x}\right] du + K - \frac{2}{n}\log[P_n(u_0)].$$

For finite u_0 the last term is bounded as $n \to \infty$. Thus

$$\log(P_n(U))^{1/n} = \frac{1}{2}\int_{-1}^{1} \log(U - x)\, dx + \frac{1}{2}\int_{u_0}^{U} E\left[\frac{1}{u - x}\right] du + O(1).$$

Since $|x| \leqq 1$ and $f(x)$ can be approximated uniformly by polynomials in x for $|u| > 1 + \epsilon$, by the previous theorem the second integral goes to zero with increasing n. Taking limits, we get (a). Exponentiating first, we get (b). This convergence theorem and its corollary are interesting not only for their own sake, but also because they mirror theorems about real quadrature formulas. Thus Krylov [9] has shown that Theorem 3 is true for a general class of interpolatory quadrature formulas with real nodes, and Shohat [7] proves the corollary in the case where the $x^{(n)}$ are real and asymptotically uniformly distributed on $[-1, 1]$.

Computation. Although we know that the quadrature scheme converges for functions analytic in a compact set including Γ, we are only able to obtain error estimates for functions analytic in a somewhat larger set G:

$$E(f) \leqq \frac{L(G)}{2\pi} \max_{\partial G} |f(t)| \left(\frac{D}{d}\right)^{n+1} \frac{2}{\delta},$$

$$L(G) = \text{length of } \partial G,$$

$$D = \max\{\text{dist}(x_1^{(n)}, [-1, 1]), \cdots, \text{dist}(x_n^{(n)}, [-1, 1])\} \approx .52,$$

$$d = \min\{\text{dist}(x_1^{(n)}, \partial G), \cdots, \text{dist}(x_n^{(n)}, \partial G)\},$$

$$\delta = \min_{-1\leqq x\leqq 1} \{\text{dist}(x, G)\}.$$

Thus, in particular if $f(z)$ is analytic in $|z| = \frac{3}{2}$, we get geometrical convergence.

Moreover, because of the nature of Γ, we get similar convergence for functions analytic in the rectangle R centered at the origin of height 1.05 and width 3.0. Finally, for finite n, we get geometrical decrease in the error initially, even for functions analytic in much shorter rectangles, because of the slow convergence of the z_k to Γ. For example, when $n = 30$ the largest imaginary part of any z_k is .36.

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