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Unitary Amicable Numbers

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Abstract. Two positive integers are said to be unitary amicable if the sum of the unitary divisors of each is equal to their sum. In this paper a table of such numbers is given, and some theorems concerning them are proved.

1. Introduction. In this paper, lower-case letters will be used to denote positive integers, with p and q always representing primes. If $cd = n$ and $(c, d) = 1$, then d is said to be a unitary divisor of n. We shall use the symbol $\sigma^*(n)$ to denote the sum of the unitary divisors of n. It is immediate that $\sigma^*(1) = 1$, while

$$\sigma^*(n) = (1 + p_1^{a_1})(1 + p_2^{a_2}) \cdots (1 + p_s^{a_s}), \tag{1}$$

if $n = p_1^{a_1} p_2^{a_2} \cdots p_s^{a_s}$. From (1) we see that $\sigma^*(n)$ is a multiplicative function; and $\sigma^*(n) = \sigma(n)$ (where $\sigma(n)$ represents the sum of *all* the positive divisors of n) if and only if n is square-free. We also obtain the following result from (1).

LEMMA. *$\sigma^*(n)$ is odd if and only if n is a power of* 2.

Subbarao and Warren [10] have defined n to be a unitary perfect number if $\sigma^*(n) = 2n$. Similarly, we shall say that a pair of positive integers m and n are unitary amicable numbers if

$$\sigma^*(m) = \sigma^*(n) = m + n. \tag{2}$$

To the best of my knowledge, such numbers have not been studied, or even defined, before. In this paper, some elementary theorems concerning pairs of unitary amicable numbers are proved and a short table of such pairs is given. Some questions are also posed.

2. Some Theorems. In what follows, m and n are assumed to be members of a pair of unitary amicable numbers. It is obvious that neither m nor n is one.

THEOREM 1. *Neither m nor n is of the form p^a.*

Proof. Assume that $n = p^a$. Then from (1) and (2), we have $m = 1$. This is impossible.

THEOREM 2. *m and n are of the same parity.*

Proof. Assume the contrary. Then $\sigma^*(n)$ is odd and therefore, by our lemma, $n = 2^a$. This contradicts Theorem 1.

THEOREM 3. *If $m = 2^a M$ and $n = 2^b N$ where $b > a > 0$ and M and N are odd integers having, respectively, s and t distinct prime divisors, then $s \leqq a$ and $t \leqq a$.*

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Proof. We see immediately that $m + n = 2^a r$ where r is odd. If $M = p_1^{a_1} \cdots p_s^{a_s}$, then from (1) and (2), $2^a r = (1 + 2^a)(1 + p_1^{a_1}) \cdots (1 + p_s^{a_s})$. Since each factor, except the first, on the right-hand side is even, we have $s \leqq a$. By the same argument $t \leqq a$.

COROLLARY 3.1. *If* $m = 2M$ *and* $n = 2^b N$ *where* $b > 1$, *then* $M = p^a$ *and* $N = q^h$.

COROLLARY 3.2. *If* M *and* N *are squares and* $b > a$, *then* $s = t = a$.

COROLLARY 3.3. *If* M *and* N *are squares and* $b = a$, *then* $s = t = a + 1$.

(For Corollary 3.2, recall that $1 + p^{2e} = 2 + 4k$ if p is odd; for Corollary 3.3 recall that $u^2 + v^2 = 2 + 8k$ if u and v are odd.)

THEOREM 4. *If* $m = 2M$ *and* $n = 2N$, *where* $(10, MN) = 1$, *then* $10m$ *and* $10n$ *also constitute a pair of unitary amicable numbers.*

Proof. From the multiplicative property of σ^*, we have

$$\sigma^*(10m) = \sigma^*(4)\sigma^*(5)\sigma^*(M) = 30\sigma^*(M)$$

$$= 10 \cdot \sigma^*(2)\sigma^*(M) = 10\sigma^*(m) = 10m + 10n.$$

In the same way, $\sigma^*(10n) = 10m + 10n$.

A similar argument establishes Theorem 5, while our final theorem follows from the remark made just after (1).

THEOREM 5. *If* $m = 12M$ *and* $n = 12N$ *where* $(6, MN) = 1$, *then the pair* $18M$ *and* $18N$ *are also unitary amicable numbers.*

THEOREM 6. *If* m *and* n *are both square-free, then the pair* m, n *is unitary amicable if and only if the pair* m, n *is amicable.*

It is, perhaps, of some interest to compare the above results with our state of knowledge concerning (ordinary) amicable numbers.

Remark 1. Whether it is possible for a number of the form p^a to be one of a pair of amicable numbers is not known. It *is* known that an affirmative answer would imply $p^a > 10^{1500}$ and $a > 1400$ (see [6]).

Remark 2. Whether or not a pair of amicable numbers of opposite parity exists is also an open question. Some necessary conditions for the existence of such a pair may be found in [5] and [7].

Remark 3. Theorem 3 is false for amicable numbers. A counterexample is provided by the pair $2 \cdot 5 \cdot 11^2$ and $2^5 \cdot 37$.

Remark 4. To my knowledge, no results analogous to Theorems 4 and 5 are known concerning *even* amicable numbers. However, the referee has pointed out that many "isomeric" pairs of *odd* amicable numbers exist (for example, see the pairs 60 and 61, 65 and 66, 138 and 139 in [3]), and that theorems similar to Theorems 4 and 5 could be (although apparently they have not been) stated concerning them.

3. A List of Unitary Amicable Pairs.

Using the CDC 6400 at the Temple University Computing Center, a search was made for all unitary amicable pairs m, n such that $m < n$ and $m \leqq 10^6$. The search required about two hours and ten minutes of computer time, and 19 pairs were found. These comprise the first 19 entries in the list at the end of this section. Of these, three (pairs 16, 18, 19) satisfied the hypotheses of Theorem 5 *and* had $m > 666666$. An application of this theorem yielded the next three pairs in our list. The eighth pair, 142310 and 168730, is square-free and therefore, by Theorem 6, is amicable. A search of the published pairs of amicable numbers (see [1], [2], [3], [4], [8], [9] and [11]) yielded 76 square-free pairs including the one just mentioned. Nine of these pairs are odd, the smallest being 8619765 and 9627915. Of the remaining 67 pairs, eleven satisfy the conditions of Theorem 4 and therefore, upon multiplication by 10, yield "new" unitary pairs. These comprise the last eleven pairs in our list. Thus, a total of 108 unitary amicable number pairs are known to date, 76 being square-free and therefore simultaneously amicable, the remaining 32 being "new".

Unitary Amicable Pairs

$114 = 2\cdot 3\cdot 19$	$126 = 2\cdot 3^2\cdot 7$
$1140 = 2^2\cdot 3\cdot 5\cdot 19$	$1260 = 2^2\cdot 3^2\cdot 5\cdot 7$
$18018 = 2\cdot 3^2\cdot 7\cdot 11\cdot 13$	$22302 = 2\cdot 3^3\cdot 7\cdot 59$
$32130 = 2\cdot 3^3\cdot 5\cdot 7\cdot 17$	$40446 = 2\cdot 3^3\cdot 7\cdot 107$
$44772 = 2^2\cdot 3\cdot 7\cdot 13\cdot 41$	$49308 = 2^2\cdot 3\cdot 7\cdot 587$
$56430 = 2\cdot 3^3\cdot 5\cdot 11\cdot 19$	$64530 = 2\cdot 3^3\cdot 5\cdot 239$
$67158 = 2\cdot 3^2\cdot 7\cdot 13\cdot 41$	$73962 = 2\cdot 3^2\cdot 7\cdot 587$
$142310 = 2\cdot 5\cdot 7\cdot 19\cdot 107$	$168730 = 2\cdot 5\cdot 47\cdot 359$
$180180 = 2^2\cdot 3^2\cdot 5\cdot 7\cdot 11\cdot 13$	$223020 = 2^2\cdot 3^3\cdot 5\cdot 7\cdot 59$
$197340 = 2^2\cdot 3\cdot 5\cdot 11\cdot 13\cdot 23$	$286500 = 2^2\cdot 3\cdot 5^3\cdot 191$
$241110 = 2\cdot 3^3\cdot 5\cdot 19\cdot 47$	$242730 = 2\cdot 3^3\cdot 5\cdot 29\cdot 31$
$296010 = 2\cdot 3^2\cdot 5\cdot 11\cdot 13\cdot 23$	$429750 = 2\cdot 3^2\cdot 5^3\cdot 191$
$308220 = 2^2\cdot 3\cdot 5\cdot 11\cdot 467$	$365700 = 2^2\cdot 3\cdot 5^2\cdot 23\cdot 53$
$462330 = 2\cdot 3^2\cdot 5\cdot 11\cdot 467$	$548550 = 2\cdot 3^2\cdot 5^2\cdot 23\cdot 53$
$591030 = 2\cdot 3^3\cdot 5\cdot 11\cdot 199$	$618570 = 2\cdot 3^3\cdot 5\cdot 29\cdot 79$
$669900 = 2^2\cdot 3\cdot 5^2\cdot 7\cdot 11\cdot 29$	$827700 = 2^2\cdot 3\cdot 5^2\cdot 31\cdot 89$
$671580 = 2^2\cdot 3^2\cdot 5\cdot 7\cdot 13\cdot 41$	$739620 = 2^2\cdot 3^2\cdot 5\cdot 7\cdot 587$
$785148 = 2^2\cdot 3\cdot 7\cdot 13\cdot 719$	$827652 = 2^2\cdot 3\cdot 7\cdot 59\cdot 167$
$815100 = 2^2\cdot 3\cdot 5^2\cdot 11\cdot 13\cdot 19$	$932100 = 2^2\cdot 3\cdot 5^2\cdot 13\cdot 239$
$1004850 = 2\cdot 3^2\cdot 5^2\cdot 7\cdot 11\cdot 29$	$1241550 = 2\cdot 3^2\cdot 5^2\cdot 31\cdot 89$
$1177722 = 2\cdot 3^2\cdot 7\cdot 13\cdot 719$	$1241478 = 2\cdot 3^2\cdot 7\cdot 59\cdot 167$
$1222650 = 2\cdot 3^2\cdot 5^2\cdot 11\cdot 13\cdot 19$	$1398150 = 2\cdot 3^2\cdot 5^2\cdot 13\cdot 239$
$27287260 = 2^2\cdot 5\cdot 7\cdot 11\cdot 13\cdot 29\cdot 47$	$30773540 = 2^2\cdot 5\cdot 7\cdot 19\cdot 23\cdot 503$
$307246940 = 2^2\cdot 5\cdot 7\cdot 11\cdot 13\cdot 103\cdot 149$	$321745060 = 2^2\cdot 5\cdot 7\cdot 13\cdot 17\cdot 10399$
$342562220 = 2^2\cdot 5\cdot 7\cdot 11\cdot 13\cdot 71\cdot 241$	$359973460 = 2^2\cdot 5\cdot 7\cdot 11\cdot 23\cdot 10163$
$353613260 = 2^2\cdot 5\cdot 7\cdot 11\cdot 13\cdot 17\cdot 1039$	$401177140 = 2^2\cdot 5\cdot 7\cdot 13\cdot 53\cdot 4159$
$4141957820 = 2^2\cdot 5\cdot 7\cdot 11\cdot 13\cdot 19\cdot 10889$	$4639738180 = 2^2\cdot 5\cdot 7\cdot 11\cdot 83\cdot 36299$
$11620749140 = 2^2\cdot 5\cdot 7\cdot 11\cdot 13\cdot 43\cdot 13499$	$12329330860 = 2^2\cdot 5\cdot 7\cdot 11\cdot 29\cdot 359\cdot 769$
$11939455220 = 2^2\cdot 5\cdot 7\cdot 11\cdot 19\cdot 53\cdot 7699$	$12010624780 = 2^2\cdot 5\cdot 7\cdot 11\cdot 17\cdot 149\cdot 3079$
$22709666980 = 2^2\cdot 5\cdot 7\cdot 11\cdot 13\cdot 191\cdot 5939$	$23274486620 = 2^2\cdot 5\cdot 7\cdot 11\cdot 19\cdot 307\cdot 2591$
$145363958740 = 2^2\cdot 5\cdot 7\cdot 11\cdot 13\cdot 43\cdot 131\cdot 1289$	$156726383660 = 2^2\cdot 5\cdot 7\cdot 11\cdot 43\cdot 139\cdot 17027$
$1810211983580 = 2^2\cdot 5\cdot 7\cdot 11\cdot 13\cdot 181\cdot 499559$	$1855679190820 = 2^2\cdot 5\cdot 7\cdot 13\cdot 17\cdot 181\cdot 229\cdot 1447$

$5921980188328820 = 2^2\cdot 5\cdot 7\cdot 11\cdot 19\cdot 61\cdot 853\cdot 3889679$

$5940784546135180 = 2^2\cdot 5\cdot 7\cdot 17\cdot 19\cdot 37\cdot 61\cdot 853\cdot 68239$

4. Some Questions. An examination of the list in Section 3 suggests the following questions.

Question 1. Are there an infinite number of unitary amicable numbers?

Question 2. Does there exist a relatively prime pair of unitary amicable numbers?

Question 3. Is every odd pair of unitary amicable numbers simultaneously amicable?

Question 4. If $m = 2^a M$ and $n = 2^b N$, where MN is odd, is it always the case that $a = b$?

Question 5. Are there an infinite number of "twin" pairs of unitary amicable numbers (such as the tenth and eleventh pairs or the twenty-eighth and twenty-ninth pairs in our list) with the property that $m_1 + n_1 = m_2 + n_2$?

I would conjecture that the answer to the first question is "Yes", while the answer to the second, third and fourth questions is "No". I have no opinion to express on the final question.

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