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Convergence of Difference Methods for Initial and Boundary Value Problems with Discontinuous Data

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Abstract. This paper extends the classical convergence theory for numerical solutions to initial and boundary value problems with continuous data (the right-hand side) to problems with Riemann integrable data. Order of convergence results are also obtained.

1. Introduction. The purpose of this paper is to show that difference methods for solving initial and boundary value problems will converge in a variety of cases where the data (the right-hand side) is not well behaved in the classical sense.

To illustrate this basic idea, we consider two problems:

1. First, we look at Euler's method for solving initial-value problems on [0, 1]:

$$y'(t) = f(t, y(t)), \qquad y(0) = y_0. \tag{1.1}$$

Classically, for convergence, it is assumed that f satisfies a Lipschitz condition in the second variable and $y(t)$ is continuously differentiable (see e.g., Henrici [4], [5]) or, at least, piecewise continuously differentiable (see Goodman [3] or Zverkina [7]). We obtain the following results: If f is a bounded Riemann integrable function along the trajectory and satisfies a Lipschitz condition in its second argument, then Euler's method converges; and, if f is of bounded variation along the solution trajectory, then the convergence is of order h.

2. Second, we look at the standard simple difference method for solving the boundary value problem on [0, 1]:

$$y''(t) = f(t, y(t)), \qquad y(0) = a, \qquad y(1) = b. \tag{1.2}$$

In general, for convergence, it is assumed that $y(t) \in C^2[0, 1]$ (see e.g. Lees [6]). We obtain convergence results under assumptions on f of the same type as in (1).

2. Convergence Results 1. Let N be some positive integer and $h = 1/N$. Set $t_n = nh$, $n = 0, 1, \cdots, N$; then Euler's method for solving (1.1) on [0, 1] is given by

$$y_{n+1} = y_n + hf(t_n, y_n), \qquad n = 0, \cdots, N-1. \tag{2.1}$$

The following theorem gives the convergence properties of (2.1) to solutions of (1.1) as $h \to 0$, where, by a solution to (1.1), we mean an absolutely continuous $y(t)$ on [0, 1] which satisfies the initial condition, and the derivative $y'(t)$ equals f every-

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where except on a set of Lebesgue measure zero (see e.g. Coddington and Levinson [1, p. 42]).

THEOREM 2.1. *Suppose the solution of* (1.1) *exists, where f is a bounded Riemann integrable function along the solution trajectory, and there exists $k < \infty$, such that**** *for all $t \in [0, 1]$, a and b real*

$$|f(t, a) - f(t, b)| \leqq k|a - b|. \tag{2.2}$$

Then

$$|y_n - y(t_n)| \to 0 \text{ uniformly}, \quad \text{as } h \to 0,\ hn \to t \in [0, 1], \tag{2.3}$$

where $y(t)$ is the exact solution of (1.1). *If, in addition, f is of bounded variation along the solution trajectory, then*

$$|y_n - y(t_n)| = O(h). \tag{2.4}$$

Proof. Define the local truncation error τ_n by

$$h\tau_{n+1} = y(t_{n+1}) - y(t_n) - hf(t_n, y(t_n)), \qquad n = 0, \cdots, N - 1, \tag{2.5}$$

where $y(t)$ is the exact solution of (1.1). Observe

$$|y_1 - y(t_1)| = h\,|\tau_1|$$

and by (2.2),

$$|y_n - y(t_n)| \leqq (1 + hk)\,|y_{n-1} - y(t_{n-1})| + h\,|\tau_n|, \qquad n = 2, \cdots, N.$$

Hence,

$$\begin{aligned} |y_n - y(t_n)| &\leqq h \sum_{j=0}^{n-1} (1 + hk)^j\,|\tau_{n-j}| \leqq e^k \sum_{j=0}^{n-1} h\,|\tau_{n-j}| \\ &\leqq e^k \sum_{j=0}^{n-1} \left| \int_{t_j}^{t_{j+1}} f(t, y(t))\,dt - hf(t_j, y(t_j)) \right| \\ &\leqq e^k \sum_{j=0}^{n-1} h\,|f_j^* - f(t_j, y(t_j))|, \qquad n = 1, \cdots, N, \end{aligned} \tag{2.6}$$

where

$$m_j \equiv \inf_{t \in [t_j, t_{j+1}]} f(t, y(t)) \leqq f_j^* \leqq \sup_{t \in [t_j, t_{j+1}]} f(t, y(t)) \equiv M_j.$$

Thus,

$$|y_n - y(t_n)| \leqq e^k h \sum_{j=0}^{n-1} (M_j - m_j) \leqq e^k (S_1 - S_2), \tag{2.7}$$

where S_1 and S_2 are, respectively, the upper and lower Riemann sums for $\int_0^1 f(t, y(t))\,dt$ over the partition $\{t_0, \cdots, t_N\}$. Since f is Riemann integrable on the solution curve, (2.3) follows. If, in addition, f is of bounded variation on the solution curve, then

*** G. Dahlquist has pointed out that the results of this theorem hold when this condition is replaced by the weaker one-sided Lipschitz condition $|a - b + h(f(t, a) - f(t, b))| \leqq (1 + k)\,|a - b|$; (concerning one-sided Lipschitz conditions, see Dahlquist [2]).

(2.4) follows immediately as a consequence of the uniform boundedness of the sum in (2.7) over all partitions of [0, 1].

3. Convergence Results 2. With the notation as in Section 2, a simple difference approximation to (1.2) is

$$u_{n+1} - 2u_n + u_{n-1} = h^2 f(t_n, u_n), \qquad n = 1, \cdots, N-1, \tag{3.1}$$
$$u_0 = a, \qquad u_N = b.$$

The following theorem gives the convergence properties of a solution of (3.1) to a solution of (1.2), as $h \to 0$. By a solution of the differential equation, we mean a function $y(t)$ which has an absolutely continuous first derivative on [0, 1] satisfies the boundary condition, and $y''(t)$ equals f, except on a set of Lebesgue measure zero. The proof of the theorem is similar to that of Theorem 2.1; we only sketch the differences.

THEOREM 3.1. *Suppose the solution to* (1.2) *exists, where f is a bounded Riemann integrable function along the solution trajectory. Assume, in addition, there exists* $K < 8$ *such that for all* $t \in [0, 1]$ *and x, y real*

$$|f(t, x) - f(t, y)| \leqq K\, |x - y|. \tag{3.2}$$

Then

$$|u_n - y(t_n)| \to 0 \text{ uniformly}, \quad \text{as } h \to 0,\ nh \to t \in [0, 1], \tag{3.3}$$

where $y(t)$ is the exact solution of the differential equation and u_n is the solution to (3.1). *If, in addition, f is of bounded variation along the solution trajectory, then*

$$|u_n - y(t_n)| = O(h). \tag{3.4}$$

Proof. Define the local truncation error τ_n by

$$h^2 \tau_n = y(t_{n+1}) - 2y(t_n) + y(t_{n-1}) - h^2 f(t_n, y(t_n)), \tag{3.5}$$

$n = 1, \cdots, N-1$. Set v to be the vector with components $v_n = y(t_n) - u_n$, $n = 1, \cdots, N-1$. Then, from (3.5), v satisfies

$$v = -h^2 A^{-1} F - h^2 A^{-1} \tau, \tag{3.6}$$

where $A^{-1} = (r_{ij})$, $i, j = 1, \cdots, N-1$, has elements

$$r_{ij} = \frac{i(N-j)}{N}, \qquad i \leqq j,$$
$$= r_{ji}, \qquad i > j,$$

F has components $F_n = f(t_n, y(t_n)) - f(t_n, u_n)$, $n = 1, \cdots, N-1$, and τ has components τ_n, $n = 1, \cdots, N-1$. From (3.6), it follows, by using (3.2), that

$$\|v\|_\infty \leqq \frac{2}{8 - K} \sum_{j=1}^{N-1} h\, |\tau_j|.$$

Integration by parts shows that

$$\frac{y(t_{n+1}) - 2y(t_n) + y(t_{n-1})}{h} = \int_0^h [y''(t_n + \theta) + y''(t_n - \theta)]\left(1 - \frac{\theta}{h}\right) d\theta.$$

Then using this, (3.5) and (1.2), we obtain

$$\sum_{n=1}^{N-1} h\,|\tau_n| \leqq \frac{h}{2} \sum_{n=1}^{N-1} (M_n - m_n) + \frac{h}{2} \sum_{n=1}^{N-1} (M_{n-1} - m_{n-1}),$$

where

$$m_n = \inf_{t \in [0,h]} f(t_n + t, y(t_n + t)), \qquad n = 0, 1, \cdots, N - 1,$$

and

$$M_n = \sup_{t \in [0,h]} f(t_n + t, y(t_n + t)), \qquad n = 0, 1, \cdots, N - 1.$$

Then, as in Theorem 2.1, the conclusions follow.

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