MATHEMATICS OF COMPUTATION, VOLUME 25, NUMBER 116, OCTOBER, 1971

A Finite-Difference Method for Parabolic Differential Equations with Mixed Derivatives

By Jan Krzysztof Kowalski

Abstract. In a recent paper, P. Jamet constructed a positive difference operator for a parabolic differential operator whose coefficients are singular on the boundary, and proved the existence of a unique solution of the boundary-value problem for the differential equation using discrete barriers. In the present paper, Jamet's results are extended to the parabolic operator with mixed derivatives.

I. Introduction. Let G be a bounded domain in R^{n+1} and $P = (x_1, \cdots, x_n, t)$ denote an element of G. Let L be a differential operator of the form

$$(1.1)\qquad \begin{aligned} Lu(P) \equiv \sum_{i,j=1}^{n} a_{ij}(P) \frac{\partial^2 u}{\partial x_i\, \partial x_j}(P) \\ + \sum_{i=1}^{n} b_i(P) \frac{\partial u}{\partial x_i}(P) - c(P)u(P) - d(P)\frac{\partial u}{\partial t}(P). \end{aligned}$$

The coefficients $a_{ij} = a_{ji}$, b_i, c and d are smooth functions in the interior of G, but they may be singular as P approaches the boundary ∂G of G. The existence of the solution and the convergence of its approximations depend on the type of the singularities. We assume that the operator L is parabolic, i.e.

$$(1.2)\qquad \begin{aligned} \forall P \in G\ \forall(\xi_1, \cdots, \xi_n) \neq (0, \cdots, 0) \qquad \sum_{i,j=1}^{n} a_{ij}(P)\xi_i\xi_j > 0, \\ c(P) \geqq 0, \qquad d(P) > 0. \end{aligned}$$

Let Γ_1 be a nonempty subset of ∂G; $\Gamma_2 = \partial G - \Gamma_1$; f be a bounded function defined on $\bar{G}$ which is smooth in the interior of $\bar{G}$, and let $g \in C(\bar{G})$. We consider the boundary-value problem

$$(1.3)\qquad Lu(P) = f(P), \quad P \in G, \qquad u(P) = g(P), \quad P \in \Gamma_1.$$

We want the solution u to be continuous in $G \cup \Gamma_1$, bounded in G and of the class $C^2(G)$.

In [3], P. Jamet investigated problem (1.3), however, without mixed derivatives. In the present work, Jamet's fundamental theorem (Theorem 2.1) is applied to the problem with mixed derivatives.

II. Finite-Difference Operators of Positive Type. Let $h = (h_1, \cdots, h_n, \tau)$ be a parameter, m_i-integer, and for each h,

Received January 18, 1971.

AMS 1970 *subject classifications.* Primary 65N10, 65N05, Secondary 35K20.

Key words and phrases. Parabolic differential operator, boundary-value problem, mixed derivatives, positive difference operator, consistency of operators, convergence on the mesh, discrete barrier.

$$\bar{G}_h = \{(x_1, \cdots, x_n, t) \in \bar{G}: x_i = m_i h_i, i = 1, \cdots, n; t = m_0 \tau\}.$$

Let G_h and ∂G_h be two complementary nonempty subsets of $\bar{G}_h$. We assume that

$$\max_{P \in \partial G_h} d(P, \partial G) \to 0 \quad \text{as } h \to 0.$$

(We denote by $d(B, B')$ the distance between two sets B and B' in R^{n+1}.)

To each point $P \in G_h$ we associate a set $\mathfrak{N}(P) \subset \bar{G}_h$ which satisfies

$$P \in \mathfrak{N}(P) \quad \text{and} \quad \max_{P \in G_h} \max_{P' \in \mathfrak{N}(P)} d(P, P') \to 0 \quad \text{as } h \to 0,$$

and which is called the mesh-neighborhood of P in $\bar{G}_h$.

We say that $\bar{G}_h$ is simply connected, if $\forall P \in G_h \ \exists$ a sequence of points $P_0, \cdots, P_k$, such that $P_0 = P$; $P_i \in G_h$, $0 \leqq i \leqq k - 1$; $P_k \in \partial G_h$ and $P_{i+1} \in \mathfrak{N}(P_i)$ for $0 \leqq i \leqq k - 1$.

Let v be a function defined on $\bar{G}_h$. We define the finite-difference operator

$$L_h v(P) = \sum_{P' \in \mathfrak{N}(P)} A(P, P') v(P'). \tag{2.1}$$

If, for all $P \in G_h$,

$$A(P, P') > 0 \quad \text{for } P' \neq P; \qquad E(P) \equiv \sum_{P' \in \mathfrak{N}(P)} A(P, P') \leqq 0, \tag{2.2}$$

then the operator L_h is said to be "of positive type" or "positive".

The following maximum principle holds:

Let L_h be of positive type, $\bar{G}_h$ be connected and v be any function defined on $\bar{G}_h$ and such that $\forall P \in G_h$, $L_h v(P) \geqq 0$; then

$$\max_{P \in G_h} v(P) \leqq \max\left(0, \max_{P \in \partial G_h} v(P)\right).$$

Now, we introduce some notations and definitions. For any given subdomain G' of G, we define:

$$\bar{G}'_h = \bar{G}_h \cap \bar{G}', \qquad G'_h = \{P \in G_h \cap \bar{G}': \mathfrak{N}(P) \subset \bar{G}'_h\}, \qquad \partial G'_h = \bar{G}'_h - G'_h.$$

Definition 2.1. Let $G' \subset G$. We say that L_h is consistent with L in the norm $C_h(G'_h)$, if

$$\forall \varphi \in C^2(\bar{G}'), \qquad \max_{P \in G_h'} |L_h \varphi(P) - L\varphi(P)| \to 0 \quad \text{as } h \to 0.$$

Definition 2.2. Let $G' \subset G$, H be any set of parameters h, $\{\bar{G}_h\}_{h \in H}$ be a family of nets and $\mathfrak{F} = \{v(P, h)\}$ be a family of mesh-functions defined for each h on $\bar{G}_h \in \{\bar{G}_h\}$. We say that the family $\mathfrak{F}$ is equicontinuous in G', if

$$\forall \epsilon > 0 \ \exists \eta > 0 \ \forall h \in H \ \forall P, P' \in \bar{G}'_h,$$

$$d(P, P') < \eta \Rightarrow |v(P, h) - v(P', h)| < \epsilon.$$

Definition 2.3. Let $G' \subset G$. Let $\{v(P, h)\}$ be a family of mesh-functions defined on $\bar{G}_h \in \{\bar{G}_h\}$, and let u be a function defined on $\bar{G}'$. We say that $v(P, h)$ converges uniformly to $u(P)$ on G', if

$$\max_{P \in G_h'} |v(P, h) - u(P)| \to 0 \quad \text{as } h \to 0.$$

Now, let us consider an infinite set $H = \{h\}$ of vectors h with zero as an accumulation point and the corresponding family $\{L_h\}$ of operators.

Definition 2.4 Let $Q \in \partial G$. A function $B(P, Q)$ is a strong (local) discrete barrier at the point Q relative to the family $\{L_h\}$, if there exists a neighborhood N_Q of the point Q in the relative topology of $\bar{G}$ such that:

(2.3a) $B(\cdot, Q) \in C(N_Q)$,

(2.3b) $B(Q, Q) = 0$, $B(P, Q) < 0 \ \forall P \in N_Q - \{Q\}$,

(2.3c) $\forall P \in N_{Qh}$ $L_h B(P, Q) + E(P) \geqq 1$ for h small enough.

Now, we consider the following system of linear equations

$$(2.4) \qquad L_h v(P, h) = f(P), \quad P \in G_h, \qquad v(P, h) = g(P), \quad P \in \partial G_h.$$

It follows from the maximum principle that, if L_h is positive and $\bar{G}_h$ is simply connected, then the system (2.3) has a unique solution $v(P, h)$.

We shall assume that L_h is positive and $\bar{G}_h$ is connected. With these assumptions, P. Jamet proved the following theorem, [3].

THEOREM 2.1. *Let* $\mathfrak{F} = \{v(P, h)\}$ *be the family of the solutions of* (2.3) *for all* h *small enough. Let us assume*

(i) *There exists a function* $\varphi \in C(\bar{G})$ *such that* $L_h\varphi(P) \geqq 1$, $\forall P \in G_h$ *and for all* h.

(ii) *For any* $G' \subset \bar{G}' \subset G$ *and for any sequence* $\{v(P, h_n);\ h_n \to 0\} \subset \mathfrak{F}$, *there exists a subsequence which converges uniformly on* G' *to a solution of the equation* $Lu = f$.

(iii) *At each point* $Q \in \Gamma_1$, *there exists a strong discrete barrier relative to the family* $\{L_h\}$.

Then, problem (1.3) *has at least one solution* $u(P)$. *Moreover, if this solution is unique,* $v(P, h)$ *converges to* $u(P)$ *as* $h \to 0$, *uniformly in* $\bar{G} - N(\Gamma_2)$, *where* $N(\Gamma_2)$ *is an arbitrary neighborhood of* Γ_2.

In the subsequent sections, we investigate when the assumptions of Theorem 2.1 are satisfied.

III. Construction of the Finite-Difference Schemes for the Problem with Mixed Derivatives. Let h, τ be positive numbers and $\bar{G}_h$ be the rectangular net with the step h for the space variables $(x_1, \cdots, x_n)$ and τ for the time t. At each point $P \in \bar{G}_h$ we define a vector of positive integers $[m_i]_{i=1,\cdots,n}$.

At the point $P_0 \in \bar{G}_h$ we define a set

$$\begin{aligned}\mathfrak{N}_0(P_0) = & \bigcup_{i=1}^{n} \{P = P_0 + e_i m_i h,\ P = P_0 - e_i m_i h\} \\ & \cup \bigcup_{i=1}^{n} \bigcup_{j=i+1}^{n} \{P = P_0 + e_i m_i h + e_j m_j h \cdot \operatorname{sgn} a_{ij}(P_0), \\ & \qquad\qquad P = P_0 - e_j m_j h \cdot \operatorname{sgn} a_{ij}(P_0) - e_i m_i h\} \\ & \cup \{P_0, P_0 - e_{n+1}\tau\},\end{aligned}$$

where e_i is the versor of the x_i-axis $(1 \leqq i \leqq n)$ in R^{n+1}, and e_{n+1} the versor of t-axis.

By $\bar{\mathfrak{N}}_0(P_0)$ we denote the sum of all segments joining the point P_0 to each of the points of $\mathfrak{N}_0(P_0)$. Let

$$G_h^0 = \{P \in \bar{G}_h : \bar{\mathfrak{N}}_0(P) \subset \bar{G}\}, \qquad M_i = \max_{P \in G_h^0} m_i(P),$$

$$\Gamma_{1h} = \Big\{P_0 = (x_1^0, \cdots, x_n^0, t^0) \in \bar{G}_h - G_h^0 : \min_{P=(x_1,\ldots,x_n,t)\in\Gamma_1} |t - t^0| < \tau \text{ and } \forall i \min_{P \in \Gamma_1} |x_i - x_i^0| < hM_i\Big\}. \tag{3.1}$$

We choose the sets G_h and ∂G_h arbitrarily, provided $\Gamma_{1h} \subset \partial G_h$ and $G_h^0 \subset G_h$. At each point $P \in G_h^0$ we take $\mathfrak{N}(P) = \mathfrak{N}_0(P)$, and at the points $P \in G - G_h^0$ we define $\mathfrak{N}(P)$ arbitrarily, provided $\mathfrak{N}(P) \cap G_h^0 \neq \emptyset$; this choice guarantees the connectedness of $\bar{G}_h$ for h small. At each point $P \in G_h - G_h^0$ we define the operator L_h arbitrarily, provided conditions (2.2) are satisfied at that point. For $P \in G_h^0$ we take

$$\begin{aligned} L_h v(P) = {} & \sum_{i=1}^n \alpha_i(P) v_{,i,i} + \sum_{i=1}^n \sum_{j=i+1}^n a_{ij}^+(P) v_{,i+j} - \sum_{i=1}^n \sum_{j=i+1}^n a_{ij}^-(P) v_{,i-j} \\ & + \sum_{i=1}^n \beta_i(P)(v_{,i} + v_{,i})/2 - c(P)v - d(P)v_{\bar{t}}, \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} v_{,i}(P) &= [v(P + e_i m_i h) - v(P)]/m_i h, \\ v_{,i}(P) &= [v(P) - v(P - e_i m_i h)]/m_i h, \qquad v_{,i,j} = (v_{,i})_{,j}; \\ v_{,i+j}(P) &= [v(P + e_i m_i h + e_j m_j h) - 2v(P) + v(P - e_i m_i h - e_j m_j h)]/h^2 m_i m_j, \\ v_{,i-j}(P) &= [v(P + e_i m_i h - e_j m_j h) - 2v(P) + v(P - e_i m_i h + e_j m_j h)]/h^2 m_i m_j, \\ v_{\bar{t}}(P) &= [v(P) - v(P - e_{n+1}\tau)]/\tau, \\ a_{ij}^+(P) &= [a_{ij}(P) + |a_{ij}(P)|]/2, \quad a_{ij}^-(P) = a_{ij}(P) - a_{ij}^+(P); \quad i, j = 1, \cdots, n; \end{aligned}$$

and

$$\alpha_i(P) = A_i(P) \equiv a_{ii}(P) - \sum_{j=1; j\neq i}^n \frac{m_i}{m_j} |a_{ij}(P)|, \qquad \beta_i(P) = b_i(P). \tag{3.3}$$

If the operator L_h is positive, then its coefficients satisfy the following system of inequalities

$$A_i(P) - \frac{m_i h}{2} |b_i(P)| > 0.$$

If $m_i h \; |b_i(P)| = o(A_i(P))$ for P near the boundary ∂G, then the upper system is equivalent to the system

$$a_{ii}(P) - \sum_{j=1; j\neq i}^n \frac{m_i}{m_j} |a_{ij}(P)| > 0. \tag{3.4}$$

Now, we shall prove the existence of the solution of system (3.4). Let $B = [b_{lr}]_{l,r=1,\cdots,n}$ be an arbitrary matrix and assume that $0 \leqq k \leqq n - 1$; $k < i$, $j \leqq n$. We denote

$$B^k = \begin{vmatrix} b_{11} & \cdots & b_{1k} \\ \cdots & \cdots & \cdots \\ b_{k1} & \cdots & b_{kk} \end{vmatrix}, \qquad B^k_{.j} = \begin{vmatrix} b_{11} & \cdots & b_{1,k-1} & b_{1j} \\ \cdots & \cdots & \cdots & \cdots \\ b_{k-1,1} & \cdots & b_{k-1,k-1} & b_{k-1,j} \\ b_{k1} & \cdots & b_{k,k-1} & b_{kj} \end{vmatrix},$$

$$B^k_{i.} = \begin{vmatrix} b_{11} & \cdots & b_{1,k-1} & b_{1k} \\ \cdots & \cdots & \cdots & \cdots \\ b_{k-1,1} & \cdots & b_{k-1,k-1} & b_{k-1,k} \\ b_{i1} & \cdots & b_{i,k-1} & b_{ik} \end{vmatrix}, \qquad B^k_{ij} = \begin{vmatrix} b_{11} & \cdots & b_{1,k-1} & b_{1j} \\ \cdots & \cdots & \cdots & \cdots \\ b_{k-1,1} & \cdots & b_{k-1,k-1} & b_{k-1,j} \\ b_{i1} & \cdots & b_{i,k-1} & b_{ij} \end{vmatrix},$$

where $|B| = \det B$.

Let $B^k(m, p)$ be the minor of B^k after striking out the mth column and pth row, let $B^k(m) = B^k(m, k)$. We introduce the analogous notation for the minors of $B^k_{i.}$, $B^k_{.j}$ and B^k_{ij}.

LEMMA 3.1. *Using this notation, the following equality is valid*:

$$B^l B^l_{ij} - B^l_{i.} B^l_{.j} = B^{l-1} B^{l+1}_{ij}. \tag{3.5}$$

Proof. We carry out the proof by induction. For $l = 1$ the formula (3.5) is valid (we take $B^0 = 1$). Suppose that the theorem is true for $l = k - 1 \geqq 1$.

We compute the left and right side of the formula (3.5) for $l = k$.

$$\begin{aligned} \text{Left} &= B^k\left[\sum_{m=1}^{k-1} (-1)^{k+m} b_{im} B^k_{.j}(m) + b_{ij} B^{k-1}\right] \\ &\quad - B^k_{.j}\left[\sum_{m=1}^{k-1} (-1)^{k+m} b_{im} B^k(m) + b_{ik} B^{k-1}\right] \\ &= \sum_{m=1}^{k-1} (-1)^{k+m} b_{im} [B^k B^k_{.j}(m) - B^k(m) B^k_{.j}] + b_{ij} B^k B^{k-1} - b_{ik} B^k_{.j} B^{k-1}. \end{aligned}$$

We compute now the term in square brackets, using the Laplace formula.

$$\begin{aligned} B^k B^k_{.j}(m) - B^k(m) B^k_{.j} &= B^k_{.j}(m)\left[\sum_{p=1}^{k-1} (-1)^{m+p} b_{pm} B^k(m, p) + (-1)^{m+k} b_{km} B^k(m)\right] \\ &\quad - B^k(m)\left[\sum_{p=1}^{k-1} (-1)^{m+p} b_{pm} B^k_{.j}(m, p) + (-1)^{m+k} B^k_{.j}(m)\right] \\ &= \sum_{p=1}^{k-1} (-1)^{m+p} b_{pm} [B^k(m, p) B^k_{.j}(m) - B^k_{.j}(m, p) B^k(m)]. \end{aligned}$$

We introduce a matrix $C(m, p) = [c_{rs}]$ with the elements:

	$1 \leqq r < p$	$p \leqq r \leqq k - 1$	$r = i$
$1 \leqq s < m$	b_{rs}	$b_{r+1,s}$	b_{ps}
$m \leqq s \leqq k - 1$	$b_{r,s+1}$	$b_{r+1,s+1}$	$b_{p,s+1}$
$s = j$	b_{rj}	$b_{r+1,j}$	b_{pj}

Then

$$B^k_{\cdot j}(m) = (-1)^{k-1-p}[C(m,p)]^{k-1}_{ij}, \qquad B^k_{\cdot j}(m,p) = [C(m,p)]^{k-1}_{\cdot j},$$
$$B^k(m,p) = [C(m,p)]^{k-1}, \qquad B^k(m) = (-1)^{k-1-p}[C(m,p)]^{k-1}_{i\cdot}$$

Using the inductive assumption we get

$$B^k_{\cdot j}(m)B^k(m,p) - B^k(m)B^k_{\cdot j}(m,p) = -B^{k-1}(m,p)B^{k+1}_{\cdot j}(m).$$

Hence,

$$\text{Left} = \sum_{m=1}^{k-1}(-1)^{k+m}b_{im}\sum_{p=1}^{k-1}(-1)^{m+p+1}b_{pm}B^{k-1}(m,p)B^{k+1}_{\cdot j}(m)$$
$$+ b_{ij}B^kB^{k-1} - b_{ik}B^k\, B^{k-1}.$$

But

$$\text{Right} = B^{k-1}\left[\sum_{m=1}^{k-1}(-1)^{m+k+1}b_{im}B^{k+1}_{\cdot j}(m) - b_{ik}B^k_{\cdot j} + b_{ij}B^k\right]$$
$$= \sum_{m=1}^{k-1}(-1)^{k+m+1}b_{im}B^{k+1}_{\cdot j}(m)\sum_{p=1}^{k-1}(-1)^{p+m}b_{pm}B^{k-1}(m,p)$$
$$- b_{ik}B^{k-1}B^k_{\cdot j} + b_{ij}B^kB^{k-1} = \text{Left},$$

which concludes the proof of the lemma.

For each $P \in G$ we set

$$b_{ij}(P) = a_{ij}(P), \qquad i = j,$$
$$= -|a_{ij}(P)|, \qquad i \neq j.$$

THEOREM 3.1. *If*

$$\exists\gamma > 0\ \exists M > 0\ \forall P \in G\ \forall(\xi_1, \cdots, \xi_n) \qquad \sum_{i,j=1}^n b_{ij}(P)\xi_i\xi_j \geqq \gamma\sum_{i=1}^n \xi_i^2$$
$$\text{and} \quad |b_{ij}(P)| < M,$$

then the system of inequalities (3.4) *has an integer solution and* M_i *given by* (3.1) *are bounded.*

Proof. Let P be a fixed point. We transform (3.4) to the form

$$\sum_{j=1}^n b_{ij}\mu_j > 0, \quad \text{where } \mu_j = 1/m_j.$$

For $i = 2, 3, \cdots, n$, we multiply the first inequality by b_{i1} and the ith by b_{11} and sum them. Using the inequalities $b_{ii} > 0$ and $b_{ij} \leqq 0$ for $i \neq j$, we get

$$\sum_{j=2}^n \mu_j(b_{11}b_{ij} - b_{1j}b_{i1}) > 0.$$

Let $b^{(2)}_{ij} = (b_{11}b_{ij} - b_{i1}b_{1j})/b_{11}$ and, for $k = 2, 3, \cdots, n-1$ and $i, j \geqq k+1$, let $b^{(k+1)}_{ij} = (b^{(k)}_{kk}b^{(k)}_{ij} - b^{(k)}_{kj}b^{(k)}_{ik})/b^{(k)}_{kk}$. Using Lemma 3.1 we deduce that

$$b^{(k+1)}_{ij} = B^{k+1}_{ij}\left[\prod_{l=1}^k b^{(l)}_{ll}\right]^{-1}. \tag{3.6}$$

For $k = 1$ formula (3.6) is true. Suppose that it is valid for $k = m - 1 \geqq 1$. Then

$$b_{ij}^{(m+1)} = [B_{ij}^m B^m - B_{\cdot j}^m B_{i\cdot}^m] \cdot \left[\prod_{l=1}^{m-1} b_{ll}^{(l)}\right]^{-2} \Big/ b_{mm}^{(m)} = B^{m-1} B_{ij}^{m+1} \left[\prod_{l=1}^{m-1} b_{ll}^{(l)}\right]^{-2} \Big/ b_{mm}^{(m)}$$

$$= \frac{b_{m-1,m-1}^{(m-1)} \prod_{l=1}^{m-2} b_{ll}^{(l)}}{b_{mm}^{(m)} \left[\prod_{l=1}^{m-1} b_{ll}^{(l)}\right]^2} \cdot B_{ij}^{m+1} = \left[\prod_{l=1}^{m} b_{ll}^{(l)}\right]^{-1} B_{ij}^{m+1},$$

therefore the formula (3.6) is true.

It follows from the assumption of the theorem that there exist two positive numbers M_0 and C_0 (independent of P) such that $\forall k\ \forall i,\ j \geqq k,\ b_{ii}^{(k)} > C_0$, $|b_{ij}^{(k)}| < M_0$. Moreover, for $i \neq j$, $b_{ij}^{(k+1)} = (b_{kk}^{(k)} b_{ij}^{(k)} - b_{ik}^{(k)} b_{kj}^{(k)})/b_{kk}^{(k)} \leqq 0$, because $b_{kk}^{(k)} > 0$ and the other three numbers are nonpositive. Therefore, for each k, we get the following system of inequalities:

$$\sum_{j=k}^{n} b_{ij}^{(k)} \mu_j > 0, \qquad i = k, k+1, \cdots, n.$$

For $k = n - 1$ the system consists of two inequalities:

$$b_{n-1,n-1}^{(n-1)} \mu_{n-1} + b_{n-1,n}^{(n-1)} \mu_n > 0; \qquad b_{n,n-1}^{(n-1)} \mu_{n-1} + b_{nn}^{(n-1)} \mu_n > 0. \tag{3.7}$$

We put $\mu_n' = 1$ and set out to find the rational numbers $\mu_1', \cdots, \mu_{n-1}'$ and C, such that (3.4) is satisfied for $\mu_i = C\mu_i'$ and $1/\mu_i$ integer. From (3.7),

$$\frac{|b_{n-1,n}^{(n-1)}|}{b_{n-1,n-1}^{(n-1)}} < \mu_{n-1}' < \frac{b_{nn}^{(n-1)}}{|b_{n,n-1}^{(n-1)}|}$$

and

$$\frac{b_{nn}^{(n-1)}}{|b_{n,n-1}^{(n-1)}|} - \frac{|b_{n-1,n}^{(n-1)}|}{b_{n-1,n-1}^{(n-1)}} = \frac{b_{nn}^{(n)}}{|b_{n,n-1}^{(n-1)}|} \geqq \frac{C_0}{M_0}.$$

For

$$\rho = E\left[\frac{|b_{n-1,n}^{(n-1)}|}{b_{n-1,n-1}^{(n-1)}} E\left(3\frac{M_0}{C_0} + 1\right) + 1\right] \Big/ E\left(3\frac{M_0}{C_0} + 1\right),$$

the following inequalities are valid:

$$\rho \geqq \left[\frac{|b_{n-1,n}^{(n-1)}|}{b_{n-1,n-1}^{(n-1)}} E\left(3\frac{M_0}{C_0} + 1\right) + 1\right] \Big/ E\left(3\frac{M_0}{C_0} + 1\right) \geqq \frac{|b_{n-1,n}^{(n-1)}|}{b_{n-1,n-1}^{(n-1)}} + \frac{C_0}{3M_0 + C_0};$$

$$\rho \leqq \left[\frac{|b_{n-1,n}^{(n-1)}|}{b_{n-1,n-1}^{(n-1)}} E\left(3\frac{M_0}{C_0} + 1\right) + 2\right] \Big/ E\left(3\frac{M_0}{C_0} + 1\right)$$

$$\leqq \frac{|b_{n-1,n}^{(n-1)}|}{b_{n-1,n-1}^{(n-1)}} + \frac{2C_0}{3M_0} \leqq \frac{b_{nn}^{(n-1)}}{|b_{n,n-1}^{(n-1)}|} - \frac{C_0}{3M_0}.$$

We can take $\mu_{n-1}' = \rho$. Then

$$\mu_{n-1}' > \frac{C_0}{3M_0 + C_0} \equiv \nu_{n-1} \quad \text{and} \quad \mu_{n-1}' < \frac{M_0}{C_0} + \frac{2C_0}{3M_0} \equiv \nu_{n-1}'.$$

Moreover,

$$b_{n-1,n-1}^{(n-1)}\mu_{n-1}' + b_{n-1,n}^{(n-1)}\mu_n' \geqq |b_{n-1,n}^{(n-1)}| + b_{n-1,n-1}^{(n-1)}\frac{C_0}{3M_0+C_0} - |b_{n-1,n}^{(n-1)}| > \frac{C_0^2}{3M_0+C_0},$$

and

$$b_{n,n-1}^{(n-1)}\mu_{n-1}' + b_{nn}^{(n-1)}\mu_n' \geqq -|b_{n,n-1}^{(n-1)}|\cdot\left[\frac{b_{n-1,n}^{(n-1)}}{b_{n-1,n-1}^{(n-1)}} + \frac{2C_0}{3M_0}\right] + b_{nn}^{(n-1)}$$

$$= b_{nn}^{(n)} - |b_{n,n-1}^{(n-1)}|\frac{2C_0}{3M_0} > \frac{C_0}{3}.$$

Let

$$K_{n-1} = \min\left(\frac{C_0^2}{3M_0+C_0}, \frac{C_0}{3}\right).$$

Then

$$\sum_{s=n-1}^{n} b_{ls}^{(n-1)}\mu_s' > K_{n-1}, \qquad l = n-1, n.$$

Suppose that we have defined $\mu_{n-1}', \cdots, \mu_{k+1}'$ such that $\sum_{s=k+1}^{n} b_{ls}^{(k+1)}\mu_s' > K_{k+1}$ and $\nu_l' > \mu_l' > \nu_l > 0$ for $l = k+1, \cdots, n$, where K_l, ν_l, ν_l' depend only on l, M_0 and C_0. Now, we must define $\mu_k', K_k, \nu_k, \nu_k'$ such that

$$\sum_{s=k}^{n} b_{ls}^{(k)}\mu_s' > K_k, \qquad l = k, \cdots, n \quad \text{and} \quad \nu_k' > \mu_k' > \nu_k > 0.$$

This system is equivalent to the system

$$-\sum_{s=k+1}^{n}\frac{b_{ks}^{(k)}}{b_{kk}^{(k)}}\mu_s' < \mu_k' < -\sum_{s=k+1}^{n}\frac{b_{ls}^{(k)}}{b_{lk}^{(k)}}\mu_s', \qquad l = k+1, \cdots, n.$$

The following inequalities hold

$$\sigma_k \equiv \sum_{s=k+1}^{n}\frac{|b_{ks}^{(k)}|}{b_{kk}^{(k)}}\mu_s' < \frac{M_0}{C_0}\sum_{s=k+1}^{n}\nu_s';$$

$$-\sum_{s=k+1}^{n}\left(\frac{b_{ls}^{(k)}}{b_{lk}^{(k)}} - \frac{b_{ks}^{(k)}}{b_{kk}^{(k)}}\right)\mu_s' = -\sum_{s=k+1}^{n}\frac{b_{ls}^{(k+1)}}{b_{lk}^{(k)}}\mu_s' > \frac{K_{k+1}}{M_0}.$$

Let

$$\mu_k' = E\left[\sigma_k E\left(3\frac{M_0}{K_{k+1}} + 1\right) + 2\right] \Big/ E\left(3\frac{M_0}{K_{k+1}} + 1\right).$$

Then

$$\sigma_k + \frac{K_{k+1}}{3M_0 + K_{k+1}} \leqq \mu_k' \leqq \sigma_k + \frac{2K_{k+1}}{3M_0} < -\sum_{s=k+1}^{n}\frac{b_{ls}^{(k)}}{b_{lk}^{(k)}}\mu_s' - \frac{K_{k+1}}{3M_0},$$

therefore

$$\nu_k = \frac{K_{k+1}}{3M_0 + K_{k+1}}, \qquad \nu_k' = \frac{M_0}{C_0}\sum_{s=k+1}^{n}\nu_s' + \frac{2K_{k+1}}{3M_0},$$

and

$$\sum_{s=k}^{n} b_{ks}^{(k)}\mu_s' \geqq b_{kk}^{(k)}\left[\sum_{s=k+1}^{n} \frac{|b_{ks}^{(k)}|}{b_{kk}^{(k)}} \mu_s' + \frac{K_{k+1}}{3M_0 + K_{k+1}}\right] - \sum_{s=k+1}^{n} |b_{ks}^{(k)}|\, \mu_s' \geqq \frac{C_0 K_{k+1}}{3M_0 + K_{k+1}},$$

and, for $l > k$,

$$\sum_{s=k}^{n} b_{ls}^{(k)}\mu_s' \geqq b_{lk}^{(k)}\left[\sum_{s=k+1}^{n} \frac{|b_{ks}^{(k)}|}{b_{kk}^{(k)}} \mu_s' + \frac{2K_{k+1}}{3M_0}\right] + \sum_{s=k+1}^{n} b_{ls}^{(k)}\mu_s' \geqq \frac{K_{k+1}}{3}.$$

We can then take

$$K_k = \min\left(\frac{K_{k+1}}{3}, \frac{C_0 K_{k+1}}{3M_0 + K_{k+1}}\right).$$

We have the estimates

$$C_0 > K_k > C_0^{n-k+1}/(3M_0 + 3C_0)^{n-k}, \qquad \nu_k > [C_0/(3M_0 + 3C_0)]^{n-k+1},$$

$$\nu_k' < \frac{5}{3}\frac{(n-k)^2(n-k+1)}{2}\left(\frac{M_0}{C_0}\right)^k.$$

For $\mu_1', \cdots, \mu_n'$, defined as before, we take

$$m_i = \frac{1}{\mu_i'} \operatorname*{lcm}_{k=1,\ldots,n-1} \left[E\left(\sigma_k E\left(\frac{3M_0}{K_{k+1}} + 1\right)\right) + 2\right],$$

where lcm denotes the least common multiple of the numbers in brackets for $k = 1, \cdots, n-1$; $K_n = C_0$, $\sigma_n = |b_{n-1,n}^{(n-1)}|/b_{n-1,n-1}^{(n-1)}$. Then the numbers m_i satisfy the inequality

$$m_i \leqq \frac{1}{\nu_i}\prod_{k=1}^{n-1} E\left[\frac{M_0}{C_0}\sum_{s=k+1}^{n} \nu_s' E\left(\frac{3M_0}{K_{k+1}} + 1\right) + 2\right]$$

$$\leqq \left(\frac{3M_0 + 3C_0}{C_0}\right)^{n-i+1} \prod_{k=1}^{n-1} E\left[\frac{5}{6}\frac{M_0}{C_0} n^3(n+1)\left(\frac{M_0}{C_0}\right)^n \cdot E\left(\frac{3M_0(3M_0 + 3C_0)^{n-k-1}}{C_0^{n-k}} + 1\right) + 2\right].$$

The estimate is independent of P, and the theorem is proved.

In the particular case, if we can take for each point $P \in G_h$ the same numbers m_i, satisfying (3.4), then we can consider, instead of the square net, a rectangular net with steps $h_i = m_i h$ $(i = 1, 2, \cdots, n)$. Then the mesh-neighborhood of each point P consists of the mesh-points which lie nearest to P. In this case, the operator L need not be uniformly elliptic, as the matrix $[b_{ij}]$ need not be positive definite near the boundary.

However, if the coefficients of L are singular on the boundary, then the operator L_h given by (3.3) is not always of positive type. In this case we define following P. Jamet:

$$\alpha_i^+(P, h) = \exp\int_{x_i}^{x_i+h_i/2} \frac{b_i(x_1, \cdots, x_{i-1}, y, x_{i+1}, \cdots, x_n, t)}{A_i(x_1, \cdots, x_{i-1}, y, x_{i+1}, \cdots, x_n, t)}\, dy,$$

$$\alpha_i^-(P, h) = \exp\int_{x_i}^{x_i-h_i/2} \frac{b_i(x_1, \cdots, x_{i-1}, y, x_{i+1}, \cdots, x_n, t)}{A_i(x_1, \cdots, x_{i-1}, y, x_{i+1}, \cdots, x_n, t)}\, dy$$

and

$$(3.8)\qquad \begin{aligned} \alpha_i(P, h) &= A_i(P)[\alpha_i^+(P, h) + \alpha_i^-(P, h)]/2, \\ \beta_i(P, h) &= A_i(P)[\alpha_i^+(P, h) - \alpha_i^-(P, h)]/h_i. \end{aligned}$$

We substitute the α_i and β_i as defined in (3.2) for those i, for which A_i is singular. The operator corresponding to (3.8) is always positive, because

$$\alpha_i v_{,i,\bar{i}} + \beta_i(v_{,i} + v_{,\bar{i}})/2 = A_i[\alpha_i^+ v_{,i} - \alpha_i^- v_{,\bar{i}}]/h_i$$

and the coefficients α_i^+ and α_i^- are positive.

LEMMA 3.2. *The operator* (3.2) *with the coefficients* (3.8) *is consistent with L in the norm $C_h(G_h')$ for any $G' \subset \bar{G}' \subset G$.*

Proof. For G' there exist the numbers $N > 0$, $\epsilon > 0$ such that $|b_i(P)/A_i(P)| \leqq N$ and $A_i(P) \geqq \epsilon$ for $P \in G'$. Therefore

$$\alpha_i^+(P, h) = \exp\left(\frac{h_i}{2}\frac{b_i(P)}{A_i(P)} + O(h_i^2)\right) = 1 + \frac{h_i}{2}\frac{b_i(P)}{A_i(P)} + O(h_i^2);$$

$$\alpha_i^-(P, h) = 1 - \frac{h_i}{2}\frac{b_i(P)}{A_i(P)} + O(h_i^2).$$

Hence,

$$(3.9)\qquad \alpha_i(P, h) = A_i(P) + O(h_i^2), \qquad \beta_i(P, h) = b_i(P) + O(h_i).$$

Because

$$v_{,i,j} + v_{,\bar{i},\bar{j}} = v_{,i+j} - \frac{m_i}{m_j} v_{,i,\bar{i}} - \frac{m_j}{m_i} v_{,j,\bar{j}}$$

and

$$v_{,i,\bar{j}} + v_{,\bar{i},j} = \frac{m_i}{m_j} v_{,i,\bar{i}} + \frac{m_j}{m_i} v_{,j,\bar{j}} - v_{,i-j},$$

we have

$$\begin{aligned} L_h v(P) = {} & \sum_{i=1}^n [A_i(P) + O(h_i^2)] v_{,i,\bar{i}} + \sum_{i=1}^n \sum_{j=i+1}^n a_{ij}^+(P) v_{,i+j} \\ & - \sum_{i=1}^n \sum_{j=i+1}^n a_{ij}^-(P) v_{,i-j} + \sum_{i=1}^n [b_i(P) + O(h_i)] \frac{v_{,i} + v_{,\bar{i}}}{2} - c(P)v - d(P)v_t \\ = {} & \sum_{i=1}^n [a_{ii}(P) + O(h_i^2)] v_{,i,\bar{i}} \\ & + \sum_{i=j}^n \sum_{j=i+1}^n \{a_{ij}^+(P)(v_{,i,j} + v_{,\bar{i},\bar{j}}) + a_{ij}^-(P)(v_{,i,\bar{j}} + v_{,\bar{i},j})\} \\ & + \sum_{i=1}^n [b_i(P) + O(h_i)] \frac{v_{,i} + v_{,\bar{i}}}{2} - c(P)v - d(P)v_t. \end{aligned}$$

Using this equation, we deduce that for $v \in C^2(G')$

$$\max_{P \in G_h'} |L_h v(P) - Lv(P)| = O(h + \tau).$$

We take h_i/τ = const, therefore $\max_{P \in G_h'} |L_h v(P) - Lv(P)| = O(h)$.

Moreover, if a_{ij}, b_i, c and $d \in C^p(G)$, then in any $G' \subset \bar{G}' \subset G$ the difference quotients of the order p of α_i, β_i, c and d are uniformly bounded for all $P \in G_h'$ and for all h sufficiently small.

IV. Sufficient Conditions for Uniform Boundedness of the Solutions. In this section we study the existence of a function $\varphi(P)$ which satisfies condition (i) of Theorem 2.1. The existence of such a function guarantees the uniform boundedness of the approximations $v(P, h)$. The following criteria are given in [3] (we assume that $G_h = G_h^0$ and L_h is defined by formula (3.2) together with (3.3) or (3.8)):

1. Suppose $c(P) > m > 0$ in G, then we can take $\varphi(P) = -1/m$.

2. Suppose $d(P) > m > 0$ in G, then we take $\varphi(P) = -(K + t/m)$, where $K > 0$ is chosen so large that $\varphi(P) < 0$ in G.

3. If there exists an i such that $A_i(P) > m > 0$ and $|b_i(P)| < M$ in G, then $\varphi(P) = K(\exp(\rho x_i) - K')$, with $\rho > M/m$ and K, K' sufficiently large, satisfies condition (i) of Theorem 2.1.

V. Estimates of the Solutions of the Finite-Difference Problem. Let L_h be a finite-difference operator of positive type which has the form (3.2) for all $P \in G_h^0$. Let $\mathfrak{F} = \{v(P, h)\}$ be a family of mesh-functions defined for each h on $\bar{G}_h \in \{\bar{G}_h\}$ and such that $L_h v(P, h) = f(P)$, $\forall P \in G_h^0$; $\mathfrak{F}^{(p)}$ be the family of all difference quotients of order p of the functions of $\mathfrak{F}$; G' be an arbitrary interior subdomain of G (i.e., $G' \subset \bar{G}' \subset G$). Let the numbers m_i be the same for all $P \in G$. Let the coefficients $a_{ij}, b_i, c, d \in C^{(n+1)}(G)$ and their derivatives of order $(n + 1)$ be Lipschitz-continuous in G'. We intend to show that the condition (ii) of Theorem 2.1 is satisfied.

We shall firstly prove the uniform boundedness of the sums $h^{n+1} \sum_{G_h} w^2(P, h)$, where w are difference quotients of order $\leqq n + 1$ of the functions of $\mathfrak{F}$, $\mathfrak{F}^{(1)}$, $\mathfrak{F}^{(2)}$. To avoid complications in the proof, we will develop the argument only in the case $n = 2$.

Let h be so small that $G_h' \subset G_h^0$. Then, at each point $P \in G_h'$, we have

$$\begin{aligned} L_h v \equiv{}& \alpha_{11} v_{,1,\bar{1}} + \alpha_{+12}(v_{,1,2} + v_{,\bar{1},\bar{2}}) + \alpha_{-12}(v_{,1,\bar{2}} + v_{,\bar{1},2}) + \alpha_{22} v_{,2,\bar{2}} \\ &+ \beta_1(v_{,1} + v_{,\bar{1}})/2 + \beta_2(v_{,2} + v_{,\bar{2}})/2 - cv - dv_{\bar{t}} = f, \end{aligned} \tag{5.1}$$

where

$$\alpha_{11} = \alpha_1 + \frac{h_1}{h_2}|a_{12}|, \qquad \alpha_{22} = \alpha_2 + \frac{h_2}{h_1}|a_{12}|, \qquad \alpha_{+12} = a_{12}^+, \qquad \alpha_{-12} = a_{12}^-,$$

and α_i, β_i are defined by formula (3.3) or (3.8).

Moreover, we shall assume that $d \equiv 1$ in G'. There exist constants m and M, such that for all $P \in G_h'$ we have

$$\begin{aligned} &0 < m < \alpha_i(P, h) < M, \qquad |\alpha_{ij}(P)| < M, \qquad |\beta_i(P)| < M, \qquad i, j = 1, 2; \\ &|c(P)| < M, \qquad |f(P)| < M, \qquad |v(P, h)| < M \quad \text{for } v \in \mathfrak{F} \end{aligned} \tag{5.2}$$

for h small enough. We assume that M is also an upper bound for any of the difference quotients of α_{ij}, β_i, c, f of order $\leqq n + 2$. We denote $L_h^0 v = L_h v + v_{\bar{t}}$.

Let $h = (h_1, h_2, \tau)$ be fixed. We consider an expanding sequence of concentric rectangles in G'_h, say $\{Q_0, \cdots, Q_N\}$, such that

$$Q_l = \{P = (ih_1, jh_2, k\tau) \in G'_h\colon i_l \leqq i \leqq i'_l, j_l \leqq j \leqq j'_l, k_l \leqq k \leqq k'_l\};$$
$$i_{l+1} = i_l - 1, i'_{l+1} = i'_l + 1, j_{l+1} = j_l - 1, j'_{l+1} = j'_l + 1,$$
$$k_{l+1} = k_l - 1, k'_{l+1} = k'_l + 1.$$

We define

$$S_l = Q_l - Q_{l-1}, \qquad R_{1l} = \{P = (ih_1, jh_2, k\tau) \in Q_l\colon i = i_l \vee i = i'_l\},$$
$$R_{2l} = \{P = (ih_1, jh_2, k\tau) \in Q_l\colon j = j_l \vee j = j'_l\},$$
$$T_l = (R_{1l} \cup R_{1,l-1}) \cap (R_{2l} \cup R_{2,l-1})$$

and $h_2/h_1 = \lambda$, $\tau/h_1 = \mu$, $\Lambda = (\frac{5}{2} + \frac{5}{2}\lambda + 2(1 + \lambda^2)^{1/2})M$.

LEMMA 5.1. *For every function w defined on $\bar{G}_h$, the following inequality holds*:

$$(5.3)\quad \begin{aligned}
&(m - \tfrac{3}{2}M\kappa)\tau h_1 h_2 \sum\sum_{Q_l}\sum (w_{,1}^2 + w_{,\bar{1}}^2 + w_{,2}^2 + w_{,\bar{2}}^2) \\
&\leqq 2\tau h_1 h_2 \sum\sum_{Q_l}\sum |w|\cdot|L_h w| + \tau\lambda\Big(\sum_{R_{1,l+1}}\sum \alpha_{11}w^2 - \sum_{R_{1l}}\sum \alpha_{11}w^2\Big) \\
&+ \frac{\tau}{\lambda}\Big(\sum_{R_{2,l+1}}\sum \alpha_{22}w^2 - \sum_{R_{2l}}\sum \alpha_{22}w^2\Big) + \tau h_1\Big(\Lambda + \frac{\lambda}{\mu}\Big)\Big(\sum_{S_l}\sum w^2 + \sum_{S_{l+1}}\sum w^2\Big) \\
&+ 2M\Big(1 + \frac{3}{\kappa}\Big)\tau h_1 h_2 \sum\sum_{Q_{l+1}}\sum w^2 + M\tau \sum_{T_{l+1}} w^2.
\end{aligned}$$

Proof. For any function w defined on $\bar{G}_h$, we denote

$$w(ih_1, jh_2, k\tau) = w; \qquad w((i+1)h_1, jh_2, k\tau) = w^{+i};$$
$$w((i-1)h_1, jh_2, k\tau) = w^{-i}; \qquad w(i_1h_1, jh_2, k\tau) = w|_{i=i_1};$$
$$w(i_1h_1, jh_2, k\tau) - w(i_0h_1, jh_2, k\tau) = w|_{i=i_0}^{i=i_1},$$

and analogous for the indices j and k. Let

$$\begin{aligned}
C(w) = {}& \alpha_{11}^{+i} w_{,1}^2 + \alpha_{11,1} ww_{,1} + [(\alpha_{+12}^{-i} + \alpha_{+12}^{+j})w_{,\bar{1}}w_{,2} + \alpha_{+12,\bar{1}}ww_{,2} + \alpha_{+12,2}ww_{,\bar{1}}] \\
&+ [(\alpha_{-12}^{+i} + \alpha_{-12}^{+j})w_{,1}w_{,2} + \alpha_{-12,1}ww_{,2} + \alpha_{-12,2}ww_{,1}] + \alpha_{22}^{+j}w_{,2}^2 \\
&+ \alpha_{22,2}ww_{,2} + \frac{w}{2}(\beta_1 w)_{,1} - \frac{\beta_1}{2}ww_{,1} + \frac{w}{2}(\beta_2 w)_{,2} - \frac{\beta_2}{2}ww_{,2} + cw^2;
\end{aligned}$$

$$\begin{aligned}
\bar{C}(w) = {}& \alpha_{11}^{-i} w_{,\bar{1}}^2 + \alpha_{11,\bar{1}} ww_{,\bar{1}} + [(\alpha_{+12}^{+i} + \alpha_{+12}^{-j})w_{,1}w_{,\bar{2}} + \alpha_{+12,1}ww_{,\bar{2}} + \alpha_{+12,\bar{2}}ww_{,1}] \\
&+ [(\alpha_{-12}^{-i} + \alpha_{-12}^{-j})w_{,\bar{1}}w_{,\bar{2}} + \alpha_{-12,\bar{1}}ww_{,\bar{2}} + \alpha_{-12,\bar{2}}ww_{,\bar{1}}] + \alpha_{22}^{-j}w_{,\bar{2}}^2 + \alpha_{22,\bar{2}}ww_{,\bar{2}} \\
&+ \frac{w}{2}(\beta_1 w)_{,\bar{1}} - \frac{\beta_1}{2}ww_{,\bar{1}} + \frac{w}{2}(\beta_2 w)_{,\bar{2}} - \frac{\beta_2}{2}ww_{,\bar{2}} + cw^2.
\end{aligned}$$

Let $j_0 \leqq j \leqq j'_0$, $k_0 \leqq k \leqq k'_0$. By summation by parts with respect to i, we get

$$h_1h_2 \sum_{i=i_0}^{i_0'} [C(w) + \bar{C}(w)]$$

$$= h_1h_2 \sum_{i=i_0}^{i_0'} \{-\alpha_{11} w w_{,1,\bar{1}} - \alpha_{11} w w_{,1,\bar{1}} + w_{,1}(\alpha_{-12}w)_{,2}$$

$$+ w_{,\bar{1}}(\alpha_{-12}w)_{,\bar{2}} - \alpha_{-12} w w_{,\bar{1},\bar{2}} - \alpha_{-12} w w_{,1,\bar{2}} - \alpha_{+12} w w_{,1,2}$$

$$- \alpha_{+12} w w_{,\bar{1},\bar{2}} + w_{,\bar{1}}(\alpha_{+12}w)_{,2} + w_{,1}(\alpha_{+12}w)_{,\bar{2}} + w_{,2}(\alpha_{22}w)_{,2}$$

$$+ w_{,\bar{2}}(\alpha_{22}w)_{,\bar{2}} - (\beta_1 w w_{,\bar{1}} + \beta_1 w w_{,1} + \beta_1 w w_{,1} + \beta_1 w w_{,\bar{1}})/2$$

$$+ (w(\beta_2 w)_{,2} + w(\beta_2 w)_{,\bar{2}})/2 - (\beta_2 w w_{,2} + \beta_2 w w_{,\bar{2}})/2 + 2cw^2\}$$

$$+ h_2\{\alpha_{11}^{+i} w^{+i} w_{,1} + \alpha_{11} w w_{,\bar{1}}^{+i} + \alpha_{-12}^{+i} w^{+i} w_{,2} + \alpha_{-12} w w_{,\bar{2}}^{+i}$$

$$+ \alpha_{+12} w w_{,2}^{+i} + \alpha_{+12}^{+i} w^{+i} w_{,\bar{2}} + [\beta_1^{+i} w^{+i} w + \beta_1 w w^{+i}]/2\} \Big|_{i=i_1}^{i=i_0'}.$$

Hence,

$$h_1h_2 \sum_{i=i_0}^{i_0'} \sum_{j=j_0}^{j_0'} [C(w) + \bar{C}(w)]$$

$$= -2h_1h_2 \sum_{i=i_0}^{i_0'} \sum_{j=j_0}^{j_0'} \{\alpha_{11} w w_{,1,\bar{1}} + \alpha_{-12} w(w_{,1,\bar{2}} + w_{,\bar{1},2}) + \alpha_{+12} w(w_{,1,2} + w_{,\bar{1},\bar{2}})$$

$$+ \alpha_{22} w w_{,2,\bar{2}} + [\beta_1 w(w_{,1} + w_{,\bar{1}}) + \beta_2 w(w_{,2} + w_{,\bar{2}})]/2 - cw^2\}$$

$$+ h_2 \sum_{j=j_0}^{j_0'} \left\{\frac{1}{h_1} [\alpha_{11}^{+i}(w^{+i})^2 - \alpha_{11} w^2 + (\alpha_{11} - \alpha_{11}^{+i}) w w^{+i}] \right.$$

$$+ [\alpha_{-12}^{+i} w^{+i} w_{,2} + \alpha_{-12} w w_{,\bar{2}}^{+i} + \alpha_{+12}^{+i} w^{+i} w_{,\bar{2}} + \alpha_{+12} w w_{,2}^{+i}]$$

$$\left. + w w^{+i}(\beta_1^{+i} + \beta_1)/2\right\}\Bigg|_{i=i_1}^{i=i_0'}$$

$$+ h_1 \sum_{i=i_0}^{i_0'} \left\{\frac{1}{h_2} [\alpha_{22}^{+j}(w^{+j})^2 - \alpha_{22} w^2 + (\alpha_{22} - \alpha_{22}^{+j}) w w^{+j}] \right.$$

$$+ [\alpha_{-12}^{+j} w^{+j} w_{,1} + \alpha_{-12} w w_{,\bar{1}}^{+j} + \alpha_{+12} w w_{,1}^{+j} + \alpha_{+12}^{+j} w^{+j} w_{,\bar{1}}]$$

$$\left. + w w^{+j}(\beta_2^{+j} + \beta_2)/2\right\}\Bigg|_{j=j_1}^{j=j_0'}.$$

The following inequalities hold:

$$h_1 \sum_{i=i_0}^{i_0'} [\alpha_{-12}^{+j} w^{+j} w_{,1} + \alpha_{-12} w w_{,\bar{1}}^{+j}]\Big|_{j=j_1}^{j=j_0'}$$

$$= \left\{\sum_{i=i_0}^{i_0'} [(\alpha_{-12}^{+j} - \alpha_{-12}^{+i}) w^{+i} w^{+j} + (\alpha_{-12} - \alpha_{-12}^{+j}) w^{+j} w] + \alpha_{-12}^{+i} w^{+i} w^{+j}\Big|_{i=i_1}^{i=i_0'}\right\}\Bigg|_{j=j_1}^{j=j_0'}$$

$$\leqq \frac{M}{2} \sum_{i=i_0}^{i_0'} \sum_{j=j_1,j_0'} \{(h_1^2 + h_2^2)^{1/2}[(w^{+i})^2 + (w^{+j})^2] + h_2[w^2 + (w^{+j})^2]\}$$

$$+ \frac{M}{2} \sum_{i=i_1,i_0'} \sum_{j=j_1,j_0'} [(w^{+i})^2 + (w^{+j})^2];$$

$$h_1 \sum_{i=i_0}^{i_0'} [\alpha_{+12} w w_{,1}^{+j} + \alpha_{+12}^{+j} w^{+j} w_{,\bar{1}}]|_{j=j_1}^{j=j_0'}$$

$$\leq \frac{M}{2} \sum_{i=i_0}^{i_0'} \sum_{j=j_1,j_0'} \{(h_1^2 + h_2^2)^{1/2}[(w^{+i+j})^2 + w^2] + h_2[w^2 + (w^{+j})^2]\}$$

$$+ \frac{M}{2} \sum_{i=i_1,i_0'} \sum_{j=j_1,j_0'} [(w^{+i+j})^2 + w^2].$$

Therefore,

$$h_1 h_2 \sum_{i=i_0}^{i_0'} \sum_{j=j_0}^{j_0'} [C(w) + \bar{C}(w)]$$

$$\leq -2h_1 h_2 \sum_{i=i_0}^{i_0'} \sum_{j=j_0}^{j_0'} w L_h^0 w + \frac{h_2}{h_1} \sum_{j=j_0}^{j_0'} [\alpha_{11}^{+i}(w^{+i})^2 - \alpha_{11} w^2]|_{i=i_1}^{i=i_0'}$$

$$+ \frac{h_1}{h_2} \sum_{i=i_0}^{i_0'} [\alpha_{22}^{+j}(w^{+j})^2 - \alpha_{22} w^2]|_{j=j_1}^{j=j_0'} + h_2 \sum_{j=j_0}^{j_0'} \frac{3}{2} M \sum_{i=i_1,i_0'} [w^2 + (w^{+i})^2]$$

$$+ h_1 \sum_{i=i_0}^{i_0'} \frac{3}{2} M \sum_{j=j_1,j_0'} [w^2 + (w^{+j})^2]$$

$$+ \frac{1}{2} M \sum_{i=i_0}^{i_0'} \sum_{j=j_1,j_0'} \{(h_1^2 + h_2^2)^{1/2}[(w^{+i})^2 + (w^{+j})^2 + (w^{+i+j})^2 + w^2]$$
$$+ 2h_2[w^2 + (w^{+j})^2]\}$$

$$+ \frac{1}{2} M \sum_{j=j_0}^{j_0'} \sum_{i=i_1,i_0'} \{(h_1^2 + h_2^2)^{1/2}[(w^{+i})^2 + (w^{+j})^2 + (w^{+i+j})^2 + w^2]$$
$$+ 2h_1[w^2 + (w^{+i})^2]\}$$

$$+ M \sum_{i=i_1,i_0'} \sum_{j=j_1,j_0'} [(w^{+j+i})^2 + w^2].$$

Summing from $k = k_0$ to $k = k_0'$, we get

$$\tau h_1 h_2 \sum \sum_{Q_0} \sum [C(w) + \bar{C}(w)]$$

(5.4)
$$\leq -2\tau h_1 h_2 \sum \sum_{Q_0} \sum w L_h^0 w + \tau\lambda\Big(\sum_{R_{11}} \sum \alpha_{11} w^2 - \sum_{R_{10}} \sum \alpha_{11} w^2\Big)$$
$$+ \frac{\tau}{\lambda}\Big(\sum_{R_{21}} \sum \alpha_{22} w^2 - \sum_{R_{20}} \sum \alpha_{22} w^2\Big)$$
$$+ \tau h_1 \Lambda \sum_{S_1} \sum w^2 + \tau h_1 \Lambda \sum_{S_0} \sum w^2 + M\tau \sum_{T_1} w^2.$$

Now, let $i_0 \leq i \leq i_0'$, $j_0 \leq j \leq j_0'$. By summation by parts with respect to k, we get

(5.5)
$$\tau^2 \sum_{k=k_0}^{k_0'} (w_t^2 + w_{\bar{t}}^2) = -2\tau^2 \sum_{k=k_0}^{k_0'} w w_{t\bar{t}} + [(w^{+k})^2 - w^2]|_{k=k_1}^{k=k_0'}.$$

Using the identity $\tau w_{t\bar{t}} = (w_t + w_{\bar{t}}) - 2w_{\bar{t}}$, we deduce, for $i_0 \leq i \leq i_0'$, $j_0 \leq j \leq j_0'$,

$$\tau^2 \sum_{k=k_0}^{k_0'} ww_{t\bar t} = ww^{+k}\Big|_{k=k_1}^{k=k_0'} - 2\tau \sum_{k=k_0}^{k_0'} ww_{\bar t}$$

$$\geqq -\frac{1}{2} \sum_{k=k_1,k_0'} [w^2 + (w^{+k})^2] - 2\tau \sum_{k=k_0}^{k_0'} ww_{\bar t}.$$

Taking this inequality into (5.5), we get

$$\tau^2 \sum_{k=k_0}^{k_0'} (w_t^2 + w_{\bar t}^2) \leqq 4\tau \sum_{k=k_0}^{k_0'} ww_{\bar t} + 2 \sum_{k=k_1,k_1'} w^2,$$

hence

$$\tau h_1 h_2 \sum \sum_{Q_0} \sum (w_t^2 + w_{\bar t}^2) \leqq 4h_1 h_2 \sum \sum_{Q_0} \sum ww_{\bar t} + \frac{2}{\mu} h_2 \sum_{S_1} \sum w^2.$$

Multiplying this inequality by $\tau/2$ and adding (5.4), we get

$$\tau h_1 h_2 \sum \sum_{Q_0} \sum \left[C(w) + \bar C(w) + \frac{\tau}{2} (w_t^2 + w_{\bar t}^2) \right]$$

(5.6)
$$\leqq 2\tau h_1 h_2 \sum \sum_{Q_0} \sum w(L_h^0 w - w_{\bar t}) + \tau\lambda\Big(\sum_{R_{11}} \sum \alpha_{11} w^2 - \sum_{R_{10}} \sum \alpha_{11} w^2 \Big)$$
$$+ \frac{\tau}{\lambda} \Big(\sum_{R_{21}} \sum \alpha_{22} w^2 - \sum_{R_{20}} \sum \alpha_{22} w^2 \Big)$$
$$+ \tau h_1 \Big(\Lambda + \frac{\lambda}{\mu} \Big) \Big(\sum_{S_1} \sum w^2 + \sum_{S_0} \sum w^2 \Big) + M\tau \sum_{T_1} w^2.$$

The next step of the proof is to estimate $\tau h_1 h_2 \sum \sum \sum_{Q_0} (w_{,1}^2 + w_{,\bar 1}^2 + w_{,2}^2 + w_{,\bar 2}^2)$ in terms of $\tau h_1 h_2 \sum \sum \sum_{Q_0} [C(w) + \bar C(w)]$. We have

(5.7)
$$\tau h_1 h_2 \sum \sum_{Q_0} \sum [C(w) + \bar C(w)] \equiv D + E + F + H,$$

where

$$D = \tau h_1 h_2 \sum \sum_{Q_0} \sum [\alpha_{11}^{+i} w_{,1}^2 + (\alpha_{-12}^{+i} + \alpha_{-12}^{+j}) w_{,1} w_{,2} + (\alpha_{+12}^{-i} + \alpha_{+12}^{+j}) w_{,\bar 1} w_{,2}$$
$$+ \alpha_{22}^{+j} w_{,2}^2 + \alpha_{11}^{-i} w_{,\bar 1}^2 + (\alpha_{-12}^{-i} + \alpha_{-12}^{-j}) w_{,\bar 1} w_{,\bar 2}$$
$$+ (\alpha_{+12}^{+i} + \alpha_{+12}^{-j}) w_{,1} w_{,\bar 2} + \alpha_{22}^{-j} w_{,\bar 2}^2];$$

$$E = \tau h_1 h_2 \sum \sum_{Q_0} \sum w[\alpha_{11,1} w_{,1} + \alpha_{-12,1} w_{,2} + \alpha_{-12,2} w_{,1} + \alpha_{+12,\bar 1} w_{,2} + \alpha_{+12,2} w_{,\bar 1}$$
$$+ \alpha_{22,2} w_{,2} + \alpha_{11,\bar 1} w_{,\bar 1} + \alpha_{-12,\bar 1} w_{,\bar 2} + \alpha_{-12,\bar 2} w_{,\bar 1}$$
$$+ \alpha_{+12,1} w_{,\bar 2} + \alpha_{+12,\bar 2} w_{,1} + \alpha_{22,\bar 2} w_{,\bar 2}];$$

$$F = \tfrac{1}{2} \tau h_1 h_2 \sum \sum_{Q_0} \sum w[\beta_{1,1} w^{+i} + \beta_{1,\bar 1} w^{-i} + \beta_{2,\bar 2} w^{-j} + \beta_{2,2} w^{+j}];$$

$$H = 2\tau h_1 h_2 \sum \sum_{Q_0} \sum c w^2 \geqq 0.$$

Using (5.2), we deduce

$$D \geqq \tau h_1 h_2 \sum \sum_{Q_0} \sum \Big[\alpha_{11}^{+i} w_{,1}^2 + \tfrac{1}{2}(\alpha_{-12}^{+i} + \alpha_{-12}^{+j})\Big(\frac{1}{\lambda} w_{,1}^2 + \lambda w_{,2}^2\Big)$$

$$- \tfrac{1}{2}(\alpha_{+12}^{-i} + \alpha_{+12}^{+j})\Big(\frac{1}{\lambda} w_{,\bar{1}}^2 + \lambda w_{,2}^2\Big)$$

$$+ \alpha_{22}^{+j} w_{,2}^2 + \alpha_{11}^{-i} w_{,\bar{1}}^2 + \tfrac{1}{2}(\alpha_{-12}^{-i} + \alpha_{-12}^{-j})\Big(\frac{1}{\lambda} w_{,\bar{1}}^2 + \lambda w_{,\bar{2}}^2\Big)$$

$$- \tfrac{1}{2}(\alpha_{+12}^{+i} + \alpha_{+12}^{-j})\Big(\frac{1}{\lambda} w_{,1}^2 + \lambda w_{,\bar{2}}^2\Big) + \alpha_{22}^{-j} w_{,\bar{2}}^2 \Big]$$

$$\geqq \tau h_1 h_2 m \sum \sum_{Q_0} \sum (w_{,1}^2 + w_{,\bar{1}}^2 + w_{,2}^2 + w_{,\bar{2}}^2),$$

because for h small enough, $\alpha_{11}^{+i} - (\alpha_{+12}^{-i} - \alpha_{-12}^{+i})/\lambda > m$;

$$|E| \leqq 3M\tau h_1 h_2 \sum \sum_{Q_0} \sum |w| \, (|w_{,1}| + |w_{,2}| + |w_{,\bar{1}}| + |w_{,\bar{2}}|)$$

$$\leqq \frac{6M}{\kappa} \tau h_1 h_2 \sum \sum_{Q_0} \sum w^2 + \tfrac{3}{2} M\kappa \tau h_1 h_2 \sum \sum_{Q_0} \sum (w_{,1}^2 + w_{,2}^2 + w_{,\bar{1}}^2 + w_{,\bar{2}}^2)$$

for any positive number κ;

$$|F| \leqq 2M\tau h_1 h_2 \sum \sum_{Q_1} \sum w^2.$$

Using those estimates we deduce from (5.7)

$$(m - \tfrac{3}{2}M\kappa)\tau h_1 h_2 \sum \sum_{Q_0} \sum (w_{,1}^2 + w_{,2}^2 + w_{,\bar{1}}^2 + w_{,\bar{2}}^2)$$
$$\leqq \tau h_1 h_2 \sum \sum_{Q_0} \sum [C(w) + \bar{C}(w)] + 2M\Big(1 + \frac{3}{\kappa}\Big)\tau h_1 h_2 \sum \sum_{Q_1} \sum w^2. \tag{5.8}$$

Lemma 5.1 follows directly from (5.6) and (5.8) and the obvious fact that the preceding argument is valid for any l and not only $l = 0$.

LEMMA 5.2. *Let $G'' \subset \bar{G}'' \subset G'$. Suppose that functions y and z defined on G_h' satisfy for any rectangle $Q_l \subset G_h'$ an inequality of the form:*

$$\tau h_1 h_2 \sum \sum_{Q_{l-1}} \sum y^2$$
$$\leqq \tau M_0\Big(\sum_{R_{1l}} \sum \varphi_2 z^2 - \sum_{R_{1,l-1}} \sum \varphi_1 z^2\Big) + \tau M_1\Big(\sum_{R_{2l}} \sum \varphi_2 z^2 - \sum_{R_{2,l-1}} \sum \varphi_2 z^2\Big) \tag{5.9}$$
$$+ \tau h_1 M_2\Big(\sum_{S_l} \sum z^2 + \sum_{S_{l-1}} \sum z^2\Big) + \tau h_1 h_2 M_3 \sum \sum_{Q_l} \sum z^2 + M_4,$$

where M_0, M_1, M_2, M_3, M_4 are positive constants and where φ_1 and φ_2 are positive bounded functions defined on G_h'. Then, we have the estimate

$$\tau h_1 h_2 \sum \sum_{G_h''} \sum y^2 \leqq K\tau h_1 h_2 \sum \sum_{G_h'} \sum z^2 + K', \tag{5.10}$$

where the constants K and K' depend only on the constants M_i, on the bound of the functions φ_1 and φ_2 and on the domains G' and G''.

Proof. The proof of this lemma is a simple modification of the proof which is contained in Courant, Friedrichs and Lewy [1]. It is based on a double summation of inequality (5.9).

LEMMA 5.3. *If conditions* (5.2) *hold for* $n = 2$, *then the sums* $\tau h_1 h_2 \sum\sum\sum_{G_h'} w^2$ *for all* w, *which are difference quotients of order* $\leqq 5$ *of the functions of* $\mathfrak{F}$, *are uniformly bounded.*

Proof. We will study separately each of these sums.

1. $\tau h_1 h_2 \sum\sum\sum_{G_h'} v^2_{,i}$. We put $w = v$ in formula (5.3). Since $|v(P, h)| < M$ and $|L_h v(P, h)| = |f(P)| < M$, it follows from Lemma 5.1 that

$$(m - \tfrac{3}{2}M\kappa)\tau h_1 h_2 \sum \sum_{Q_l} \sum (w^2_{,1} + w^2_{,2} + w^2_{,\bar{1}} + w^2_{,\bar{2}})$$

$$\leqq 2M^2 V(G) + \tau\lambda\Big(\sum_{R_{1,l+1}} \sum \alpha_{11} w^2 - \sum_{R_{1l}} \sum \alpha_{11} w^2\Big)$$

$$+ \frac{\tau}{\lambda}\Big(\sum_{R_{2,l+1}} \sum \alpha_{22} w^2 - \sum_{R_{2l}} \sum \alpha_{22} w^2\Big) + \tau h_1\Big(\Lambda + \frac{\lambda}{\mu}\Big)\Big(\sum_{S_l} \sum w^2 + \sum_{S_{l+1}} \sum w^2\Big)$$

$$+ 2M\Big(1 + \frac{3}{\kappa}\Big)\tau h_1 h_2 \sum \sum_{Q_{l+1}} \sum w^2 + 16M^3\varphi,$$

where $V(G)$ denotes volume of G, φ—diameter of G. Taking $y^2 = w^2_{,1} + w^2_{,2} + w^2_{,\bar{1}} + w^2_{,\bar{2}}$ and $z = w$, we get the inequality of the form (5.9). Applying Lemma 5.2, we deduce that the sums $\tau h_1 h_2 \sum\sum\sum_{G_h'} v^2_{,i}$ are uniformly bounded.

2. $\tau h_1 h_2 \sum\sum\sum_{G_h'} v^2_{,i,j}$. We take $w_1 = v_{,1}$. It follows from (5.1) that

$$L_h w_1 = f_{,1} - \alpha_{11,1} w_{1,1} - \alpha_{+12,1}(w^{+i}_{1,2} + w_{1,\bar{2}}) - \alpha_{-12,1}(w^{+i}_{1,\bar{2}} + w_{1,2}) - \alpha_{22,1} v^{+i}_{,2,\bar{2}}$$

$$- \beta_{1,1}(w^{+i}_1 + w_1)/2 - \beta_{2,1}(v^{+i}_{,2} + v^{+i}_{,\bar{2}})/2 + c_{,1} v^{+i}.$$

Therefore, using (5.2), we have

$$|L_h w_1| \cdot |w_1| \leqq M\,|w_1| \cdot [1 + M + |w_{1,1}| + |w^{+i}_{1,2}| + |w_{1,2}| + |w^{+i}_{1,\bar{2}}|$$

$$+ |w_{1,\bar{2}}| + |v^{+i}_{,2,\bar{2}}| + \tfrac{1}{2}(|w_1| + |w^{+i}_1| + |v^{+i}_{,2}| + |v_{,2}|)].$$

Since

$$|w_1| \leqq (1 + w_1^2)/2, \qquad |w_1| \cdot |w_{1,r}| \leqq \frac{w_1^2}{2\kappa} + \frac{\kappa}{2} w^2_{1,r},$$

$$|w_1| \cdot |w^{+i}_1| \leqq \tfrac{1}{2}(w_1^2 + (w^{+i}_1)^2)$$

and $\tau h_1 h_2 \sum\sum\sum_{G_h'} w_1^2$, $\tau h_1 h_2 \sum\sum\sum_{G_h'} v^2_{,2}$ are bounded, we have the inequality

$$\tau h_1 h_2 \sum \sum_{G_h'} \sum |L_h w_1| \cdot |w_1| \leqq \mathfrak{M}_1(\kappa)$$

$$+ \frac{\kappa}{2} M\tau h_1 h_2 \sum \sum_{G_h'} \sum [v^2_{,1,1} + (v^{+i}_{,1,2})^2 + v^2_{,1,2} + (v^{+i}_{,1,\bar{2}})^2 + v^2_{,1,\bar{2}} + (v^{+i}_{,2,\bar{2}})^2].$$

Likewise,

$$\tau h_1 h_2 \sum \sum_{G_h'} \sum |L_h v_{,2}| \cdot |v_{,2}| \leqq \mathfrak{M}_2(\kappa)$$

$$+ \frac{\kappa}{2} M\tau h_1 h_2 \sum \sum_{G_h'} \sum [v_{,2,2}^2 + (v_{,2,1}^{+i})^2 + v_{,2,1}^2 + (v_{,2,\bar{1}}^{+i})^2 + v_{,2,\bar{1}}^2 + (v_{,1,\bar{1}}^{+i})^2].$$

If we substitute these two inequalities in (5.3) and take

$$z^2 = v_{,1}^2 + v_{,2}^2, \qquad y^2 = v_{,1,1}^2 + v_{,1,\bar{1}}^2 + v_{,1,2}^2 + v_{,1,\bar{2}}^2 + v_{,2,1}^2 + v_{,2,\bar{1}}^2 + v_{,2,2}^2 + v_{,2,\bar{2}}^2,$$

we get the inequality

$$(m - \tfrac{7}{2}M\kappa)\tau h_1 h_2 \sum \sum_{Q_l} \sum y^2$$

$$- M\tau h_1 h_2 \sum_{S_{l+1}} \sum (v_{,1,2}^2 + v_{,1,\bar{2}}^2 + v_{,2,\bar{2}}^2 + v_{,1,2}^2 + v_{,2,\bar{1}}^2 + v_{,1,\bar{1}}^2)$$

$$\leqq \mathfrak{M}(\kappa) + \tau\lambda\left(\sum_{R_{1,l+1}} \sum \alpha_{11} z^2 - \sum_{R_{1l}} \sum \alpha_{11} z^2\right) + \frac{\tau}{\lambda}\left(\sum_{R_{2,l+1}} \sum \alpha_{22} z^2 - \sum_{R_{2l}} \sum \alpha_{22} z^2\right)$$

$$+ \tau h_1\left(\Lambda + \frac{\lambda}{\mu}\right)\left(\sum_{S_l} \sum z^2 + \sum_{S_{l+1}} \sum z^2\right) + 2M\left(1 + \frac{3}{\kappa}\right)\tau h_1 h_2 \sum \sum_{Q_{l+1}} \sum z^2.$$

For τ small enough,

$$M\tau h_1 h_2 \sum_{S_{l+1}} \sum (v_{,1,2}^2 + v_{,1,\bar{2}}^2 + v_{,2,\bar{2}}^2 + v_{,1,2}^2 + v_{,2,\bar{1}}^2 + v_{,1,\bar{1}}^2)$$

$$\leqq \frac{m}{2} \tau h_1 h_2 \sum \sum_{Q_l} \sum y^2,$$

therefore, we get an inequality of the form (5.9). Applying Lemma 5.2, we deduce that the sums $\tau h_1 h_2 \sum \sum \sum_{G_h'} v_{,i,j}^2$ are uniformly bounded for any $G' \subset \bar{G}' \subset G$.

3. $\tau h_1 h_2 \sum \sum \sum_{G_h'} v_t^2$. Formula (5.1) yields

$$|v_{\bar{t}}| \leqq |f| + M[|v_{,1,\bar{1}}| + |v_{,1,2}| + |v_{,\bar{1},\bar{2}}| + |v_{,\bar{1},2}| + |v_{,1,\bar{2}}| + |v_{,2,\bar{2}}|$$

$$+ \tfrac{1}{2}(|v_{,1}| + |v_{,\bar{1}}| + |v_{,2}| + |v_{,\bar{2}}|) + |v|].$$

Therefore, the boundedness of the sums $\tau h_1 h_2 \sum \sum \sum_{G_h'} v_{,i}^2$ and $\tau h_1 h_2 \sum \sum \sum_{G_h'} v_{,i,j}^2$ implies the boundedness of the sums $\tau h_1 h_2 \sum \sum \sum_{G_h'} v_t^2$.

The uniform boundedness of all sums $\tau h_1 h_2 \sum \sum \sum_{G_h'} w^2$ can be proved in the same way, after differencing Eq. (5.1).

LEMMA 5.4 (SOBOLEV'S THEOREM). *If the sums $\tau h_1 \cdots h_n \sum_{G_h'} w^2(P, h)$ are uniformly bounded for all $w(P, h)$ which are difference quotients of order $\leqq n + 1$ of the functions of $\mathfrak{F}$, then the family $\mathfrak{F}$ is equicontinuous in any subdomain $G'' \subset \bar{G}'' \subset G'$.*

Proof. The proof is a modification of the proof of Sobolev's theorem which is contained in [4].

We denote

$$P_0 = (i_1^0 h_1, \cdots, i_n^0 h_n, k^0 \tau), \qquad P_1 = (i_1^0 h_1, \cdots, i_n^0 h_n, k'' \tau),$$

$$R(P_0) = \{P = (i_1 h_1, \cdots, i_n h_n, k\tau): i_j^0 \leqq i_j \leqq i_j'' \ (j = 1, \cdots, n), k^0 \leqq k \leqq k''\},$$

$$(i_j'' - i_j^0)h_j = b_j, \qquad (k'' - k^0)\tau = a.$$

We take b_j and a such that for each $P_0 \in G_h''$ is $R(P_0) \subset G_h'$. Let $i_1^0 \leqq i_1' \leqq i_1''$. For any function w defined on G_h',

$$w\Big|_{i_1=i_1^0}^{i_1=i_1'} = h_1 \sum_{i_1=i_1^0}^{i_1'-1} w_{,1}.$$

Applying Schwarz's inequality, we get

$$\Big[w\Big|_{i_1=i_1^0}^{i_1=i_1'}\Big]^2 \leqq \frac{b_1}{h_1} h_1^2 \sum_{i_1=i_1^0}^{i_1'-1} w_{,1}^2 ,$$

therefore

$$|w|_{i_1=i_1^0}| \leqq |w|_{i_1=i_1'}| + (b_1 h_1)^{1/2}\Big(\sum_{i_1=i_1^0}^{i_1'-1} w_{,1}^2\Big)^{1/2}.$$

Squaring both sides of this inequality and applying the inequality $(a+b)^2 \leqq 2a^2 + 2b^2$, we have

$$(w|_{i_1=i_1^0})^2 \leqq 2(w|_{i_1=i_1'})^2 + 2b_1 h_1 \sum_{i_1=i_1^0}^{i_1'-1} w_{,1}^2.$$

Summing these inequalities for $i_1^0 \leqq i_1' \leqq i_1''$, we obtain

$$\frac{b_1}{h_1}(w|_{i_1=i_1^0})^2 \leqq 2 \sum_{i_1=i_1^0}^{i_1''-1} w^2 + 2b_1^2 \sum_{i_1=i_1^0}^{i_1''-1} w_{,1}^2.$$

Hence

$$(w|_{i_1=i_1^0})^2 \leqq 2\frac{h_1}{b_1}\Big[\sum_{i_1=i_1^0}^{i_1''-1} w^2 + b_1^2 \sum_{i_1=i_1^0}^{i_1''-1} w_{,1}^2\Big].$$

By induction we get

$$\begin{aligned}(w|_{i_1=i_1^0,i_2=i_2^0})^2 &\leqq 2\frac{h_1}{b_1}\Big[\sum_{i_1=i_1^0}^{i_1''-1}(w|_{i_2=i_2^0})^2 + b_1^2 \sum_{i_1=i_1^0}^{i_1''-1}(w_{,1}^2)|_{i_2=i_2^0}\Big]\\ &\leqq 4\frac{h_1h_2}{b_1b_2}\sum_{i_1=i_1^0}^{i_1''-1}\sum_{i_2=i_2^0}^{i_2''-1}(w^2 + b_1^2 w_{,1}^2 + b_2^2 w_{,2}^2 + b_1^2 b_2^2 w_{,1,2}^2),\end{aligned}$$

and, for n,

$$\begin{aligned}&w(i_1^0 h_1, \cdots, i_n^0 h_n, k\tau)^2\\ &\quad\leqq 2^n \frac{h_1 \cdots h_n}{b_1 \cdots b_n}\sum_{i_1=i_1^0}^{i_1''-1}\cdots\sum_{i_n=i_n^0}^{i_n''-1}\Big[w^2 + \sum_{p=1}^{n} b_p^2 w_{,p}^2 + \cdots + b_1^2 \cdots b_n^2 w_{,1,2,\dots,n}^2\Big].\end{aligned}$$

For any function $v \in \mathfrak{F}$,

$$v(P_1) - v(P_0) = \tau \sum_{k=k^0}^{k''-1} v_t(i_1^0 h_1, \cdots, i_n^0 h_n, k\tau),$$

therefore

$$\begin{aligned}|v(P_1) - v(P_0)| &\leqq (a\tau)^{1/2}\Big(\sum_{k=k^0}^{k''-1} v_t^2(i_1^0 h_1, \cdots, i_n^0 h_n, k\tau)\Big)^{1/2}\\ &\leqq \Big(a\tau 2^n \frac{h_1 \cdots h_n}{b_1 \cdots b_n}\sum_{R(P_0)}\Big[v_t^2 + \sum_{p=1}^{n} b_p^2 v_{t,p}^2 + \cdots + b_1^2 \cdots b_n^2 v_{t,1,2,\dots,n}^2\Big]\Big)^{1/2}.\end{aligned}$$

The assumption of the lemma implies the equicontinuity of the functions $v \in \mathfrak{F}$ with respect to t. In the same way, we can show the equicontinuity with respect to each variable; therefore the functions of $\mathfrak{F}$ are equicontinuous.

THEOREM 5.1. *Let $G \subset R^3$ and let the coefficients of the operator L be of the class $C^3(G)$ and their third derivatives be Lipschitz-continuous in any $G' \subset \bar{G}' \subset G$, and let $\forall h \leqq h_0 \; \forall P \in G_h \; L_h v(P, h) = f(P)$. Then, any sequence $\{v(P, h_n); h_n \to 0\} \subset \mathfrak{F}$ admits a subsequence which converges uniformly in G' to a solution of the differential equation $Lu = f$.*

Proof. If the assumptions of the theorem are satisfied, we can apply Lemma 5.3. Then, Lemma 5.4 shows that v, $v_{,i}$, $v_{,i,j}$, v_t are equicontinuous in G'_h for h small enough. G' is covered by cubic cells of the mesh; by linear interpolation in these cells, we can extend the equicontinuous family $\mathfrak{F}$ of the mesh-functions into an equicontinuous family defined on all of $\bar{G}'$.

The theorem follows by application of Ascoli's theorem to the families $\mathfrak{F}$, $\mathfrak{F}^{(1)}$ and $\mathfrak{F}^{(2)}$ and because of conditions (3.9).

VI. Existence of Discrete Barriers. Throughout this section, we study various types of local conditions on G and on L_h which guarantee the existence of a strong discrete barrier.

Let $Q = (x_1^0, x_2^0, t^0) \in \Gamma_1$ and assume that there exists a neighborhood N_Q of Q such that $G_h \cap N_Q \subset G_h^0$ for h small enough.

1. Assume that: the coefficients of the operator L are uniformly continuous in N_Q; $\lim_{P \to Q} [a_{11}(P)a_{22}(P) - a_{12}^2(P)] > 0$ and there exists a nondegenerate sphere through Q whose intersection with $\bar{G}$ is the single point Q and whose center is not on the straight line $x_1 = x_1^0$, $x_2 = x_2^0$.

Then, there exists a strong discrete barrier at Q.

Proof. Let us take the origin of the coordinates at the center of the sphere and let

$$s = x_1^2 + x_2^2 + t^2, \qquad s_0 = s(Q) = (x_1^0)^2 + (x_2^0)^2 + (t^0)^2.$$

Let k and p be positive constants and $B(P, Q) = k(s^{-p} - s_0^{-p})$. This is the barrier defined by Jamet [3] for the operator without mixed derivatives, but it can also be defined in the more general case.

This function satisfies condition (2.3a, b). Moreover, we have

$$LB(P, Q) = 2kps^{-p-2}\{2(p+1)(a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2) - s(a_{11} + a_{22} + b_1x_1 + b_2x_2 - dt)\} - cB(P, Q). \tag{6.1}$$

In a certain neighborhood of Q we have $x_1^2 > \frac{1}{2}(x_1^0)^2$, $x_2^2 > \frac{1}{2}(x_2^0)^2$ and there exists α_0 such that $\forall \xi, \eta$, $a_{11}\xi^2 + 2a_{12}\xi\eta + a_{22}\eta^2 \geqq \alpha_0(\xi^2 + \eta^2)$. Therefore,

$$LB(P, Q) \geqq 2kps^{-p-2}\{(p+1)\alpha_0[(x_1^0)^2 + (x_2^0)^2] - s(a_{11} + a_{22} + b_1x_1 + b_2x_2 - dt)\}.$$

It follows that $LB(P, Q)$ can be made arbitrarily large in N_Q, provided we choose k and p large enough. In particular, we can choose k and p such that

$$L_hB(P, Q) + E(P) = LB(P, Q) - c(P) + O(h) > 1$$

in N_Q, for h small enough. Thus, $B(P, Q)$ is a strong discrete barrier at Q.

2. If the coefficients of the operator L are uniformly continuous in N_Q and L_h is consistent with L in the norm $C_h(N_{Q_h})$, $\lim_{P \to Q} [a_{11}(P)a_{22}(P) - a_{12}^2(P)] = 0$ (but not

all coefficients a_{ij} vanish on the boundary) and there exists a sphere through Q whose intersection with $\bar{G}$ is the single point Q and whose center is not in the plane $a_{11}(Q)(x_1 - x_1^0) + a_{22}(Q)(x_2 - x_2^0) = 0$, then B defined as before is the discrete barrier in Q.

Proof. Suppose $a_{11}(Q) \neq 0$. In a certain neighborhood of Q we have

$$a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2 > \tfrac{1}{2}[a_{11}(Q)(x_1^0)^2 + 2a_{12}(Q)x_1^0x_2^0 + a_{22}(Q)(x_2^0)^2]$$
$$= [a_{11}(Q)x_1^0 + a_{22}(Q)x_2^0]^2/2a_{11}(Q) > 0.$$

From this inequality and from (6.1) we deduce that B is the discrete barrier.

3. Assume that the coefficients of the operator L are uniformly continuous and that L_h is consistent with L in the norm $C_h(N_{Qh})$. Assume $d(Q) > 0$ and that there exists a nondegenerate sphere through Q with radius $R > (a_{11}(Q) + a_{22}(Q))/d(Q)$ whose intersection with $\bar{G} \cap N_Q$ is the single point Q and whose center lies on the half-line $x_1 = x_1^0$, $x_2 = x_2^0$, $t < t^0$.

Then, B, defined as in 1, is a strong discrete barrier.

Proof.

$$LB(P, Q) = 2kps^{-p-2}[2(p + 1)(a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2)$$
$$- s(a_{11} + a_{22} + b_1x_1 + b_2x_2 - dt)] - cB(P, Q)$$
$$> 2kps^{-p-1}[dt - (a_{11} + a_{22} + b_1x_1 + b_2x_2)]$$
$$\xrightarrow[P \to Q]{} 2kps_0^{-p-1}[Rd(Q) - a_{11}(Q) - a_{22}(Q)] > 0.$$

Then, B is a strong discrete barrier.

The two following sufficient conditions are contained in [3].

4. Assume that there exists a neighborhood N_Q of Q such that $G \cap N_Q$ lies in the half-space $t > t^0$. Assume that the coefficients of the operator L are bounded, except d which may be unbounded, $d(P) > k(t - t^0)^\sigma$, $\sigma < 1$, $k > 0$. Let L_h be the operator corresponding to formulas (3.3) or (3.8). Then, there exists a strong discrete barrier at Q.

5. Suppose that there exists a neighborhood N_Q of Q such that $G \cap N_Q$ is a cylinder parallel to the t-axis. Let us write $L = L_0 - d(\partial/\partial t)$; L_0 is an elliptic operator in space variables. Suppose that there exists a function $B_0(P, Q)$ which does not depend on t and which is a strong discrete barrier for the family of operators L_{0h} for any t such that $|t - t^0| < \eta$, where $\eta > 0$ is a constant independent of h_1, h_2. Suppose $d(P)$ is bounded.

Then, the function $B(P, Q) = KB_0(P, Q) - (t - t^0)^2$ is a strong discrete barrier for the family $\{L_h\}$.

Example 1. Let $\psi(x_1)$ be a convex function defined for all real x_1 and such that $|\psi(x_1') - \psi(x_1'')|/|x_1' - x_1''| < M$ for all x_1' and $x_1'' \neq x_1'$, where M is a positive constant. Let $\mathcal{C}$ be the curve $Y \equiv x_2 - \psi(x_1) = 0$ in the plane $t = 0$. Let G_0 be a bounded simply-connected plane domain whose boundary consists of a portion of $\mathcal{C}$ and of a smooth curve which lies entirely in the region $Y > 0$. Let $G = G_0 \times (0, T)$, and $G_\epsilon = G \cap \{P = (x_1, x_2, t)\colon Y > \epsilon\}$. Let $\Gamma_2 = \{P = (x_1, x_2, T) \in \partial G\}$ and $\Gamma_1 = \partial G - \Gamma_2$. Let

$$L = a_{11}\frac{\partial^2}{\partial x_1^2} + 2a_{12}\frac{\partial^2}{\partial x_1\,\partial x_2} + a_{22}\frac{\partial^2}{\partial x_2^2} + b_1\frac{\partial}{\partial x_1} + b_2\frac{\partial}{\partial x_2} - \frac{\partial}{\partial t}, \tag{6.2}$$

where

$$\begin{aligned} &a_{11} - h_1\,|a_{12}|/h_2 > q, \qquad a_{22} - h_2\,|a_{12}|/h_1 > q\,h_2^2/h_1^2\,,\\ &b_1,\ b_2 \in C^4(G), \qquad b_1,\ b_2 \in C(\bar{G}_\epsilon),\\ &[b_1^2(P) + h_1^2 b_2^2(P)/h_2^2]^{1/2} < qk/Y + K,\\ &0 < k < h_2/h_1, \qquad K > 0. \end{aligned} \tag{6.3}$$

Let L_h be the operator defined by formulas (3.2) and (3.3). Then, the problem (1.3) has a unique solution $u(P)$ and $v(P, h)$ converges uniformly to $u(P)$ in G as $h \to 0$.

The proof will be performed for a square net $h_1 = h_2 = h$; by the transformation of the variables $\bar{x}_2 = (h_2/h_1)x_2$, $\bar{\psi}(x_1) = (h_2/h_1)\psi(x_1)$, one obtains the general case.

Under our assumptions, L_h given by (3.2) and (3.3) is positive. For instance, the coefficient

$$\begin{aligned} A(P, P + e_1 h) &= [a_{11}(P) - |a_{12}(P)|]\frac{1}{h^2} + b_1(P)\frac{1}{2h}\\ &\geqq \frac{q}{h^2} - \frac{qk}{2Yh} = \frac{q}{h^2}\left(1 - \frac{kh}{2Y}\right) > 0, \end{aligned}$$

since at each interior mesh-point there is $Y > h$.

The existence of discrete barriers at the points $Q \in \Gamma_1 - \mathcal{C} \times [0, T]$ follows from our third sufficient condition. The discrete barrier for $\{L_{0h}\}$ at $Q = (x_1^0, x_2^0, t^0) \in \mathcal{C} \times [0, T]$ is

$$B_0(P, Q) = -(x_1 - x_1^0)^2 - Y^{1-k'}, \quad \text{where } k < k' < 1.$$

This function has the properties required for the application of our fifth sufficient condition.

The existence of a function $\varphi(P)$ satisfying condition (i) of Theorem 2.1 follows from the second sufficient condition in Section IV. Theorem 7.1 implies that the solution of problem (1.3) with the operator (6.2) is unique. Therefore, we can apply Theorem 2.1, which concludes the proof.

Example 2. Let G_0 be a convex domain in the plane $t = 0$ such that in the neighborhood of any point $Q_0 \in \partial G_0$, ∂G_0 admits a representation of the form $x_2 = \varphi(x_1)$ or of the form $x_1 = \psi(x_2)$, where φ and ψ are convex functions. Let $G = G_0 \times (0, T)$, $\Gamma_2 = \{P = (x_1, x_2, T) \in \partial G\}$ and $\Gamma_1 = \partial G - \Gamma_2$. Let L be the operator (6.2), where

$$\begin{aligned} &a_{11} - \frac{h_1}{h_2}|a_{12}| > q, \qquad a_{22} - \frac{h_2}{h_1}|a_{12}| > q\frac{h_2^2}{h_1^2}, \qquad b_1, b_2 \in C^4(G),\\ &\forall\ P \in G, \qquad \left[b_1^2(P) + \frac{h_1^2}{h_2^2}\,b_2^2(P)\right]^{1/2} < qk/d(P, \partial G) + K,\\ &0 < k < h_2/(h_1^2 + h_2^2)^{1/2}, \qquad K > 0. \end{aligned} \tag{6.4}$$

Let L_h be defined by formulas (3.2) and (3.3) and let $v(P, h)$ be a solution of problem (2.3). Then, the problem (1.3) has a unique solution $u(P)$ and $v(P, h) \to u(P)$ uniformly in G as $h \to 0$.

Proof. Same as in Example 1.

VII. Uniqueness of the Solution of the Differential Problem. We denote by Γ' the set of all points $Q = (x_1^0, \cdots, x_n^0, t^0) \in \partial G$ which admit a neighborhood N_Q such that $\partial G \cap N_Q$ lies in the plane $t = t^0$, and $G \cap N_Q$ lies in the half-space $t < t^0$. For any $Q \in G$ we denote by $S(Q)$ the set of all points $P \in G$ which can be joined with Q by a continuous curve lying entirely in G along which the coordinate t does not decrease from P to Q.

LEMMA 7.1 (THE MAXIMUM PRINCIPLE FOR PARABOLIC OPERATORS). *Let L be a parabolic operator* (*satisfying conditions* (1.2)) *whose coefficients are continuous in G. If $Lu \geqq 0$ ($Lu \leqq 0$) in G and u has a positive maximum* (*negative minimum*) *in G which is attained at the point P_0, then $u(P) = u(P_0)$ for all points $P \in S(P_0)$.*

This theorem is proved in [2].

We deduce at once from the maximum principle the following

THEOREM 7.1. *If $\Gamma_2 \subset \Gamma'$, then problem* (1.3) *has at most one solution.*

THEOREM 7.2. *A necessary condition for the existence of a solution of problem* (1.3) *for arbitrary $g \in C(\bar{G})$ is*

$$\Gamma_1 \cap \bigcup_{Q \in \Gamma_1 \cap \Gamma'} [S(Q)]^- = \emptyset. \tag{7.1}$$

Proof. If (7.1) does not hold, then there exists a point $Q_0 \in \Gamma_1 \cap \Gamma'$ for which $\Gamma_1 \cap [S(Q_0)]^- \neq \emptyset$. Suppose that g_1 and g_2 are functions such that $g_1(Q_0) - g_2(Q_0) > g_1(Q) - g_2(Q)$ for $Q \neq Q_0$ and $g_1(Q_0) - g_2(Q_0) > 0$. If

$$Lu_1 = f \qquad\qquad Lu_2 = f$$
$$\text{and}$$
$$u_1|_{\Gamma_1} = g_1|_{\Gamma_1} \qquad\qquad u_2|_{\Gamma_1} = g_2|_{\Gamma_1},$$

then $L(u_1 - u_2) = 0$ and $(u_1 - u_2)|_{\Gamma_1} = (g_1 - g_2)|_{\Gamma_1}$. It follows from our assumptions that $u_1(Q) - u_2(Q) = g_1(Q_0) - g_2(Q_0)$ for $Q \in \Gamma_1 \cap [S(Q_0)]^-$. This is a contradiction.

If $\Gamma' \subset \Gamma_2$, then the condition (7.1) holds. From now on we will assume $\Gamma' \subset \Gamma_2$ and we define $\Gamma'' = \Gamma_2 - \Gamma'$.

THEOREM 7.3. *Suppose Γ'' is closed and suppose that there exists a neighborhood N of Γ'' and a function $U(P)$ such that*

$$U \in C(G_0 - \Gamma''), \qquad U \in C^2(G_0), \quad \textit{where } G_0 = G \cap N;$$
$$LU(P) \leqq 0, \qquad P \in G_0; \tag{7.2}$$
$$U(P) \to +\infty \quad \textit{as } P \to Q,\ \forall\ Q \in \Gamma'',\ P \in \bar{G}_0 - \Gamma''.$$

Then, the problem (1.3) *has at most one solution.*

The proof is contained in [3].

We give two examples as applications of Theorem 7.3.

Example 3. Let G lie in the intersection of the half-space $\sum_{i=1}^n \alpha_i x_i > 0$ and the slab $t_1 < t < t_2$. Let L be the operator (1.1) and assume that there exists a constant $K \geqq 0$ such that

$$\sum_{i=1}^n \alpha_i b_i(P) \Big/ \sum_{i,j=1}^n \alpha_i \alpha_j a_{ij}(P) > \left(\sum_{i=1}^n \alpha_i x_i \right)^{-1} - K$$

for all $P \in G$, and for $\sum_{i=1}^n \alpha_i x_i$ small enough.

Let $\Gamma'' = \partial G \cap \{P = (x_1, \cdots, x_n, t) : \sum_{i=1}^{n} \alpha_i x_i = 0\}$. Then, problem (1.3) has at most one solution.

Proof. Let

$$U(P) = -K_1\left(\sum_{i=1}^{n} \alpha_i x_i\right) - \ln\left(\sum_{i=1}^{n} \alpha_i x_i\right), \quad \text{where } K_1 > K.$$

We have

$$LU(P) = \sum_{i,j=1}^{n} a_{ij}(P)\alpha_i\alpha_j\left(\sum_{i=1}^{n} \alpha_i x_i\right)^{-2} - \sum_{i=1}^{n} b_i(P)\alpha_i\left(K_1 + 1\Big/\sum_{i=1}^{n} \alpha_i x_i\right) - cU$$

$$\leqq \sum_{i,j=1}^{n} a_{ij}(P)\alpha_i\alpha_j\left[KK_1 + (K - K_1)\left(\sum_{i=1}^{n} \alpha_i x_i\right)^{-1}\right] < 0,$$

if $\sum_{i=1}^{n} \alpha_i x_i \leqq (K_1 - K)/KK_1$. Then, the assumptions of Theorem 7.3 are satisfied.

Example 4. Let G lie in the half-space $t > 0$, $\Gamma'' = \partial G \cap \{P = (x_1, \cdots, x_n, 0)\}$. Suppose that there exists a number $\sigma \leqq 1$ such that $d(P) \leqq t^{\sigma}$ and that there exists i such that $a_{ii}(P) > \epsilon$ for t small enough.

Then, problem (1.3) has at most one solution.

Proof. Let $U(P) = -x_i^2 - \ln t$. Then

$$LU(P) = -2a_{ii}(P) + d(P)t^{-1} \leqq -2\epsilon + t^{\sigma-1} < 0$$

for t sufficiently small. Thus, the assumptions of Theorem 7.3 are satisfied and the solution of problem (1.3) is unique.

Acknowledgment. The author would like to thank Dr. Andrzej Wakulicz for suggesting this problem, for reading the manuscript and for helpful discussions.

Institute of Mathematics
Polish Academy of Sciences
Sniadeckich 8
Warsaw 1, Poland

1. R. Courant, K. Friedrichs & H. Lewy, "Über die Differenzengleichungen der mathematischen Physik," *Math. Ann.*, v. 100, 1928, pp. 32–74; English transl., *IBM J.*, v. 11, 1967, pp. 215–234.

2. A. Friedman, *Partial Differential Equations of Parabolic Type*, Prentice-Hall, Englewood Cliffs, N.J., 1964; Russian transl., "Mir," Moscow, 1968. MR **31** #6062.

3. P. Jamet, "Numerical methods and existence theorems for parabolic differential equations whose coefficients are singular on the boundary," *Math. Comp.*, v. 22, 1968, pp. 721–743. MR **40** #8291.

4. F. John, *Lectures on Advanced Numerical Analysis*, Gordon and Breach, New York, 1967. MR **36** #4773.