

A Useful Approximation to e^{-t}

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Abstract. Using differential approximation, we obtain a remarkably accurate representation of e^{-t} as a sum of three exponentials.

1. Introduction. The function e^{-t} occurs in many important contexts in mathematics. In some of these, it is quite useful to replace it by an approximation of some type, such as, for example, a Padé approximation. In this note, we wish to exhibit a surprisingly good approximation as a sum of three exponentials. This is obtained using differential approximation, [1]. The approximation obtained here holds for $0 \leq t \leq 1$.

2. Differential Approximation. Given a function $k(t)$ for $0 \leq t \leq T$, we determine the coefficients a_1, a_2, \dots, a_N which minimize the quadratic expression

$$(2.1) \quad J(a_i) = \int_0^T \left[k^{(N)} + \sum_{i=1}^N a_i k^{(N-i)} \right]^2 dt,$$

where $k^{(i)}$ denotes the i th derivative. We then expect that the solution of the linear differential equation

$$(2.2) \quad u^{(N)} + a_1 u^{(N-1)} + \dots + a_N u = 0,$$

with suitable boundary conditions, will yield an approximation to $k(t)$. This is a question in stability theory.

The procedure is most useful when N can be taken small. In this case, $N = 3$ and 5 yield excellent results for $k(t) = e^{-t}$, as is demonstrated below.

3. Numerical Results. It turns out that good results are obtained by choosing as initial conditions in (2.2): $u^{(i)}(0) = k^{(i)}(0)$, $i = 0, 1, \dots, N - 1$. The coefficients a_i are listed in the first column of Table 1.

For the case $N = 3$, the calculated values of u, u', u'', \dots agree to eight figures with the exact values k, k', k'', \dots , respectively. The accuracy is even better for $N = 5$.

If we express the solution of the linear differential equation as a sum of exponentials, we obtain the expression

$$(3.1) \quad u(t) = \sum_{i=1}^N b_i \exp(-\lambda_i t),$$

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where b_i and λ_i can have complex values. These values are calculated and listed in Table 1. The numerical values of function $u(t)$ of the above equation at different time intervals ($0 \leq t \leq 1$) are listed in Table 2. In the same table, the absolute errors are also shown.

TABLE 1

N	a_i	b_i	λ_i
3	2.7403	.7853	.9180
	7.9511	.1074 + i .1963	.9111 + i 2.334
	5.7636	.1074 - i .1963	.9111 - i 2.334
5	4.7471	.6509	.9509
	27.9415	.1795 + i .2204	.9503 + i 1.866
	62.5129	.1795 - i .2204	.9503 - i 1.866
	109.1101	-.0049 + i .0163	.9478 + i 3.930
	68.1498	-.0049 - i .0163	.9478 - i 3.930

TABLE 2

Time	$N = 3$		$N = 5$	
	Calculated Value	Absolute Error	Calculated Value	Absolute Error
.1	.990020	.30 $\times 10^{-4}$.990049	.4 $\times 10^{-8}$
.3	.913676	.255 $\times 10^{-3}$.913931	.2 $\times 10^{-6}$
.5	.778679	.122 $\times 10^{-3}$.778800	.2 $\times 10^{-7}$
.8	.527665	.372 $\times 10^{-3}$.527292	.2 $\times 10^{-6}$
1.0	.367951	.72 $\times 10^{-4}$.367879	.2 $\times 10^{-6}$

4. Discussion. If desired, we can improve the accuracy of the approximation by taking the values of $u^{(i)}(0)$, the initial conditions, as parameters, $u^{(i)}(0) = c_i$ and then, by determining these values by the minimization of the quadratic expression,

$$(4.1) \quad J(c_i) = \int_0^T \left[k(t) - \sum_{i=1}^N c_i u_i \right]^2 dt,$$

where u_1, \dots, u_N are N linearly independent solutions of (2.2).

The integrals which arise are evaluated by using the differential equation (2.2) plus the auxiliary equations

$$(4.2) \quad \frac{dv_{ij}}{dt} = u_i u_j, \quad v_{ij}(0) = 0, \quad \frac{dw_i}{dt} = u_i k, \quad w_i(0) = 0.$$

Then,

$$(4.3) \quad v_{ii}(T) = \int_0^T u_i u_i dt, \quad w_i(T) = \int_0^T u_i k dt.$$

The same technique can often be used in the determination of the coefficients a_i ,

when the function $k(t)$ satisfies a differential equation, linear or nonlinear. In this case, $k' = -t^2k$, $k(0) = 1$.

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