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The Finite Element Method for Infinite Domains. I

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Abstract. Numerical methods (finite element methods) for the approximate solution of elliptic partial differential equations on unbounded domains are considered, and error bounds, with respect to the number of unknowns which have to be determined, are proven.

1. Introduction. The finite element method, its theory and practice, has recently become of interest in numerical analysis, see e.g. [1]-[13], and the papers of Aubin, Birkhoff, Bramble, Ciarlett, Schatz, Schultz, Varga, etc.

The theoretical analyses of the finite element method have been concerned with bounded domains. Strang and Fix, [1], [2], have, however, analyzed the finite element method with respect to an infinite domain (the space R_n), but their procedure requires the solution of an infinite system of linear algebraic equations.

This paper will deal with the problem of finding, by the finite element method, an approximate solution of a boundary value problem for elliptic partial differential equations on an infinite domain by solving only a finite system of linear algebraic equations.

The approach will be shown on a model problem. Our task will be to find the solution of the equation (weak solution)

$$-\Delta u + u = f \tag{1.1}$$

on R_n, where $u \in W_2^1(R_n)$ and $f \in W_2^k(R_n)$, $k \geqq 0$.

We will show that the rate of convergence on compact sets of R_n is practically the same as the rate of convergence for boundary value problems on bounded domains. The rate of convergence will turn out to be determined by the number of unknowns in the system of linear algebraic equations.

Our approach may be easily generalized to the case of an elliptic differential equation of order $2m$, provided that the coefficient of the zero order term of the equation is bounded above and below by positive constants.

We will analyze only the case when $\Omega = R_n$. By combining the approach described above with the results concerning bounded domains (see e.g. [6]-[11]), it is easy to get the corresponding results for unbounded domains with bounded boundary.

Throughout this paper, let x denote the n-dimensional vector in R_n, i.e. $x \equiv (x_1, \cdots, x_m)$ where $x_i \in R_1$, $i = 1, 2, \cdots, n$.

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Let $||x||^2 = \sum_{i=1}^n x_i^2$ and $|x| = \sum_{i=1}^n |x_i|$. If $a \in R_1$, let $x + a_i \equiv (x_1, \cdots, x_{i-1}, x_i + a, x_{i+1}, \cdots, x_n)$. Let k denote a multi-integer, i.e. $k \equiv (k_1, \cdots, k_n)$, where k_i is an integer, $i = 1, 2, \cdots, n$. Let $||k||^2 = \sum_{i=1}^n k_i^2$ and $|k| = \sum_{i=1}^n |k_i|$. Define the inequality $k \geqq 0$ to mean $k_i \geqq 0$, $i = 1, 2, \cdots, n$. Let C denote a generic constant which may have different values wherever it appears in the text.

2. The Spaces. In this section, we shall introduce the spaces which will be used in the paper.

Definition 2.1. Let the space $W_{2,\mu}^l(R_n)$, with $l \geqq 0$ an integer, μ real, be the Banach space of all functions u such that

$$||u||^2_{W_{2,\mu}^l(R_n)} = \int_{R_n} e^{2\mu|x|} \sum_{|k| \leq l; k \geq 0} (D^k u)^2 \, dx < \infty, \tag{2.1}$$

where $D^k = \partial^{k_1+k_2+\cdots+k_n}/\partial x_1^{k_1} \cdots \partial x_n^{k_n}$. For $\mu = 0$, we get the usual Sobolev space. For $\mu \neq 0$, we get a weighted Sobolev space. Note that $L_2(R_n) = W_{2,0}^0(R_n)$.

Let us now introduce the so called B splines. For $x \in R_1$, let

$$\begin{aligned} \varphi_1(x) &= 1, \quad |x| < \tfrac{1}{2}, \\ &= 0, \quad |x| \geqq \tfrac{1}{2}. \end{aligned} \tag{2.2}$$

Let t be an integer $\geqq 2$. Starting with $t = 2$, we recursively define $\varphi_t(x)$ as

$$\varphi_t(x) = \varphi_1(x) * \varphi_{t-1}(x), \qquad t \geqq 2, \tag{2.3}$$

where $*$ denotes convolution.

Now, for $x \in R_n$, $x \equiv (x_1, \cdots, x_n)$, define

$$\varphi_t(x) = \prod_{j=1}^n \varphi_t(x_j), \qquad t \geqq 1, \tag{2.4}$$

and

$$\varphi_{t,i}(x) = \varphi_{t-1}(x_i - \tfrac{1}{2}) \prod_{j=1; j \neq i}^n \varphi_t(x_j), \qquad t \geqq 2, \ i = 1, \cdots, n. \tag{2.5}$$

Let us mention some of the well-known properties of these functions which will be important later on:

(1) $\varphi_t(x) \geqq 0$, $\varphi_{t,i}(x) \geqq 0$ for all $x \in R_n$.

(2) $\varphi_t(x)$ and $\varphi_{t,i}(x)$ have compact support.

(3) Denoting by $F(\varphi)(\sigma)$, the Fourier transform of $\varphi(x)$, we have, with $\sigma = (\sigma_1, \cdots, \sigma_n)$,

$$F(\varphi_t)(\sigma) = \prod_{j=1}^n \left(\frac{\sin \frac{1}{2}\sigma_j}{\frac{1}{2}\sigma_j}\right)^t, \tag{2.6}$$

$$F(\varphi_{t,i})(\sigma) = \left(\frac{\sin \frac{1}{2}\sigma_i}{\frac{1}{2}\sigma_i}\right)^{t-1} e^{i\sigma_i/2} \prod_{j=1; j \neq i}^n \left(\frac{\sin \frac{1}{2}\sigma_j}{\frac{1}{2}\sigma_j}\right)^t, \tag{2.7}$$

$$\frac{\partial \varphi_t(x)}{\partial x_i} = -\varphi_{t,i}(x) + \varphi_{t,i}(x + 1_i). \tag{2.8}$$

In the following, let U and V denote the functions defined on the set of all multi-integers k.

Definition 2.2. Let the space $Q_\mu^{l,h}$, $l = 0, 1, h > 0$, μ real, be the space of all functions U such that

$$\|U\|^2_{Q_\mu^{0,h}} = h^n \sum_k |U(k)|^2 e^{2\mu h|k|} < \infty \tag{2.9}$$

(for $l = 0$) and

(2.10)
$$\|U\|^2_{Q_\mu^{1,h}} = h^n\left[\frac{1}{h^2} \sum_k \sum_{i=1}^n |U(k + 1_i) - U(k)|^2 e^{2\mu h|k|} + \sum_k |U(k)|^2 e^{2\mu h|k|}\right] < \infty$$

(for $l = 1$).

Let $U \in Q_\mu^{l,h}$ for $l = 0$ and $l = 1$. Define the mapping S_j^h, $j \geqq 1$, by

$$S_j^h U = \sum_k U(k)\varphi_j(x/h - k), \tag{2.11}$$

and the mapping $S_j^{h,i}$, $j \geqq 2$, by

$$S_j^{h,i} U = \sum_k U(k)\varphi_{j,i}(x/h - k). \tag{2.12}$$

THEOREM 2.1. *For $|\mu|$ sufficiently small, there exist constants C_1 and C_2, $0 < C_1 < C_2 < \infty$, such that*

$$C_1\|U\|_{Q_\mu^{l,h}} \leqq \|S_j^h U\|_{W_{2,\mu}^l(R_n)} \leqq C_2\|U\|_{Q_\mu^{l,h}}, \qquad l = 0, 1, j > l, \tag{2.13}$$

and

$$C_1\|U\|_{Q_\mu^{0,h}} \leqq \|S_j^{h,i} U\|_{W_{2,\mu}^0(R_n)} \leqq C_2\|U\|_{Q_\mu^{0,h}}. \tag{2.14}$$

Proof. 1. Let us first consider (2.13) for $l = 0$ and $\mu = 0$. It is sufficient to prove this inequality for $h = 1$.

In [14] we have proved that

$$F(S_j^1 U)(\sigma) = Z_j(\sigma) \sum_k U(k) \exp(i\langle k, \sigma\rangle), \tag{2.15}$$

where

$$Z_j(\sigma) = F(\varphi_j)(\sigma) \tag{2.16}$$

and

$$\langle k, \sigma\rangle = \sum_{i=1}^n k_i\sigma_i. \tag{2.17}$$

By a well-known property of the Fourier transform, we have

$$\|F(S_j^1 U)\|^2_{L_2(R_n)} = (2\pi)^n \|S_j^1 U\|^2_{L_2(R_n)}. \tag{2.18}$$

Hence, we may write

$$\|F(S_j^1 U)\|^2_{L_2(R_n)} = \sum_l \int_{\Omega_0} \left|\sum_k U(k) \exp(i\langle k, \sigma\rangle)\right|^2 |Z_j(\sigma - l)|^2 \, d\sigma, \tag{2.19}$$

where $\Omega_0 = \{x; |x_i| < \pi\}$. Since there exists a C such that $|Z_j(\sigma)|^2 > C > 0$ on Ω_0, we have, from (2.19),

$$(2.20)\quad \|F(S_j^1 U)\|^2_{L_2(R_n)} \geqq C\int_{\Omega_0}\left|\sum_k U(k)\exp(i\langle k,\sigma\rangle)\right|^2 d\sigma \geqq C(2\pi)^n\sum_k |U(k)|^2.$$

From (2.20), (2.18) and (2.9), we obtain

$$(2.21)\qquad \|S_j^1 U\|_{L_2(R_n)} \geqq C\|U\|_{Q_0^{0,1}}.$$

From (2.19), we also have

$$(2.22)\qquad \|F(S_j^1 U)\|^2_{L_2(R_n)} \leqq \int_{\Omega_0}\left|\sum_k U(k)\exp(i\langle k,\sigma\rangle)\right|^2 d\sigma \sum_l Z^2_{j,l}$$

with

$$(2.23)\qquad Z_{j,l} = \max_{\sigma\in\Omega_0}|Z_j(\sigma - l)|.$$

From (2.6) it is clear that $\sum_l Z^2_{j,l} < \infty$. Therefore,

$$(2.24)\qquad \|S_j^1 U\|_{L_2(R_n)} \leqq C\|U\|_{Q_0^{0,1}}.$$

Inequalities (2.21) and (2.24) together prove (2.13) for the case $l = 0$ and $\mu = 0$. Inequality (2.14) can be proved in the same manner.

2. Let us now prove (2.13) for $l = 0$ and $\mu \neq 0$. Let $U \in Q_\mu^{0,1}$ and $U_\mu(k) = U(k)e^{\mu|k|}$. Using (2.13) for $\mu = 0$ and U_μ, we obtain, because of (2.9),

$$(2.25)\qquad \int_{R_n}\left|\sum_k |U(k)|\, e^{\mu|k|}\varphi_j(x-k)\right|^2 dx \leqq C\,\|U\|^2_{Q_\mu^{0,1}}.$$

On the other hand, we have

$$(2.26)\qquad \|S_j^1 U\|^2_{W_{2,\mu}^0(R_n)} = \int_{R_n}\left|\sum_k U(k)\varphi_j(x-k)\right|^2 e^{2\mu|x|}\,dx.$$

Now since $\varphi_j(x) \geqq 0$ for all $x \in R_n$ and since φ_j has compact support, it follows from (2.26) that

$$(2.27)\qquad \begin{aligned}\|S_j^1 U\|_{W_{2,\mu}^0(R_n)} &\leqq \int_{R_n}\left|\sum_k |U(k)|\,\varphi_j(x-k)\right|^2 e^{2\mu|x|}\,dx\\ &\leqq C\int_{R_n}\left|\sum_k |U(k)|\, e^{\mu|k|}\varphi_j(x-k)\right|^2 dx.\end{aligned}$$

From (2.25) and (2.27) we obtain

$$\|S_j^1 U\|_{W_{2,\mu}^0(R_n)} \leqq C\|U\|_{Q_\mu^{0,1}}.$$

Now from (2.13) for $\mu = 0$, we have with $C > 0$, that

$$\begin{aligned}C\|U\|^2_{Q_\mu^{0,1}} &\leqq C\int_{R_n}\left|\sum_k |U(k)|\, e^{\mu|k|}\varphi_j(x-k)\right|^2 dx\\ &\leqq \int_{R_n}\left|\sum_k U(k)e^{\mu|k|}\varphi_j(x-k)\right|^2 dx\\ &= \int_{R_n}\left|\sum_k [U(k)e^{\mu|x|}\varphi_j(x-k) + U(k)(e^{\mu|k|} - e^{\mu|x|})\varphi_j(x-k)]\right|^2 dx\\ &\leqq 2\left[\|S_j^1 U\|^2_{W_{2,\mu}^0(R_n)} + \mu^2\int_{R_n}\left|\sum_k |U(k)|\, e^{\mu|k|}Q_k(x)\varphi_j(x-k)\right|^2 dx\right],\end{aligned}$$

where $|Q_k(x)| \leqq C$, independently of k. Hence, we have

$$C\|U\|_{Q_\mu^{0,1}} \leqq 2[\|S_j^1 U\|^2_{W_{2,\mu}{}^0(R_n)} + \mu^2 C\|U\|^2_{Q_\mu^{0,1}}]. \tag{2.28}$$

Therefore,

$$\|U\|_{Q_\mu^{0,1}} \leqq C\|S_j^1 U\|_{W_{2,\mu}{}^0(R_n)}, \tag{2.29}$$

for sufficiently small $|\mu|$.

This proves (2.13) for the case $l = 0$ and $\mu \neq 0$, μ sufficiently small in absolute value. Inequality (2.14) can be proved for the case $l = 0$ and $\mu \neq 0$, μ sufficiently small in absolute value, in the same manner.

3. Let us now prove the theorem for $l = 1$. We have

$$\begin{aligned} \frac{\partial S_j^h U}{\partial x_s} &= \frac{1}{h}\left[-\sum_k U(k)\left[\varphi_{j,s}\left(\frac{x}{h} - k\right) - \varphi_{j,s}\left(\frac{x}{h} - 1_s - k\right)\right]\right] \\ &= \frac{1}{h}\left[\sum_k (U(k + 1_s) - U(k))\varphi_{j,s}\left(\frac{x}{h} - k\right)\right] \\ &= \sum_k V(k)\varphi_{j,s}\left(\frac{x}{h} - k\right), \end{aligned} \tag{2.30}$$

where

$$V(k) = (U(k + 1_s) - U(k))/h. \tag{2.31}$$

From (2.14) with $l = 0$, we obtain

$$\begin{aligned} C_1\|V\|_{Q_\mu^{0,h}} &\leqq \left\|\frac{\partial S_j^h U}{\partial x_s}\right\|_{W_{2,\mu}{}^0(R_n)} \\ &\leqq C_2\|V\|_{Q_\mu^{0,h}(R_n)}. \end{aligned} \tag{2.32}$$

Inequality (2.14) follows immediately from (2.30), (2.31) and (2.32). This completes the proof of Theorem 2.1.

3. Some Auxiliary Theorems about a Bilinear Form. Let $\mathcal{H}_\mu = W^1_{2,\mu}(R_n) \times W^1_{2,-\mu}(R_n)$. Define on $\mathcal{H}_\mu$ the bilinear form A by

$$A(u, v) = \int_{R_n} \left(\sum_{i=1}^n \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i} + uv\right) dx. \tag{3.1}$$

We shall now prove some important properties of this bilinear form.

THEOREM 3.1. *For sufficiently small* $|\mu|$,

$$|A(u, v)| \leqq C\|u\|_{W_{2,\mu}{}^1(R_n)}\|v\|_{W_{2,-\mu}{}^1(R_n)}, \tag{3.2}$$

$$\sup_{\|u\|_{W_{2,\mu}{}^1(R_n)} \leqq 1} |A(u, v)| \geqq C\|v\|_{W_{2,-\mu}{}^1(R_n)}, \tag{3.3}$$

where $C > 0$ *and*

$$\sup_{\|v\|_{W_{2,-\mu}{}^1(R_n)} \leqq 1} |A(u, v)| \geqq C\|u\|_{W_{2,\mu}{}^1(R_n)}, \qquad C > 0. \tag{3.4}$$

Proof. 1. Inequality (3.2) is easily proved by the following inequalities:

$$|A(u,v)| = \left|\int_{R_n}\left(\sum_{i=1}^{n} e^{\mu|x|}\frac{\partial u}{\partial x_i}e^{-\mu|x|}\frac{\partial v}{\partial x_i} + e^{\mu|x|}ue^{-\mu|x|}v\right)dx\right|$$

$$\tag{3.5} \leqq \sum_{i=1}^{n}\left[\int_{R_n}\left(e^{\mu|x|}\frac{\partial u}{\partial x_i}\right)^2 dx\right]^{1/2}\left[\int_{R_n}\left(e^{-\mu|x|}\frac{\partial v}{\partial x_i}\right)^2 dx\right]^{1/2} + \left[\int_{R_n}(e^{\mu|x|}u)^2\,dx\right]^{1/2}\left[\int_{R_n}(e^{-\mu|x|}v)^2\,dx\right]^{1/2}$$

$$\leqq C\|u\|_{W_{2,\mu}^1(R_n)}\,\|v\|_{W_{2,-\mu}^1(R_n)}.$$

2. Let $v \in W^1_{2,-\mu}(R_n)$ and let $u = ve^{-2\mu|x|}$. By direct computation, we have

$$\tag{3.6} \frac{\partial(ve^{-2\mu|x|})}{\partial x_i} = e^{-2\mu|x|}\frac{\partial v}{\partial x_i} - 2\mu e^{-2\mu|x|}v\,\operatorname{sgn} x_i.$$

Hence,

$$\tag{3.7} \|u\|^2_{W_{2,\mu}^1(R_n)} \leqq 2\left[\int_{R_n}\left[\sum_{i=1}^{n} e^{-4\mu|x|}\left(\frac{\partial v}{\partial x_i}\right)^2 + 4n\mu^2v^2e^{-4\mu|x|} + v^2e^{-4\mu|x|}\right]e^{2\mu|x|}\right]dx \leqq C\|v\|^2_{W_{2,-\mu}^1(R_n)}$$

which implies that $u \in W^1_{2,\mu}(R_n)$. Furthermore,

$$\tag{3.8} |A(ve^{-2\mu|x|},v)| = \left|\int_{R_n}\sum_{i=1}^{n}\left[\frac{\partial v}{\partial x_i} - 2\mu v\,\operatorname{sgn} x_i\right]\left[\frac{\partial v}{\partial x_i}\right]e^{-2\mu|x|}\,dx + \int_{R_n} v^2e^{-2\mu|x|}\,dx\right| \geqq \|v\|^2_{W_{2,-\mu}^1(R_n)} - 2\mu n\,\|v\|^2_{W_{2,-\mu}^1(R_n)} \geqq C\|v\|^2_{W_{2,-\mu}^1(R_n)}.$$

Inequality (3.8), together with (3.7), proves (3.3).

3. Replacing μ by $-\mu$ in the above discussion we have inequality (3.4).

Let us prove a further theorem.

THEOREM 3.2. *For sufficiently small* $|\mu|$, *we have*

$$\tag{3.9} |A(S_i^hU, S_i^hV)| \leqq C\|S_i^hU\|_{W_{2,\mu}^1(R_n)}\|S_i^hV\|_{W_{2,-\mu}^1(R_n)},$$

$$\tag{3.10} \sup_{\|S_i^hU\|_{W_{2,\mu}^1(R_n)}\leq 1} |A(S_i^hU, S_i^hV)| \geqq C\|S_i^hV\|_{W_{2,-\mu}^1(R_n)}$$

and

$$\tag{3.11} \sup_{\|S_i^hV\|_{W_{2,-\mu}^1(R_n)}\leq 1} |A(S_i^hU, S_i^hV)| \geqq C\|S_i^hU\|_{W_{2,\mu}^1(R_n)}.$$

Proof. 1. Inequality (3.9) follows immediately from (3.2).

2. To prove (3.10), take $V \in Q^{1,h}_{-\mu}$ and let $U(k) = V(k)e^{-2\mu|k|h}$, then

$$\tag{3.12} \|U\|^2_{Q_\mu^{1,h}} \leqq 2h^n\left[\frac{1}{h^2}\sum_k\sum_{i=1}^{n}|V(k+1_i) - V(k)|^2e^{-2\mu|k|h} + (2\mu)^2C\sum_k|V(k)|^2e^{-2\mu|k|h} + \sum_k|V(k)|^2e^{-2\mu|k|h}\right]$$

which implies that

$$\|U\|_{Q_\mu^{1,h}} \leqq C\|V\|_{Q_{-\mu}^{1,h}}.$$

Hence, by Theorem 2.1, we have

$$\|S_i^h U\|_{W_{2,\mu}^1(R_n)} \leqq C\|S_i^h V\|_{W_{2,-\mu}^1(R_n)}. \tag{3.13}$$

Furthermore,

$$\begin{aligned} &|A(S_i^h U, S_i^h V)| \\ &= \left| \int_{R_n} \sum_{i=1}^n \left[\frac{\partial}{\partial x_i} \sum_k V(k) e^{-2\mu|k|h} \varphi_i\left(\frac{x}{h} - k\right)\right]\left[\frac{\partial}{\partial x_i} \sum_k V(k)\varphi_i\left(\frac{x}{h} - k\right)\right] dx \right. \\ &\qquad \left. + \int_{R_n} \left[\sum_k V(k) e^{-2\mu|k|h} \varphi_i\left(\frac{x}{h} - k\right)\right]\left[\sum_k V(k)\varphi_i\left(\frac{x}{h} - k\right)\right] dx \right|. \end{aligned} \tag{3.14}$$

However,

$$\begin{aligned} &\frac{\partial}{\partial x_i} \sum_k V(k) e^{-2\mu|k|h} \varphi_i\left(\frac{x}{h} - k\right) \\ &\qquad = \sum_k \frac{V(k + 1_i) - V(k)}{h} \varphi_{i,i}\left(\frac{x}{h} - k\right) e^{-2\mu|k|h} \\ &\qquad - 2\mu \sum_k V(k + 1_i)\varphi_{i,i}\left(\frac{x}{h} - k\right) Q_k e^{-2\mu|k|h} \end{aligned} \tag{3.15}$$

with Q_k uniformly bounded, independently of k. Now we obtain

$$|A(S_i^h U, S_i^h V)| \geqq C[\|V\|^2_{Q_{-\mu}^{1,h}} - 2\mu C\|V\|^2_{Q_{-\mu}^{1,h}}] \quad \text{with } C > 0. \tag{3.16}$$

In fact,

$$\begin{aligned} &\int_{R_n} \left[\sum_k V(k) e^{-2\mu|k|h} \varphi_i\left(\frac{x}{h} - k\right)\right]\left[\sum_k V(k)\varphi_i\left(\frac{x}{h} - k\right)\right] dx \\ &\quad = \int_{R_n} \Bigg[e^{-2\mu|x|}\left[\sum_k V(k)\varphi_i\left(\frac{x}{h} - k\right)\right]^2 \\ &\qquad + \left[\sum_k V(k)\varphi_i\left(\frac{x}{h} - k\right)(e^{-2\mu|k|h} - e^{-2\mu|x|})\right]\left[\sum_k V(k)\varphi_i\left(\frac{x}{h} - k\right)\right]\Bigg] dx. \end{aligned}$$

Using Theorem 2.1, we obtain

$$\begin{aligned} &\int_{R_n} \left[\sum_k V(k)\varphi_i\left(\frac{x}{h} - k\right)(e^{-2\mu|k|h} - e^{-2\mu|x|})\right]\left[\sum_k V(k)\varphi_i\left(\frac{x}{h} - k\right)\right] dx \\ &\quad \leqq \int_{R_n} \left[\sum_k |V(k)|\, \varphi_i\left(\frac{x}{h} - k\right) |Q(\mu, x, k)|\, 2\mu\right]\left[\sum_k |V(k)|\, \varphi_i\left(\frac{x}{h} - k\right)\right] e^{-2\mu|x|}\, dx \\ &\quad \leqq 2\mu C\, \|V\|^2_{Q_{-\mu}^{1,h}}, \end{aligned}$$

where $Q(\mu, x, k)$ is bounded independently of k because of the compact support of $\varphi_i(x)$. By a similar argument, using (3.15), we obtain

$$\begin{aligned} &\int_{R_n} \left[\frac{\partial}{\partial x_i} \sum_k V(k) e^{-2\mu|k|h} \varphi_i\left(\frac{x}{h} - k\right)\right]\left[\frac{\partial}{\partial x_i} \sum_k V(k)\varphi_i\left(\frac{x}{h} - k\right)\right] dx \\ &\qquad = \int_{R_n} e^{-2\mu|x|} \sum_{i=1}^n \left[\frac{\partial}{\partial x_i} \sum_k V(k)\varphi_i\left(\frac{x}{h} - k\right)\right]^2 dx + R \end{aligned} \tag{3.17}$$

with $|R| \leqq 2\mu C \, ||V||^2_{Q_{-\mu^1,h}}$. Hence, for sufficiently small $|\mu|$, we have

$$|A(S_j^h U, S_j^h V)| \geqq C||V||^2_{Q_{-\mu^1,h}} \geqq C||S_j^h V||^2_{W_{2,-\mu^1}(R_n)}. \tag{3.18}$$

From (3.18), using (3.13), we get inequality (3.10). Inequality (3.11) is proved by changing μ to $-\mu$ in the above discussion.

This completes the proof of Theorem 3.2.

4. The Finite Element Method and its Convergence. In [6] we proved the following two theorems.

THEOREM 4.1. *Let H_1 and H_2 be two Hilbert spaces with scalar product $(\cdot, \cdot)_{H_1}$ and $(\cdot, \cdot)_{H_2}$, respectively.*

Let $B(u, v)$, $u \in H_1$, $v \in H_2$, be a bilinear form on $H_1 \times H_2$ such that

$$|B(u, v)| \leqq C_1||u||_{H_1} \, ||v||_{H_2}, \tag{4.1}$$

$$\sup_{||u||_{H_1} \leq 1} |B(u, v)| \geqq C_2||v||_{H_2}, \tag{4.2}$$

and

$$\sup_{||v||_{H_2} \leq 1} |B(u, v)| \geqq C_3||u||_{H_1}, \tag{4.3}$$

with $C_1 < \infty$, $C_2 > 0$, $C_3 > 0$.

Let H_2' be the space of bounded linear functionals on H_2. Let $f \in H_2'$. Then there exists exactly one element $u_0 \in H_1$ with

$$||u_0||_{H_1} \leqq ||f||_{H_2'}/C_3, \tag{4.4}$$

such that

$$B(u_0, v) = f(v) \tag{4.5}$$

for all $v \in H_2$.

THEOREM 4.2. *Let the assumptions of the Theorem 4.1 be fulfilled. Further, let M_1 and M_2 be closed subspaces of H_1 and H_2, respectively. For every $v \in M_2$, let*

$$\sup_{||u||_{H_1} \leq 1} |B(u, v)| \geqq d_2(M_1, M_2) \, ||v||_{H_2} \tag{4.6}$$

with $d_2(M_1, M_2) > 0$, and for every $u \in M_1$ let

$$\sup_{||v||_{H_2} \leq 1} |B(u, v)| \geqq d_3(M_1, M_2) \, ||u||_{H_1} \tag{4.7}$$

with $d_3(M_1, M_2) > 0$.

Let $f \in H_2$ be given and let u_0 denote that element of H_1 for which

$$B(u_0, v) = f(v) \tag{4.8}$$

holds for all $v \in H_2$ (such an element exists and is unique by Theorem 4.1).

Assume there exists $w \in M_1$ such that

$$||u_0 - w||_{H_1} \leqq \vartheta. \tag{4.9}$$

Furthermore, let $\hat{u}_0 \in M_1$ such that

$$B(\hat{u}_0, v) = f(v) \tag{4.10}$$

for all $v \in M_2$. *Then*

$$\|u_0 - \hat{u}_0\|_{H_1} \leqq \left[1 + \frac{C_1}{d_3(M_1, M_2)}\right]\vartheta. \tag{4.11}$$

We will utilize the Theorems 4.1 and 4.2 to analyze the solution of the equation

$$-\Delta u + u = f. \tag{1.1}$$

Let us first prove Lemma 4.1.

LEMMA 4.1. *Let* $f \in W^0_{2,-\mu}(R_n)$, *then*

$$\left|\int_{R_n} \frac{\partial f}{\partial x_i} u \, dx\right| \leqq \|f\|_{W_{2,-\mu}{}^0(R_n)} \|u\|_{W_{2,\mu}{}^1(R_n)}. \tag{4.12}$$

Proof. For any $f \in C^\infty$ with compact support,

$$\left|\int_{R_n} \frac{\partial f}{\partial x_i} u \, dx\right| = \left|\int_{R_n} f \frac{\partial u}{\partial x_i} dx\right| \leqq \|f\|_{W_{2,-\mu}{}^0(R_n)} \|u\|_{W_{2,\mu}{}^1(R_n)}.$$

The functions $f \in C^\infty$ with compact support are dense in the space $W^0_{2,-\mu}(R_n)$, hence, (4.12) holds for all $f \in W^0_{2,-\mu}(R_n)$. The following theorem will complete the preparation for the main results of the paper.

THEOREM 4.3. *Let* $f \in W^l_{2,\mu}(R_n)$, $l \geqq 0$, $|\mu|$ *sufficiently small. Then there exists exactly one solution* u *of Eq.* (1.1) *in* $W^1_{2,\mu}$, *such that*

$$\|u\|_{W_{2,\mu}{}^{l+2}(R_n)} \leqq C\|f\|_{W_{2,\mu}{}^l(R_n)}. \tag{4.13}$$

Proof. Since $f \in W^0_{2,\mu}(R_n)$, we have

$$T_f(u) = \int_{R_n} fu \, dx, \qquad \|T\|_{(W_{2,-\mu}{}^1(R_n))'} \leqq \|f\|_{W_{2,\mu}{}^0(R_n)}.$$

Using Theorems 4.1 and 3.1, we have

$$\|u\|_{W_{2,\mu}{}^1(R_n)} \leqq C\|f\|_{W_{2,\mu}{}^0(R_n)} \leqq C\|f\|_{W_{2,\mu}{}^l(R_n)}.$$

Differentiating both sides of Eq. (1.1), using Theorems 4.1, 3.1 and Lemma 4.1, we obtain

$$\left\|\frac{\partial u}{\partial x_i}\right\|_{W_{2,\mu}{}^1(R_n)} \leqq C\|f\|_{W_{2,\mu}{}^0(R_n)} \leqq C\|f\|_{W_{2,\mu}{}^l(R_n)}.$$

Hence,

$$\|u\|_{W_{2,\mu}{}^2(R_n)} \leqq C\|f\|_{W_{2,\mu}{}^l(R_n)}.$$

By differentiating Eq. (1.1) $l + 1$ times and using induction, we obtain (4.13) for Eq. (1.1).

Let us now describe the finite element method. Let $f \in W^l_{2,0}(R_n)$ and let $\psi(h)$ be a decreasing function of h defined for $h > 0$. Let

$$u_{h,\psi,j}(x) = \sum_{|k| \leqq \psi(h)} C(k)\varphi_j(x/h - k). \tag{4.14}$$

Let us determine the coefficients $C(k)$ such that

$$A(u_{h,\psi,j}, v) = \int_{R_n} fv\, dx \tag{4.15}$$

for all v of the form (4.14).

We can now prove the main theorem of the paper.

THEOREM 4.4. *Let $f \in W^l_{2,0}(R_n)$ and let $\psi(h) = h^{-\epsilon}$ for any given $\epsilon > 0$. Let u_0 be the solution of* (1.1) *in $W^1_{2,0}(R_n)$.*

Then on every compact domain Ω,

$$||u_0 - u_{h,\psi,j}||_{W_2{}^1(\Omega)} \leqq Ch^{l+1}||f||_{W_{2,0}{}^l(R_n)} \tag{4.16}$$

for $j \geqq l + 2$, where

$$||u||_{W_2{}^1(\Omega)} = \int_\Omega \left[\sum_{i=1}^n \left(\frac{\partial u}{\partial x_i}\right)^2 + u^2\right] dx. \tag{4.17}$$

Proof. From Theorem 4.3, we have $u \in W^{l+2}_{2,0}(R_n)$. Therefore, $u_0 \in W^{l+2}_{2,\mu}(R_n)$ for $\mu \leqq 0$ also. Let us use a cut-off function $\chi(x) \in C^\infty$ with $\chi(x) = 1$ for $|x| \leqq 1$ and $\chi(x) = 0$ for $|x| \geqq 2$. Define $\chi_{h,\psi}(x) = \chi((\psi(h))^{-1}3x)$ and $u_{0,h}(x) = \chi_{h,\psi}(x)u_0(x)$. Then

$$u_{0,h}(x) = 0, \quad \text{for } |x| \geqq \tfrac{2}{3}\psi(h),$$

$$u_{0,h}(x) - u_0(x) = 0, \quad \text{for } |x| \leqq \tfrac{1}{3}\psi(h),$$

and

$$||u_{0,h}||_{W_{2,0}{}^{l+2}(R_n)} \leqq C||f||_{W_{2,0}{}^l(R_n)}.$$

In [14], we have shown that for $j \geqq l + 2$ there exist $d(h, k)$ such that

$$\left\|u_{0,h} - \sum_k d(h, k)\varphi_j\left(\frac{x}{h} - k\right)\right\|_{W_{2,0}{}^1(R_n)} \leqq C||u_{0,h}||_{W_{2,0}{}^{l+2}(R_n)}h^{l+1}. \tag{4.18}$$

We have also shown that the support of

$$w(x) = \sum_k d(h, k)\varphi_j\left(\frac{x}{h} - k\right)$$

lies in a Lh neighborhood of the support of $u_{0,h}$, i.e. the support of $w(x)$ lies in the set $Q^*_{h,\psi} = \{x; |x| \leqq \tfrac{2}{3}\psi(h) + Lh\}$. Therefore, for h small enough,

$$Q_{h,\psi} = \{x; |x| \leqq \psi(h)\} \supset Q^*_{h,\psi}.$$

For $\mu < 0$, $|\mu|$ sufficiently small, we obviously have

$$\begin{aligned} ||u_0 - w||_{W_{2,\mu}{}^1(R_n)} &\leqq ||u_{0,h} - w||_{W_{2,\mu}{}^1(R_n)} + ||u_{0,h} - u||_{W_{2,\mu}{}^1(R_n)} \\ &\leqq C[h^{l-1}\, ||f||_{W_{2,0}{}^l(R_n)} + e^{\mu\psi(h)/3}\, ||f||_{W_{2,0}{}^l(R_n)}], \end{aligned} \tag{4.19}$$

since $u_{0,h} - u = 0$ for $|x| \leqq \tfrac{1}{3}\psi(h)$. However, $e^{\mu\psi(h)/3} = e^{\mu h^{-\epsilon}/3} \leqq Ch^{l+1}$. Therefore,

$$||u_0 - w||_{W_{2,\mu}{}^1(R_n)} \leqq Ch^{l+1}\, ||f||_{W_{2,0}{}^1(R_n)}, \tag{4.20}$$

and hence, from Theorems 4.2 and 3.2, we have

$$||u_0 - u_{h,\psi,j}||_{W_{2,\mu}{}^1(R_n)} \leqq Ch^{l+1}\, ||f||_{W_{2,0}{}^1(R_n)}, \tag{4.21}$$

the desired result.

Let us count the number T of unknowns in (4.18) which we have to determine by solving a system of linear equations. It is clear that T is of the order $h^{-n(1+\epsilon)}$.

Now let the "effective" H (see [7]) be defined by

$$H = (T)^{-1/n}. \tag{4.22}$$

Thus $H = h^{1+\epsilon}$. Hence, instead of (4.16), we may write

$$\begin{aligned} ||u_0 - u_{h,\psi,i}||_{W_2^1(\Omega)} &\leqq C(\epsilon, \Omega)H^{l+1-\epsilon} \, ||f||_{W_{2,0}^l(R_n)} \\ &= C(\epsilon, \Omega)T^{-(l+1)/n+\epsilon} \, ||f||_{W_{2,0}^l(R_n)}. \end{aligned} \tag{4.23}$$

Now, by the same manner as in [7], we can show that the rate of convergence indicated in (4.23) is the highest possible rate of convergence on every compact domain, provided that we neglect the ϵ.

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1. G. Fix & G. Strang, "Fourier analysis of the finite element method in Ritz-Galerkin theory," *Studies in Appl. Math.*, v. 48, 1969, pp. 265–273. MR **41** #2944.

2. G. Strang & G. Fix, "A Fourier analysis of the finite element variational method." (To appear.)

3. G. Strang, *The Finite Element Method and Approximation Theory*, Numerical Solution of Partial Differential Equations. II (SYNSPADE, 1970), Academic Press, New York and London, 1971, pp. 547–585.

4. M. Zlámal, "On the finite element method," *Numer. Math.*, v. 12, 1968, pp. 394–409. MR **39** #5074.

5. L. A. Oganesjan & L. A. Ruchovec, "A study of rates of convergence of some variational difference schemes for elliptic equations of second order in a two dimensional domain with smooth boundary," *Ž. Vyčisl. Mat. i Mat. Fiz.*, v. 9, 1969, pp. 1102–1119. (Russian)

6. I. Babuška, *Error Bounds for Finite Element Method*, Technical Note BN-630, Institute for Fluid Dynamics and Appl. Math., University of Maryland, College Park, Md., 1969; *Numer. Math.*, v. 16, 1971, pp. 322–333.

7. I. Babuška, *The Rate of Convergence for the Finite Element Method*, Technical Note BN-646, Institute for Fluid Dynamics and Appl. Math., University of Maryland, College Park, Md., 1970; *SIAM J. Numer. Anal.*, v. 8, 1971, pp. 304–315.

8. I. Babuška, *Finite Element Method for Domains with Corner*, Technical Note BN-636, Institute for Fluid Dynamics and Appl. Math., University of Maryland, College Park, Md., 1970; *Computing*, v. 6, 1970, pp. 264–273.

9. I. Babuška, *Numerical Solution of Boundary Value Problems by the Perturbed Variation Principle*, Technical Note BN-624, Institute for Fluid Dynamics and Appl. Math., University of Maryland, College Park, Md., 1969.

10. I. Bauška, *The Finite Element Method for Elliptic Equations with Discontinuous Coefficients*, Technical Note BN-631, Institute for Fluid Dynamics and Appl. Math., University of Maryland, College Park, Md., 1969; *Computing*, v. 5, 1970, pp. 207–213.

11. I. Babuška, *The Finite Element Method for Elliptic Differential Equations*, Technical Note BN-653, Institute for Fluid Dynamics and Appl. Math., University of Maryland, College Park, Md., 1970, Numerical Solution of Partial Differential Equations. II (SYNSPADE, 1970), Academic Press, New York and London, 1971, pp. 69–107.

12. O. C. Zienkiewicz, *The Finite Element Method in Structural and Continuous Mechanics*, McGraw-Hill, New York and London, 1970.

13. Y. R. Rashid, *On Computational Methods in Solid Mechanics and Stress Analysis*, Conference on Effective Use of Comp. in the Nuclear Industry, Knoxville, Tenn., April 21–23, 1969.

14. I. Babuška, *Approximation by Hill Functions*, Technical Note BN-648, Institute for Fluid Dynamics and Appl. Math., University of Maryland, College Park, Md., 1970; *Comment. Math. Univ. Carolinae*, v. 11, 1970, pp. 787–811.