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Stability of Parabolic Difference Approximations to Certain Mixed Initial Boundary Value Problems*

By Stanley Osher

Abstract. We consider the equation

$$u_t - a(x, t)u_{xx} - b(x, t)u_x - c(x, t)u = f(x, t)$$

in a region $0 \leqq x \leqq 1$, $t \geqq 0$, with inhomogeneous initial and boundary data. We are concerned with stability and estimates on divided differences in the maximum norm for solutions of consistent implicit, multistep, parabolic difference approximations to this problem. Using a parametrix approach, we give sufficient conditions for certain estimates to be valid.

1. Introduction. In this paper, we shall consider invertibility, stability, and smoothness up to the boundary of the solutions of a general class of implicit multistep approximations to heat-type equations in one space variable in regions with boundaries. In an earlier paper [3], we proved a stability theorem for more general problems but allowed only constant coefficients. Here, we require mild smoothness of the coefficients, and we then estimate not only the solution, but certain of its divided differences.

The results extend those of Varah [11]. In his paper, he considered only explicit one-step approximations to the constant coefficient equation with no lower order terms. The main results in both involve the kind of normal mode analysis which was discussed in [2], [3], [4], [12].

The parametrix technique was used by Widlund [8] in his paper on approximations to the initial-value problem for more general parabolic systems. Many of the estimates for this mixed problem are similar to those he obtained for the analogous initial-value problem.

The invertibility technique modifies some of the ideas of Strang [6], [7] in his papers on difference schemes for which the solution is assumed identically zero outside the region.

Our accuracy assumptions at the boundary and the normal mode condition at $z = 1$ seem to be necessary in order to obtain the appropriate estimates on the first divided differences up to the boundary. The example (9.1) below indicates the difficulties involved in weakening these assumptions.

This last example was due to Björn Engquist, and the author would like to thank him for several stimulating discussions on these and related problems.

2. Preliminaries. We are considering the mixed problem for the equation:

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(2.1) $$u_t - a(x, t)u_{xx} - b(x, t)u_x - c(x, t)u = f(x, t)$$

in the region,

(2.2) $$0 \leqq x \leqq 1, \qquad 0 \leqq t \leqq T, \qquad T < \infty,$$

with bounded initial conditions:

(2.3) $$u(x, 0) = u_0(x),$$

and boundary conditions:

(2.4) $$\begin{aligned} \alpha_1 u(0, t) + \alpha_2 u_x(0, t) &= g_0(t), \\ \beta_1 u(1, t) + \beta_2 u_x(1, t) &= g_1(t), \end{aligned} \qquad 0 \leqq t \leqq T.$$

Each α_i, β_i is a complex number with

$$|\alpha_1^2| + |\alpha_2^2| = 1 = |\beta_1^2| + |\beta_2^2|.$$

We assume that $f(x, t)$, $u_0(x)$, $g_0(t)$, $g_2(t)$ are all bounded in their regions of definition. Moreover, the real part of $a(x, t)$ is bounded below by a positive constant c. $a(x, t)$ has the property:

(2.5) $$|a(x, t) - a(x_0, t_0)| \leqq C(|x - x_0|^\gamma + |t - t_0|^{\gamma/2}),$$

for some constants C, γ, $\gamma > 0$. (We shall often use the same letters C and c to denote different positive universal constants. C denotes a constant bounded above by $+\infty$, c denotes a constant bounded below by 0.) We also assume that $b(x, t)$ and $c(x, t)$ are uniformly bounded and measurable in x and t.

We are concerned with the following finite-difference approximation to this problem. We introduce a mesh

(2.6) $$\begin{aligned} &x_\nu = \nu h, \qquad \nu = -s, -s + 1, \cdots, 0, 1, \cdots, N, N + 1, \cdots, N + l, \\ &\qquad\qquad s, l, N \text{ are nonnegative integers, and } Nh = 1, \\ &t_n = nk, \qquad n = 1, 2, \cdots, T/k, \text{ with } \lambda = k/h^2 = \text{constant}, \end{aligned}$$

and solve

(2.7) $$\begin{aligned} &\sum_{j=-s}^{l} d_{-1,j}(x, t, h)E^j v(x, t + k, h) \\ &\qquad = \sum_{n=0}^{R} \left(\sum_{j=-s}^{l} d_{n,j}(x, t, h)E^j \right) v(x, t - nk, h) + kf(x, t). \end{aligned}$$

$Ev(\mu h) = v((\mu + 1)h)$, $x = \nu h$, and ν takes on all integer values between 0 or 1 and N or $N - 1$. For the Dirichlet problem, $\alpha_2 = 0$, we require that $\nu \geqq 1$, otherwise $\nu \geqq 0$. This is done in order to improve accuracy at the left boundary. Analogous statements are true near $x = 1$.

Also, $t = nk$, $R \leqq n \leqq T/k - 1$, and for $x = \nu h$, $v(x, 0, h) = u_0(x)$. The functions $v(x, k, h), \cdots, v(x, Rk, h)$ are given by some bounded compatible starting procedure as defined in Widlund [8]. Each finite-difference operator is of the form:

(2.8) $$\begin{aligned} \sum_{j=-s}^{l} d_{\mu,j}(x, t, h)E^j &= \alpha_\mu + \sum_{\sigma+k \geqq 2} B_{\mu,\sigma,k}(x, t)h^\sigma (hD)^k \\ &= \alpha_\mu + Q_\mu(x, t, hD, h). \end{aligned}$$

The α_μ are real constants, with $\alpha_{-1} = 1$, and hD is one of the operators hD_0, hD_+, hD_-, with

$$hD_\pm v = \pm(v(x \pm h) - v(x)), \qquad hD_0 = \tfrac{1}{2}(hD_+ + hD_-).$$

The elements of these matrices are supposed to fulfill the same conditions as the coefficients of the differential equation. Thus, we assume that they are bounded and measurable and that those with $\sigma = 0$ obey a condition like (2.5). We define $Q_\mu^{(1)}$, the principal part of Q_μ, to be the sum of those terms of Q_μ for which $\sigma = 0$.

In general, we must define the functions $v(x, t, h)$ outside the region $0 \leqq x \leqq 1$. We assume that $f(x, t)$ and $u_0(x)$ can be extended smoothly to the region $-\epsilon \leqq x \leqq 1 + \epsilon$, for some $\epsilon > 0$, and that each $v(x, t, h)$ satisfies the boundary conditions in (2.11) below.

Consider the equation

$$z^{R+1} - \alpha_0 z^R - \alpha_1 z^{R-1} - \cdots - \alpha_R = 0. \tag{2.9}$$

Denote the roots to this equation which lie on the unit circle by $e^{i\varphi_k}$, $k = 1, 2, \cdots, k_0$. Then we assume that for all t, $0 \leqq t \leqq T$, and all φ_k,

$$\begin{aligned} &\sum_{\mu=0}^{R} e^{-i\mu\varphi_k} d_{\mu,-s}(0, t, 0) \neq d_{-1,s}(0, t, 0)e^{+i\varphi_k}, \\ &\sum_{\mu=0}^{R} e^{-i\mu\varphi_k} d_{\mu,l}(0, t, 0) \neq d_{-1,l}(0, t, 0)e^{+i\varphi_k}. \end{aligned} \tag{2.10}$$

It is clear that we need additional conditions to specify v completely; this we do as follows:

(2.11) If $\alpha_2 = 0$, then we have

$$\begin{aligned} v(ph, nk, h) &- \sum_{j=1}^{q_0} b_{p,j}^{(0)}(h)v(jh, nk, h) \\ &= \left[1 - \sum_{j=1}^{q_0} b_{p,j}^{(0)}(h)\right] g_0(nk), \qquad p = 0, -1, \cdots, -s+1. \end{aligned}$$

If $\alpha_2 \neq 0$, we then have

$$\begin{aligned} v(ph, nk, h) &- \sum_{j=0}^{q_0} b_{p,j}^{(0)}(h)v(jh, nk, h) \\ &= \frac{h}{\alpha_2} g_0(nk)\left[p - \sum_{j=0}^{q_0} b_{p,j}^{(0)}(h)j\right], \qquad p = -1, -2, \cdots, -s. \end{aligned}$$

The conditions on the right boundary $x = 1$ are exactly analogous.

$$\begin{aligned} (\beta_2 = 0) \qquad v(1 + ph, nk, h) &- \sum_{j=1}^{q_1} b_{p,j}^{(1)}(h)v(1 - jh, nk, h) \\ &= \left[1 - \sum_{j=1}^{q_1} b_{p,j}^{(1)}(h)\right] g_1(nk), \qquad p = 0, 1, 2, \cdots, l-1. \end{aligned}$$

$$\begin{aligned} (\beta_2 \neq 0) \qquad v(1 + ph, nk, h) &- \sum_{j=0}^{q_1} b_{p,j}^{(1)}(h)v(1 - jh, nk, h) \\ &= \frac{h}{\beta_2}\left(p + \sum_{j=0}^{q_1} b_{p,j}^{(1)}(h)j\right) g_1(nk), \qquad p = 1, 2, \cdots, l. \end{aligned}$$

The following notations will be used:

$$(2.12)\qquad \begin{aligned} |||v(\cdot, nk, h)||| &= \sup_{\nu} |v(\nu h, nk, h)|, \\ |||v(\cdot, \cdot, h)||| &= \sup_{\nu, h} |v(\nu h, nk, h)|. \end{aligned}$$

Definition 2.1. The difference equations (2.7), (2.11) are invertible if, for arbitrary right-hand sides in all three equations, there exists a unique $v(x, t+k, h)$ satisfying the equations. Moreover, $||v(\cdot, t+k, h)||$ is bounded by a constant times the norm of the right-hand side. This is equivalent to the statement that the difference operator $I + Q_{-1}(x, t, hD, h)$ is uniformly invertible on the space of vectors obeying the boundary conditions in (2.11).

We shall obtain necessary and sufficient conditions for the difference approximation to be invertible, and sufficient conditions for certain estimates on the solutions to be valid. We think that these estimates are sharp.

3. **Statement of the Main Results.** We make the following assumptions about the difference approximation:

(a) The difference approximation (2.7) is consistent with the differential equation (2.1).

(b) If we construct the matrices $\hat{Q}_\mu^{(1)}(y, t, \xi)$ by replacing hD_0, $hD_\pm$, by $i \sin \xi$ and $(2i \sin \xi/2)e^{\pm i\xi/2}$ in $Q_\mu^{(1)}(y, t, hD)$, then the roots χ_i, $1 \leqq i \leqq R+1$, of

$$(3.1)\qquad \begin{aligned} \chi_i^{R+1}(1 + \hat{Q}_{-1}^{(1)}(y, t, \xi)) - \chi_i^R(a_0 + \hat{Q}_0^{(1)}(y, t, \xi)) \\ - \cdots - (a_R + \hat{Q}_R^{(1)}(y, t, \xi)) = 0 \end{aligned}$$

satisfy $|\chi_i| \leqq 1 - c|\xi|^2$, $-\pi < \xi \leqq \pi$, i.e. (2.7) is a parabolic difference scheme as defined in [8]. We might also equivalently define $\hat{Q}_\mu^{(1)}(y, t, wh)$ as $e^{-iwx}Q_\mu^{(1)}(y, t, hD)e^{iwx}$.

(c) The matrix

$$\begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_R \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}$$

has a simple eigenvalue equal to one, all its other eigenvalues lie on or inside the unit circle, and all its eigenvalues $e^{i\varphi_k}$ on the unit circle are simple. (The Dahlquist root condition, which is necessary for stability, is valid.)

(d) $[1 + Q_{-1}^{(1)}(y, t, \xi)] \neq 0$ for any y, t, ξ.

(e) The change in argument of $I + Q_{-1}^{(1)}(y, t, \xi)$ as ξ goes from $-\pi$ to π is zero for all y, t.

(f) The functions $b_{\mu j}^{(i)}(h)$ are C^2 in a neighborhood of 0 for $h \geqq 0$ if $\alpha_2 \neq 0$, C^1 if $\alpha_2 = 0$. Moreover, the boundary conditions (2.11) are consistent with (2.4). That is, if $\alpha_2 = 0$, then $1 - \sum_{j=1}^{q_0} b_{\mu j}^{(0)}(0) \neq 0$, and $\mu - \sum_{j=1}^{q_0} b_{\mu j}^{(0)}(0)j = 0$, for $\mu = 0, -1, \cdots, -s+1$. If $\alpha_2 \neq 0$, then for $\mu = -1, -2, \cdots, -s$.

$$\frac{1 - \sum_{j=0}^{q_0} b_{\mu j}^{(0)}(h)}{\mu - \sum_{j=0}^{q_0} b_{\mu j}^{(0)}(h)j} = \frac{\alpha_1}{\alpha_2} h + O(h^2),$$

and $\mu \neq \sum_{j=0}^{q_0} b_{\mu j}^{(0)}(0)j$.

Analogous conditions hold at $x = 1$.

Before listing the remaining assumptions, we shall consider two related problems. We define the right-half problem to be Eq. (2.1) in the region $0 \leq x < \infty$ with the same boundary conditions at zero, and the functions a, b, c, f, and u_0 extended to plus infinity, keeping all the smoothness and boundedness properties. We take the difference approximation to be (2.7) extended smoothly to the region $0 \leq x < \infty$ with the same boundary conditions at zero. The left-half problem is the complete analogue in the region $-\infty < x \leq 1$.

(g) The set of equations

$$\tag{3.2} \begin{aligned} &\sum_{j=-s}^{l} d_{-1,j}(0, t, 0)E^j v(\nu h) = 0, \qquad \nu = a, a+1, \cdots, \\ &\sum_{j=a}^{q_0} b_{\mu,j}^{(0)}(0)E^j v(0) = v(\mu h), \qquad \mu = a-1, a-2, \cdots, a-s, \end{aligned}$$

($a = 0$ or 1 depending on the problem), for all t, $0 \leq t \leq T$, has no nontrivial solution $\{v(\nu h)\}_{\nu=a-s}^{\infty}$ satisfying $\sum_{\nu=-s}^{\infty} |v(\nu h)|^2 < \infty$.

(h) The analogous statement is true for the "frozen" left-half space problem:

$$\tag{3.3} \begin{aligned} &\sum_{j=-s}^{l} d_{-1,j}(1, t, 0)E^j v(\nu h) = 0, \qquad \nu = N+a, N-1+a, \cdots, \\ &v(1+ph) = \sum_{j=a}^{q_1} b_{pj}^{(1)}(0)v(1-jh), \qquad p = a+1, a+2, \cdots, a+l, \end{aligned}$$

$a = 0$ or -1, depending on the problem.

The following two conditions involve checking for normal modes. These are the crucial stability requirements.

(i) Consider the set of equations

$$\sum_{j=-s}^{l} \left[z d_{-1,j}(0, t, 0) - d_{0,j}(0, t, 0) - \frac{d_{1,j}(0, t, 0)}{z} - \cdots - \frac{d_{R,j}(0, t, 0)}{z^R} \right] E^j \chi(\nu h) = 0,$$

$$\chi(ph) - \sum_{j=a}^{q_0} b_{p,j}^{(0)}(0)\chi(jh) = 0,$$

$$\nu = a, a+1, \cdots, \quad p = a-1, \cdots, a-s, \quad \text{for all } t.$$

We assume that for $|z| \geq 1$, $z \neq e^{i\varphi_k}$, the unique bounded solution is zero. If $\alpha_2 = 0$ and $z = e^{i\varphi_k}$, we make the same assumption. (Recall: The numbers $e^{i\varphi_k}$ are those eigenvalues of the matrix in (c) which lie on the unit circle.) We next assume that if $\alpha_2 \neq 0$, then for each $z = e^{i\varphi_k}$, $\chi(\nu h) \equiv 1$ is a solution, but there exists no other linearly independent solution having the property $|\psi(\nu h)| \leq c_1\nu + c_2$ for some positive constants c_1, c_2.

(j) The analogous statement is true for the left-half space problem.

As a crude example of what these conditions mean, we consider an approximation to the heat equation with zero boundary data in the right-half space:

$$(3.4)\quad v_j^{n+1} = v_j^n + \lambda(v_{j+1}^n - 2v_j^n + v_{j-1}^n), \quad j = 1, 2, \cdots, \quad 0 < \lambda < \tfrac{1}{2}, \quad v_0 = 2v_1.$$

The function $v_j^n = (1 + \lambda/2)^n 2^{-j}$ satisfies these equations, and hence no stability estimate can be obtained.

We may now state our main theorems.

MAIN THEOREM I. *Under the assumed smoothness hypothesis, the invertibility conditions* (d), (e), (g) *and* (h) *are necessary and sufficient for the invertibility of the two-point boundary value problem. Moreover, conditions* (d), (e) *and either* (g) *or* (h) *are necessary and sufficient for the invertibility of the right-half and left-half space problems, respectively.*

MAIN THEOREM II. *All these assumptions imply the validity of the stability estimate below, for the two-point boundary value problem. Moreover, the right-half and left-half space solutions obey the relevant estimates under the relevant assumptions.*

We shall state our estimates in several parts. If the boundary functions $g_0(t)$ and/or $g_1(t)$ are identically zero, then we have

$$(3.5)\qquad ||(t+k)^{\tau/2} D^\tau v(\cdot, t, h)|| \leqq C(t\ |||f||| + ||u_0||), \qquad \tau = 0, 1.$$

(D is applied at these points x for which we may define Du without leaving the space mesh.)

Next, suppose f and u_0 are both identically zero. Consider the right-half plane problem. If $\alpha_2 \not\equiv 0$, we have an estimate, for $x \geqq 0$,

$$(3.6)\quad \begin{aligned} |v(x, t, h)| &\leqq \sup_{0 \leq s \leq t} |g_0(s)| \sqrt{t}\, C \max\, [e^{-cx/h}, e^{-cx^2/t}]\,, \\ |Dv(x, t, h)| &\leqq \sup_{0 \leq s \leq t} |g_0(s)|\, C \max \left[\ln \frac{[x+h]}{h} e^{-cx/h}, e^{-cx/\sqrt{t}}, -\ln\left(\frac{cx^2}{t}\right) e^{-cx^2/t} \right]. \end{aligned}$$

Next, if $\alpha_2 = 0$, we have the weaker estimate, for $x \geqq h$,

$$(3.7)\quad \begin{aligned} |v(x, t, h)| \leqq{} & C\,|g_0(0)| + |g_0(t)| + Ct \sup_{0 \leq s \leq t} |g_0(s)| \\ & + C \sup_k \sum_{s=Rk}^{(t-k)/k} \left| g_0(s+k) - \sum_{\mu=0}^{R} a_\mu g_0(s - \mu k) \right|, \\ |(t+k)^{1/2} Dv(x, t, h)| \leqq{} & Ct \sup_{0 \leq s \leq t} |g_0(s)| + C\,|g_0(0)| \\ & + C \sup_k \left(\sum_{s=Rk}^{(t-k)/k} \left| \frac{g_0(s+k) - \sum_{\mu=0}^{R} a_\mu g_0(s-\mu k)}{(t-s)^{1/2}} \right| (t+k)^{1/2} \right). \end{aligned}$$

This estimate is weaker than (3.6) in that it involves the variation of g_0 rather than its maximum norm.

We have completely analogous estimates for the left-half plane and two-point boundary value problems.

Finally, we may treat the general case by letting $v(x, t, h) = v_1(x, t, h) + v_2(x, t, h)$, where v_1 satisfies the problem with homogeneous boundary conditions, and v_2 satisfies the homogeneous equation with zero initial data. We may then obtain the appropriate estimates for v and its first divided differences by adding (3.5) and one of (3.6) and using the triangle inequality.

4. Invertibility. We shall consider the right-half plane problem first. If all the $b_{\mu i}^{(0)}(h) \equiv 0$, then the work of Strang [6] guarantees the uniform invertibility for $|h|$ sufficiently small, in view of conditions (d) and (e). (Strang assumed Lipschitz continuity in x, but his proof works with our weaker smoothness assumptions.) In our more general case, we may view the difference operator plus boundary conditions as a finite-dimensional perturbation of the invertible operator T_0 of Strang. We call this operator $T_0 + S_0$, as in Osher [4]. A necessary and sufficient condition that this operator be invertible is that the finite-dimensional operator $(I + S_0T_0^{-1})$ acting on the range of S_0 be uniformly invertible as $h \to 0$. However, according to Strang's construction of T_0^{-1}, $S_0T_0^{-1}$ acting on the range of S_0 differs from the frozen coefficient operator for $x_0 = h = 0$ by terms of order h^γ. Thus, assumption (g) and the analysis in [4] imply invertibility of the right-half problem, for $|h|$ sufficiently small.

Conversely, suppose the operator is invertible. If (d) is violated for some $x_0 \neq 0$, or at ∞, then we may use Strang's argument [6] to show that the adjoint of $(T_0 + S_0)$ fails to be invertible on l_p, $1 \leqq p < \infty$, and thus, in fact, so does $(T_0 + S_0)$ on l_∞. (The boundary conditions play no role in Strang's proof if $x_0 \neq 0$.) If $x_0 = 0$, then we need only consider points centered around $x_1 = Ah + \delta$, $\delta > 0$, A is a positive fixed integer independent of h. The argument then follows in the same fashion. Thus (d) is necessary. If the index in (e) is not zero, then we may use Strang's arguments and the theory of Töplitz operators to show that T_0, and hence $T_0 + S_0$, is a completely continuous perturbation of an operator with a nonzero index, and is hence not invertible. If (g) is violated, then we let

$$v = [v((-s + a)h), \cdots, v(0), v(h), \cdots] \tag{4.1}$$

be a solution of (3.2) with l_∞ norm one. Consider

$$v_{\delta,h} = [v((-s + a)h), \cdots, v(0), \cdots, v(nh), 0, \cdots, 0, \cdots], \qquad nh = \delta, \tag{4.2}$$

as $h \to 0$, it is clear that $||v_{\delta,h}|| \to 1$, while $||(T_0 + S_0)v_{\delta,h}|| \leqq C\delta^\alpha$.

We have thus shown that conditions (d), (e) and (g) are necessary and sufficient for the right-half plane problem to be invertible. A similar statement follows for (d), (e), (h) and left-half invertibility.

We next consider the two-point problem. If each of the $b_{\mu i}^{(0)}(h) \equiv 0$, then Strang's "twisted factorization" in [7] and conditions (d) and (e) guarantee uniform invertibility. Again, we may write the total operator as $T_0 + S_0$, where S_0 is the finite-dimensional boundary perturbation. We need only verify that, if (g) and (h) are valid, then $(I + S_0T_0^{-1})$, acting on the range of S_0, is uniformly invertible as $h \downarrow 0$. This follows because of the nature of the "twisted factorization", and because the influence of each boundary on the other decays like $C\mu^{1/h}$, where μ is some fixed nonnegative constant less than one.

Conversely, suppose the two-point problem is indeed invertible. The previous arguments which involved condition (d) are still valid. Suppose (e) is violated and the index is positive. Then the right-half plane operator also has positive index, and hence has an approximate null vector:

(4.3) $$[v((-s+a)h), \cdots, v(0), \cdots, v(nh), 0, \cdots, 0]_{nh-\delta<1/2}$$

and hence, as $h \to 0$, the two-point operator is not uniformly invertible. Similarly, if the index is negative, the left-half plane problem has an approximate null vector beginning at $x = 1 + (l + a)h$ and working backwards. Thus, (e) is necessary. If condition (g) or (h) is violated, the arguments which worked for the half-plane problems are still valid.

5. Constant Coefficients. Construction of a Parametrix. We analyze the right-half plane problem first. Consider the equation on the right-half mesh:

(5.1) $$(I + Q_{-1}^{(1)}(y, t, hD))v(x, t+k, h) = \sum_{\mu=0}^{R} (a_\mu I + Q_\mu^{(1)}(y, t, hD))v(x, t-\mu k, h)$$

with homogeneous boundary conditions (2.11). Following the usual procedure, we transform the equations into one-step formulas. Introduce the $(R + 1)$ component vector:

(5.2) $$\tilde{v}(x, t, h) = \{v(x, t, h), \cdots, v(x, t - Rk, h)\}^T$$

(T denotes transpose). Then the homogeneous equations (2.7) and (2.10) transform to

(5.3) $$\tilde{v}(x, t+k, h) = \tilde{Q}(x, t, hD, h)\tilde{v}(x, t, h),$$

where

(5.4) $$\tilde{Q} = \begin{pmatrix} (I+Q_{-1})^{-1}(a_0+Q_0) & & \cdots & & (I+Q_{-1})^{-1}(a_R+Q_R) \\ 1 & 0 & & & 0 \\ \vdots & & \ddots & & \vdots \\ & & & & 0 \\ 0 & & 0 & 1 & 0 \end{pmatrix}.$$

Each component of $\tilde{v}$ obeys the boundary conditions at $x = 0$. By $(I + Q_{-1})^{-1}$, we mean the inverse acting on the space of those $\tilde{v}$ which obey the homogeneous conditions.

Definition 5.1. The Green's function for the right-half problem is an $(R + 1)$ square matrix $\Gamma(x, qk, x_0, pk)$ defined on the space mesh in x and x_0 and the time mesh for $Rk \leqq pk \leqq qk \leqq T$, satisfying

(5.5) $$\Gamma(x, pk, x_0, pk) = \delta(x, x_0)I,$$

(5.6) $$\tilde{Q}\Gamma(x, qk, x_0, pk) = \Gamma(x, (q+1)k, x_0, pk),$$

(5.7) $$\Gamma(\mu h, qk, x_0, pk) = \sum_{j=a}^{q_0} b_{\mu j}^{(0)}(h)\Gamma(jh, qk, x_0, pk),$$

for μh at the boundary points.

The Green's functions for the left-half and two-point problems are defined analogously.

Most of the remainder of this work will be devoted to obtaining appropriate estimates for this function.

We begin by considering the equation:

$$(\tilde{Q}^{(1)}(x_0, t, hD) - z)\eta(x, x_0, t) = \delta(x, x_0)I \tag{5.8}$$

for $|z| > 1$.

We wish to solve this for fixed $x_0 = \nu h$ and all $x = \nu h$, $\nu = 0, \pm 1, \pm 2, \cdots$; the solution is required to be bounded for all x, but no other boundary conditions are imposed anywhere. Let the element in the ith row and the jth column of $\eta(x, x_0, t)$ be denoted by $\eta_{ij}(x, x_0, t)$. Then (5.8) implies that if $x \neq x_0$, then

$$\eta_{ij}(x, x_0, t) = \frac{1}{z^{i-1}} \eta_{1j}(x, x_0, t).$$

Also, if $x = x_0$, then, if $i \geqq j \neq 1$, $\eta_{ij} = \eta_{1j}/z^{i-1} - 1/z^{i-j+1}$; if $i < j$ or $j = 1$ then $\eta_{ij} = \eta_{1j}/z^{i-1}$. Then (5.8) reduces to

$$\begin{aligned}(1 + Q_{-1}^{(1)})^{-1}\Bigg[[(a_0 + Q_0^{(1)}) - z(I + Q_{-1}^{(1)})] + \frac{(a_1 + Q_1^{(1)})}{z} \\ + \cdots + \frac{(a_R + Q_R^{(1)})}{z^R}\Bigg]\eta_{1j}(x, x_0, t) \\ = \delta(x, x_0), \quad \text{if } j = 1, \\ = \delta(x, x_0)(I + Q_{-1}^{(1)})^{-1}\left(\sum_{i=j-1}^{R} \frac{(a_i + Q_i^{(1)})}{z^{i-j+2}}\right), \quad \text{if } j \geqq 2.\end{aligned} \tag{5.9}$$

For our purposes, it will suffice to let the right side above be $[(I + Q_{-1}^{(1)})^{-1}/z^R]\delta(x, x_0)$. We call $\eta_{1j}(x, x_0, t) = \zeta(x, x_0, t)$. We must now solve

$$z^R\left[(I - Q_{-1}^{(1)})z - \sum_{j=0}^{R} \frac{(a_j + Q_j^{(1)})}{z^j}\right]\zeta(x, x_0, t) = \delta(x, x_0), \qquad |z| > 1. \tag{5.10}$$

This is easily done with the help of a Fourier transform. We have

$$\zeta(x, x_0, t) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\exp i\xi((x - x_0)/h)\, d\xi}{z^R\left[(I + \hat{Q}_{-1}^{(1)}(\xi))z - \sum_{j=0}^{R} \frac{(a_j + \hat{Q}_j^{(1)}(\xi))}{z^j}\right]}. \tag{5.11}$$

Next, we consider a homogeneous equation on the right-half space $x \geqq 0$ (or $x \geqq h$ if it is a Dirichlet problem):

$$[\tilde{Q}^{(1)}(0, t, hD) - zI]\Phi(x, t) = 0, \qquad |z| > 1. \tag{5.12}$$

We may again reduce the solution of this matrix equation to that of the scalar equation:

$$\left[(I + Q_{-1}^{(1)})z - \sum_{j=0}^{R} \frac{(a_j + Q_j^{(1)})}{z^j}\right]\chi(x, t) = 0. \tag{5.13}$$

For simplicity of notation, we shall assume we are not dealing with a Dirichlet problem, i.e. we have $x \geqq 0$ in the above. (The necessary modifications are simple to make it Dirichlet.) The general solution is a linear combination of s linearly independent solutions, which we may write

$$\chi(x, t) = \sum_{k=1}^{s} \frac{c_k'}{2\pi}\int_0^{2\pi} \frac{e^{i(x/h+k-s)\xi}}{\left[(I + \hat{Q}_{-1}^{(1)}(\xi))z - \sum_{j=0}^{R} \frac{(a_j + \hat{Q}_j^{(1)}(\xi))}{z^j}\right]}\, d\xi. \tag{5.14}$$

We may compute this integral using residue theory. For all but a finite number of z, we may write

$$\chi(x, t) = \sum_{j=1}^{s} c_j \tau_j^{x/h}, \tag{5.15}$$

where the τ_j are the s distinct roots of

$$0 = \sum_{j=-s}^{l} \left(z d_{-1,j}(0, t, 0)\tau^{j+s} - \sum_{\nu=0}^{R} z^{-\nu} d_{\nu,j}(0, t, 0)\tau^{j+s} \right) \tag{5.16}$$

which lie within the unit circle. See Varah [11] for a more thorough discussion of this matter.

Now we require that $\chi(x, t, z) + \zeta(x, x_0, t, z)$ obey the boundary conditions, or

$$\chi(\mu h) - \sum_{j=1}^{q_0} b_{\mu j}^{(0)}(h)\chi(jh) = -\zeta(\mu j) + \sum_{j=1}^{q_0} b_{\mu j}^{(0)}(h)\zeta(jh). \tag{5.17}$$

We may use (5.14) or (5.15) and (5.11) to see that this becomes an inhomogeneous linear system of s equations in s unknowns. We need the following important lemmas:

LEMMA 5.2. *The equation*

$$\sum_{j=-s}^{l} \left[z d_{-1,j}(x_0, t, 0)\tau^{j+s} - \sum_{\nu=0}^{R} z^{-\nu} d_{\nu,j}(x_0, t, 0)\tau^{j+s} \right] = 0 \tag{5.18}$$

in some neighborhood of $e^{i\varphi_k}$, $|z - e^{i\varphi_k}| < \epsilon$, $0 < \epsilon$, *has two roots* $\tau_1(z)$, $\tau_{s+1}(z)$, *which have the property that for* $|z| \geqq 1$, $z \neq e^{i\varphi_k}$, $|\tau_1(z)| < 1$, $|\tau_{s+1}(z)| > 1$, *and*

$$\begin{aligned} \tau_1(z) &= 1 - \left(\frac{z - e^{i\varphi_k}}{c_k}\right)^{1/2} + O(|z - e^{i\varphi_k}|), \\ \tau_{s+1}(z) &= 1 + \left(\frac{z - e^{i\varphi_k}}{c_k}\right)^{1/2} + O(|z - e^{i\varphi_k}|), \end{aligned} \tag{5.19}$$

where the real parts of $c_k e^{-i\varphi_k}$ *are bounded below by a positive constant. A branch cut is drawn along* $z = e^{i\varphi_k} + tc_k$ *from* $t = 0$ *to* $t = -\infty$, *and we choose that branch of the square root which is positive for positive values of the function.*

Proof. Parabolicity and condition (c) guarantee that for $|z| \geqq 1$, $z \neq e^{i\varphi_k}$, none of the roots τ_j lie on the unit circle. Condition (c) implies that for $z = e^{i\varphi_k}$, $\tau = 1$ is the only root on the unit circle. Consider the equation

$$\begin{aligned} z^{R+1}(I + Q_{-1}(\zeta)) - \sum_{j=0}^{R} (a_j + Q_j(\zeta))z^{R-j} = 0, \\ \zeta = \xi + i\eta \text{ near } \xi, \eta = 0, z = e^{i\varphi_k}. \end{aligned} \tag{5.20}$$

Introduce the variable $\sigma = e^{-i\varphi_k}z - 1$. The equation transforms into:

$$\begin{aligned} \sigma\left((R + 1)e^{i(R+1)\varphi_k} - \sum_{\mu=0}^{R} a_\mu(R - \mu)e^{i(R-\mu)\varphi_k}\right) + e^{i(R+1)\varphi_k}Q_{-1}(\zeta) \\ - \sum_{\mu=0}^{R} e^{i(R-\mu)\varphi_k}Q_\mu(\zeta) + O(\sigma^2 + |\zeta|^2 \sigma) = 0. \end{aligned} \tag{5.20'}$$

The second part of condition (c) guarantees that the coefficient of σ above is nonzero

Thus, by consistency, $|\sigma| = O(|\zeta|^2)$ when $\zeta \to 0$. The fact that σ is a simple root of (5.20′) for $\zeta = 0$ guarantees an expansion:

$$\sigma = \tilde{C}_2\zeta^2 + \tilde{C}_3\zeta^3 + O(\zeta^4), \quad \text{in a neighborhood of } \zeta = 0.$$

If $\zeta = \xi + i0$, then parabolicity guarantees $\operatorname{Re} \sigma \leqq -\frac{1}{2}c\, \xi^2$. This implies that $\operatorname{Re} \tilde{C}_2 < 0$. We may solve for ζ when σ is in a small neighborhood of 0 and obtain

$$\zeta_\pm = \pm\left(\frac{\sigma}{\tilde{C}_2}\right)^{1/2} + O(|\sigma|).$$

But

$$\tau = e^{i\zeta} = 1 \pm i\left(\frac{\sigma}{\tilde{C}_2}\right)^{1/2} + O(|\sigma|) = 1 \pm \left(\frac{z - e^{i\varphi_k}}{-\tilde{C}_2 e^{i\varphi_k}}\right)^{1/2} + O(|z - e^{i\varphi_k}|);$$

the result is now immediate.

Notice, we may easily show that if $e^{i\varphi_k} = 1$, then $c_1 = \lambda a(x_0, t)$.

LEMMA 5.3. *For any $\delta'' > 0$, $\delta'' > \delta_0''$ fixed, there exist positive numbers δ', C, $\rho < 1$, such that if, for each k, $|z - e^{i\varphi_k}| > \delta''$, $|z| \leqq 1 - \delta'$, then*

$$\begin{aligned} \zeta(x, x_0, t, z) &= K_{(\nu-\nu_0)}(t, z), \\ \psi(x, x_0, t, z, h) &= L_{(\nu+\nu_0)}(t, z, h), \end{aligned} \tag{5.21}$$

where both $K_{(\nu-\nu_0)}(t, z)$ and $L_{(\nu+\nu_0)}(t, z, h)$ are analytic in z in this region and

$$|K_{(\nu-\nu_0)}(t, z)| \leqq C\rho^{|\nu-\nu_0|}, \qquad |L_{(\nu+\nu_0)}(t, z, h)| \leqq C\rho^{\nu+\nu_0}.$$

Proof. If we perform the integration in (5.11) and (5.14) and keep in mind that the roots τ_i lie well inside or well outside the unit circle, then we have reduced the problem to proving that (5.17) has a unique solution in this region. This follows from assumption (i) and the smoothness of the $b_{\mu i}(h)$.

We now wish to analyze these functions in a region $\{|z - e^{i\varphi_k}| \leqq c, z \geqq 1\} = S_c^{(k)}$. We perform the integration in (5.1) and obtain

$$\begin{aligned} &\text{if } \nu \geqq \nu_0,\ \zeta(x, x_0) = \frac{\tau_1^{\nu-\nu_0}}{(\tau_1 - \tau_{s+1})} A_0(z) + B_{(\nu-\nu_0)}(z), \\ &\text{if } \nu \leqq \nu_0,\ \zeta(x, x_0) = \frac{\tau_{s+1}^{\nu-\nu_0}}{(\tau_1 - \tau_{s+1})} A_1(z) + C_{(\nu-\nu_0)}(z). \end{aligned} \tag{5.22}$$

$A_0(z)$ and $A_1(z)$ are analytic functions of $(z - e^{i\varphi_k})^{1/2}$ in $|z - e^{i\varphi_k}| < c$, $B_i(z)$ and $C_i(z)$ are analytic functions of z for $|z| \geqq 1 - c$, and

$$|C_i(z)| \leqq K\mu^{-i}, \qquad |B_i(z)| \leqq K\mu^i, \qquad K > 0,\ 0 \leqq \mu < 1;$$

and

$$\frac{A_0}{\tau_1 - \tau_{s+1}} + B_0 = \frac{A_1}{\tau_1 - \tau_{s+1}} + C_0.$$

We may use (5.14) to write

$$\chi(\nu h) = c_1\tau_1^\nu + \sum_{j=2}^{s} c_j p_j(\nu, z), \tag{5.23}$$

where again each of the $p_j(\nu, z)$ is analytic in z and decays exponentially in ν. Equation (5.17) now becomes an algebraic system:

$$BV = W, \qquad V = [c_1, \cdots, c_s]^T. \tag{5.24}$$

We shall now consider the non-Dirichlet case first. The element in the μth row, first column of B is

$$\begin{aligned} &\tau_1^{-\mu} - \sum_{j=0}^{q_0} b_{-\mu j}^{(0)}(h)\tau_1^j \\ &= (1 - \sum b_{-\mu j}^{(0)}(h)) - \left(-\mu - \sum b_{-\mu j}^{(0)}(h)j\right)\left(\frac{z - e^{i\varphi_k}}{c_k}\right)^{1/2} + O(|z - e^{i\varphi_k}|) \\ &= \left(\frac{\alpha_1 h}{\alpha_2} - \left(\frac{z - e^{i\varphi_k}}{c_k}\right)^{1/2}\right)\left(-\mu - \sum b_{-\mu j}^{(0)}(h)j\right) + O(|z - e^{i\varphi_k}| + h^2). \end{aligned} \tag{5.25}$$

If $B = \{B_{\mu j}(z, h)\}$, and $-\mu - \sum b_{-\mu j}(h)j = \tilde{c}_\mu(h)$, we claim that the matrix

$$\left[\begin{array}{c|c} \tilde{c}_1(0) & \\ \vdots & B_{\mu j}(e^{i\varphi_k}, 0) \\ \tilde{c}_s(0) & \end{array}\right] \tag{5.26}$$

has a nonvanishing determinant. If the determinant did vanish, then it is clear that the last statement in condition (i) would be violated. This, of course, implies that

$$\frac{[\det B(z, 0)]}{\prod_k (z - e^{i\varphi_k})^{1/2}} \neq 0 \quad \text{if } |z| \geqq 1.$$

We may then use Varah's argument to show that $[B(z, h)]^{-1}$ exists if $|z - e^{i\varphi_k}| \geqq ch^2$. Moreover,

$$\det B(z, h) = \left((z - e^{i\varphi_k})^{1/2} - c_k^{1/2}\frac{\alpha_1}{\alpha_2} h + O(h^2)\right) f_k(z, h) \tag{5.27}$$

with $f_k(z, h)$ and $f_k^{-1}(z, h)$ analytic functions of $(z - e^{i\varphi_k})^{1/2}$ near $z = e^{i\varphi_k}$.

Now we simplify W.

$$\begin{aligned} &\left[\tau_{s+1}^{-\mu} - \sum_{j=0}^{q_0} b_{-\mu j}^{(0)}(h)\tau_{s+1}^j\right] \\ &= 1 - \sum b_{-\mu j}^{(0)}(h) + \left(\frac{z - e^{i\varphi_k}}{c_k(x_0)}\right)^{1/2}\left[-\mu - \sum b_{-\mu j}^{(0)}(h)j\right] + O(|z - e^{i\varphi_k}|) \\ &= \left[\frac{\alpha_1}{\alpha_2} h + \left(\frac{z - e^{i\varphi_k}}{c_k(x_0)}\right)^{1/2}\right]\left[-\mu - \sum b_{-\mu j}^{(0)}(h)j\right] + O(|z - e^{i\varphi_k}| + h^2) \\ &= \frac{\frac{\alpha_1}{\alpha_2} h + \left(\frac{z - e^{i\varphi_k}}{c_k(x_0)}\right)^{1/2}}{\frac{\alpha_1}{\alpha_2} h - \left(\frac{z - e^{i\varphi_k}}{c_k(0)}\right)^{1/2}} B_{\mu 1}(z, h) \\ &\quad + \frac{\frac{\alpha_1}{\alpha_2} hO(|z - e^{i\varphi_k}| + h^2) + (z - e^{i\varphi_k})^{1/2} O(|z - e^{i\varphi_k}| + h^2)}{\frac{\alpha_1}{\alpha_2} h - \left(\frac{z - e^{i\varphi_k}}{c_k(0)}\right)^{1/2}}. \end{aligned} \tag{5.28}$$

Thus, $W = [w_{\nu_0}(1), \cdots, w_{\nu_0}(s)]^T$ has the property:

$$
\begin{aligned}
w_{\nu_0}(\mu) = &-\frac{A_1(z)}{(\tau_1 - \tau_{s+1})} \frac{\dfrac{\alpha_1}{\alpha_2} h + \left(\dfrac{z - e^{i\varphi k}}{c_k(x_0)}\right)^{1/2}}{\dfrac{\alpha_1}{\alpha_2} h - \left(\dfrac{z - e^{i\varphi k}}{c_k(0)}\right)^{1/2}} B_{\mu 1}(z, h)\tau_{s+1}^{-\nu_0} \\
&+ \frac{O(((|z - e^{i\varphi k}|)^{1/2} + h)^3)\tau_{s+1}^{-\nu_0}}{\left(\dfrac{\alpha_1}{\alpha_2} h - \left(\dfrac{z - e^{i\varphi k}}{c_k(0)}\right)^{1/2}\right)(z - e^{i\varphi k})^{1/2}} + M_{\nu_0}^{(0)}(z, h) + N_{\nu_0}^{(0)}(z, h),
\end{aligned}
\tag{5.29}
$$

where $N_{\nu_0}^{(i)}(x, h) = 0$ if $\nu_0 > q_0$ and is analytic as a function of $(z - e^{i\varphi k})^{1/2}$, $M_{\nu_0}^{(i)}(z, h)$ decays exponentially in ν_0 and is analytic for $|z| \geqq 1 - c$ near $e^{i\varphi k}$. Thus, if we solve (5.24) for the c_i, we have

$$
\begin{aligned}
c_1 = &-\tau_{s+1}^{-\nu_0} \frac{A_1(z)}{\tau_1 - \tau_{s+1}} \frac{\dfrac{\alpha_1}{\alpha_2} h + \left(\dfrac{z - e^{i\varphi k}}{c_k(x_0)}\right)^{1/2}}{\dfrac{\alpha_1}{\alpha_2} h - \left(\dfrac{z - e^{i\varphi k}}{c_k(0)}\right)^{1/2}} \\
&+ \frac{O(((|z - e^{i\varphi k}|)^{1/2} + h)^3)\tau_{s+1}^{-\nu_0}}{(z - e^{i\varphi k})^{1/2}\left(\dfrac{\alpha_1}{\alpha_2} h - \left(\dfrac{z - e^{i\varphi k}}{c_k(0)}\right)^{1/2}\right) \det B(z, h)} + \frac{\tilde{M}_{\nu_0}(z, h)}{\det B(z, h)},
\end{aligned}
\tag{5.31}
$$

$\tilde{M}_{\nu_0}(z, h)$ decays exponentially in ν_0 and is an analytic function of $(z - e^{i\varphi k})^{1/2}$. We notice that

$$
\begin{aligned}
&\left[\frac{\dfrac{\alpha_1}{\alpha_2} h - \left(\dfrac{z - e^{i\varphi k}}{c_k(0)}\right)^{1/2} + O((h + (|z - e^{i\varphi k}|)^{1/2})^2)}{\det B(z, h)}\right] \\
&\qquad\qquad = C + \frac{O((h + (|z - e^{i\varphi k}|)^{1/2})^2)}{\det B(z, h)}.
\end{aligned}
\tag{5.32}
$$

Thus, for $j = 2, \cdots, s$,

$$
\begin{aligned}
c_j = &\frac{O(((|z - e^{i\varphi k}|)^{1/2} + h)^3)}{\dfrac{\alpha_1}{\alpha_2} h - \left(\dfrac{z - e^{i\varphi k}}{c_k(0)}\right)^{1/2}} \frac{\tau_{s+1}^{-\nu_0}}{(z - e^{i\varphi k})^{1/2}} \left[1 + \frac{O((h + (|z - e^{i\varphi k}|)^{1/2})^2)}{\det B(z, h)}\right] \\
&+ [M_{\nu_0}^{(k)}(z, h) + N_{\nu_0}^{(k)}(z, h)]\left(1 + \frac{O((h + (|z - e^{i\varphi k}|)^{1/2})^2)}{\det B(z, h)}\right).
\end{aligned}
\tag{5.33}
$$

Next, we consider the Dirichlet case, $\alpha_2 = 0$. We then have

$$
\tau_1^{-\mu} - \sum_{j=1}^{q_0} b_{-\mu j}^{(0)}(h)\tau_1^j = 1 - \sum b_{-\mu j}^{(0)}(h) + O(((|z - e^{i\varphi k}|)^{1/2} + h)^{\cdot\cdot}) \tag{5.25$'$}
$$

Also, if $1 - \sum b_{-\mu j}^{(0)}(h) = d_{\mu+1}(h)$, then the matrix

$$
\left[\begin{array}{c|c} d_1(0) & \\ \vdots & B_{\mu j}(e^{i\varphi k}, 0) \\ d_s(0) & \end{array}\right]
\tag{5.26$'$}
$$

has a nonvanishing determinant. If the determinant did vanish, then condition (i) would be violated for $z = e^{i\varphi_k}$. Thus, we may write

$$(5.27') \qquad B^{-1}(z, h) = [B^{-1}_{ij}(z, h)]$$

with the property:

$$(5.28') \qquad B^{-1}_{ij}(z, h) = g_{ij}(e^{i\varphi_k}, 0) + O(((|z - e^{i\varphi_k}|)^{1/2} + h)^2).$$

We have

$$(5.29') \qquad \tau^{-\mu}_{s+1} - \sum_{j=1}^{q_0} b^{(0)}_{\mu j}(h)\tau^{j}_{j+1} = B_{\mu 1}(z, h) + O(((|z - e^{i\varphi_k}|)^{1/2} + h)^2),$$

thus

$$(5.30') \qquad w_{\nu_0}(\mu) = -\tau^{-\nu_0}_{s+1}\frac{B_{\mu 1}(z, h)A_1(z)}{\tau_1 - \tau_{s+1}} + \frac{O(((|z - e^{i\varphi_k}|)^{1/2} + h)^2)\tau^{-\nu_0}_{s+1}}{(z - e^{i\varphi_k})^{1/2}} + M^{(0)}_{\nu_0}(z, h) + N^{(0)}_{\nu_0}(z, h),$$

and

$$(5.31') \qquad c_1 = -\frac{A_1(z)\tau^{-\nu_0}_{s+1}}{\tau_1 - \tau_{s+1}} + \frac{O(((|z - e^{i\varphi_k}|)^{1/2} + h)^2)\tau^{-\nu_0}_{s+1}}{(z - e^{i\varphi_k})^{1/2}} + [M^{(1)}_{\nu_0}(z, h) + N^{(1)}_{\nu_0}(z, h)][1 + O(((|z - e^{i\varphi_k}|)^{1/2} + h)^2)],$$

while, for $j = 2, \cdots, s$,

$$(5.32') \qquad c_j = \frac{O(((|z - e^{i\varphi_k}|)^{1/2} + h)^2)\tau^{-\nu_0}_{s+1}}{(z - e^{i\varphi_k})^{1/2}} + [M^{(k)}_{\nu_0}(z, h) + N^{(k)}_{\nu_0}(z, h)][1 + O(((|z - e^{i\varphi_k}|)^{1/2} + h)^2)].$$

Now, we have

$$\alpha_2 \neq 0$$

$$(5.34) \qquad \begin{aligned}\chi(\nu h, \nu_0 h, z) = \tau_1^{\nu}\Bigg[&\frac{-\tau^{-\nu_0}_{s+1}}{\tau_1 - \tau_{s+1}}A_1(z)\frac{\dfrac{\alpha_1}{\alpha_2}h + \left(\dfrac{z - e^{i\varphi_k}}{c_k(x_0)}\right)^{1/2}}{\dfrac{\alpha_1}{\alpha_2}h - \left(\dfrac{z - e^{i\varphi_k}}{c_k(0)}\right)^{1/2}} \\ &+ \frac{O(((|z - e^{i\varphi_k}|)^{1/2} + h)^3)\tau^{-\nu_0}_{s+1}}{(z - e^{i\varphi_k})^{1/2}\left(\dfrac{\alpha_1}{\alpha_2}h - \left(\dfrac{z - e^{i\varphi_k}}{c_k(0)}\right)^{1/2}\right)\det B(z, h)} + \frac{\tilde{M}_{\nu_0}(z, h)}{\det B(z, h)}\Bigg] \\ &+ \sum_{j=2}^{s} p_j(\nu, z)\Bigg[\frac{O(((|z - e^{i\varphi_k}|)^{1/2} + h)^3}{\dfrac{\alpha_1}{\alpha_2}h - \left(\dfrac{z - e^{i\varphi_k}}{c_k(0)}\right)^{1/2}}\frac{\tau^{-\nu_0}_{s+1}}{(z - e^{i\varphi_k})^{1/2}} \\ &\cdot\left(1 + \frac{O((h + (|z - e^{i\varphi_k}|)^{1/2})^2)}{\det B(z, h)}\right) \\ &+ [M^{(j)}_{\nu_0}(z, h) + N^{(j)}_{\nu_0}(z, h)]\left(1 + \frac{O((h + (|z - e^{i\varphi_k}|)^{1/2})^2)}{\det B(z, h)}\right)\Bigg],\end{aligned}$$

$$\alpha_2 = 0$$

$$\begin{aligned} \chi(\nu h, \nu_0 h, z) = \tau_1^\nu \Big[& -A_1(z) \frac{\tau_{s+1}^{-\nu_0}}{\tau_1 - \tau_{s+1}} + \frac{O(((|z - e^{i\varphi_k}|)^{1/2} + h)^2)\tau_{s+1}^{-\nu_0}}{(z - e^{i\varphi_k})^{1/2}} \\ & + [M_{\nu_0}^{(1)}(z, h) + N_{\nu_0}^{(1)}(z, h)] \\ & \cdot [1 + O(((|z - e^{i\varphi_k}|)^{1/2} + h)^2)] \Big] \\ & + \sum_{i=2}^{s} \Big[\Big(\frac{O(((|z - e^{i\varphi_k}|)^{1/2} + h)^2)\tau_{s+1}^{-\nu_0}}{(z - e^{i\varphi_k})^{1/2}} \\ & + [M_{\nu_0}^{(i)}(z, h) + N_{\nu_0}^{(i)}(z, h)] \\ & \cdot [1 + O(((|z - e^{i\varphi_k}|)^{1/2} + h)^2)] \Big) p_i(\nu, z) \Big]. \end{aligned} \tag{5.34'}$$

We thus have expressions for $\zeta(\nu h, x_0, z)$ and $\chi(\nu h, x_0, z)$.

Definition 5.4.

$$\begin{aligned} G(x, qk, x_0, pk) &= \frac{-1}{2\pi i} \oint z^{q-p}\, dz [\Phi(x, x_0, pk, h, z) + \eta(x, x_0, pk, h, z)] \\ &= G_R(x, qk, x_0, pk) + G_i(x, qk, x_0, pk), \end{aligned} \tag{5.35}$$

where the integration is around some circle $|z| = 1 + c$.

We shall use this $G(x, qk, x_0, pk)$ as a parametrix for $\Gamma(x, qk, x_0, pk)$.

6. Construction of the Green's Function for the Half Plane. We begin by obtaining an estimate analogous to Theorem (3.1) of Widlund [8].

THEOREM 6.1. *There exist positive constants c such that, for $\tau = 0, 1, 2$ and for all $pk, qk \in [0, T]$, $p \leqq q$, and all x, x_0 on the mesh for which D_x^τ makes sense,*

$$\begin{aligned} &|D^\tau G_i(x, qk, x_0, pk)| \\ &\quad \leqq ch^{-\tau}(q - p + 1)^{-((1+\tau)/2)} \exp [C\beta^2(q - p + 1)h^2 - (x - x_0)\beta], \end{aligned} \tag{6.1}$$

and

$$\begin{aligned} &|D^\tau G_R(x, qk, x_0, pk)| \\ &\quad \leqq Ch^{-\tau}(q - p + 1)^{-((1+\tau)/2)} \exp [C\beta^2(q - p + 1)h^2 - (x + x_0)\beta] \end{aligned} \tag{6.2}$$

for all real β, $0 \leqq \beta \leqq C/h$.

Proof. We may use (5.9) and the analysis which precedes it, in order to reduce the problem to that with Φ and η replaced by χ and ζ, respectively, in (5.35). We deform the path of integration into the path

$$|z| = e^{-\delta_k^0} \quad \text{if } |\arg(ze^{-i\varphi_k})| \geqq \frac{\delta_k^0 + \beta^2 h^2}{C_k}, \qquad \delta_k^0 > 0. \tag{6.3}$$

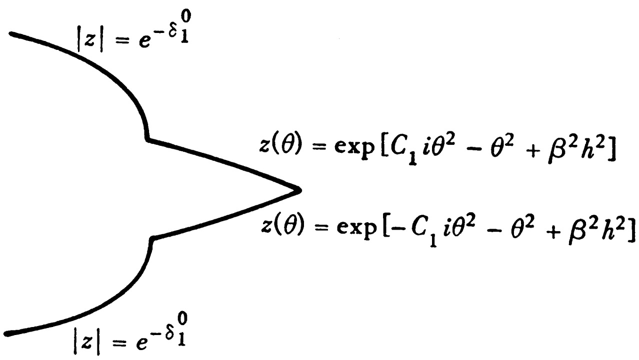
($C_k > 0$ and $\beta^2 h^2$ will be defined below.)

$$z \exp [-i\varphi_k] = \exp [C_k i\theta^2 - \theta^2 + \beta^2 h^2], \qquad 0 \leqq \theta \leqq \left(\frac{\delta_k^0 + \beta^2 h^2}{C_k}\right)^{1/2} = \theta_k,$$

$$z \exp[-i\varphi_k] = \exp[-C_k i\theta^2 - \theta^2 + \beta^2 h^2], \qquad 0 \leqq \theta \leqq \left(\frac{\delta_k^0 + \beta^2 h^2}{C_k}\right)^{1/2} = \theta_k.$$

Thus, we require $\beta^2 h^2 \leqq \min((\pi - \delta_k^0)C_k, |\varphi_k - \varphi_{k+1}|/2)$.

We illustrate what the path looks like near $z = 1 = e^{i\varphi_1}$.



We may use the expression (5.21) to analyze the part of the integral taken over Γ_0, the circumference of the circles of radius $\exp[-\delta_k^0]$. We can use the estimates in (5.21) to show that

$$\left|\int_{\Gamma_0} [D^\tau K_{(\nu-\nu_0)}(t, z)z^n]\, dz\right| \leqq h^{-\tau} e^{-n\delta^0} C\rho^{|\nu-\nu_0|} \tag{6.4}$$

and

$$\left|\int_{\Gamma_0} D^\tau L_{(\nu+\nu_0)}(t, z)z^n\, dz\right| \leqq h^{-\tau} e^{-n\delta^0} C\rho^{\nu+\nu_0}$$

for $\delta^0 = \min \delta_k^0$. Thus, $(\beta h)^2 \leqq -\ln \rho(\delta^0)$.

Thus, we need only consider the integral on the remaining paths Γ_k

LEMMA 6.2. *Along the paths Γ_k, we have*

$$\max(|\tau_1|, |\tau_{s+1}^{-1}|) \leqq e^{-\beta h C/2}, \quad C \text{ depends only on } \varphi_k. \tag{6.5}$$

Proof. By Lemma 5.2, we have

$$\begin{aligned} \tau_1 &= 1 - \left(\frac{e^{-i\varphi_k}z - 1}{e^{-i\varphi_k}C_k}\right)^{1/2} + O(|z - e^{i\varphi_k}|) \\ &= 1 - \left(\frac{C_k' i\theta^2 - \theta^2 + \beta^2 h^2}{c_k e^{-i\varphi_k}}\right)^{1/2} + O((|\theta| + \beta h)^2). \end{aligned} \tag{6.6}$$

We can easily see that the absolute value of the quantity whose square root appears is bounded below by $\beta^2 h^2 C_k'(((C_k')^2 + 1)^{1/2} |c_k|)^{-1}$. We need only require that the argument of this quantity be bounded away from any multiple of π, but

$$0 \leqq \arg(C_k' i\theta^2 - \theta^2 + \beta^2 h^2) < \tan^{-1}(-C_k'),$$

and

$$-\frac{\pi}{2} < \arg\left(\frac{1}{c_k e^{-i\varphi_k}}\right) < \frac{\pi}{2}.$$

Choose C_k' so large so that $\tan^{-1}(-C_k') + \arg(e^{+i\varphi_k}/c_k) < \pi$. The result then follows for $|\tau_1|$, and, similarly for $|\tau_{s+1}^{-1}|$. (The inverse tangent of $-C_k'$ was chosen to have an argument between $\pi/2$ and π.)

Without loss of generality, we may now perform the integration near $z = 1$; the results near any $e^{i\varphi_k}$ follow in the same fashion.

We perform the integration first for ζ, using (5.22). Multiplication by D^r is equivalent to multiplying the first terms in the expressions in (5.22) by $K_r(z)((z-1)^{1/2})^r h^{-r}$, where each $K_r(z)$ is a bounded and analytic function of $(z-1)^{1/2}$. Thus, for these terms, we must estimate

$$\int_0^{\theta_1} |((z-1)^{1/2})^{r-1} h^{-r} z^{q-p} K_r'(z)|\, |dz| \exp[-|\nu-\nu_0|\, C\beta h]. \tag{6.7}$$

This is bounded by

$$C \exp[-|\nu-\nu_0|\, c\beta h] h^{-r} \exp[(q-p+1)\beta^2 h^2] \cdot \int_0^{\theta_1} [\theta^2 + \beta^2 h^2]^{\frac{r}{2}}\, d\theta \exp[(-p+q+1)\theta^2] \tag{6.8}$$

since $dz = \theta[2C_k i - 2]z\, d\theta$. We next let $\theta = y(q-p+1)^{-1/2}$, the upper bound

$$Ch^{-r}(q-p+1)^{-r/2} \exp[-|\nu-\nu_0|\, C\beta h + (q-p+1)\beta^2 h^2], \tag{6.9}$$

for all $0 \leqq \beta h \leqq C$, then follows easily. If we replace $C\beta$ by β, we then have the estimate we seek.

D^r multiplies each of the remaining two terms in (5.22) by analytic functions of z divided by h^r. The results for these terms are thus easily obtained.

We notice that these estimates are valid for an arbitrary nonnegative integral r. This fact is not surprising, since this function is the full parametrix for the free space problem.

We now estimate the extra terms in the Dirichlet problem, using (5.34′). All the terms following the τ_1^ν may be estimated as above. Consider the terms which involve the $p_i(\nu, z)$. We may view D^r acting on these first terms as $1/h^r$ times $\tau_{s+1}^{-\nu_0}\; O(((|z-1|)^{1/2} + h)^2)/(z-1)^{1/2}$ times an analytic function of z decaying exponentially in ν_0. The result follows for $r = 0, 1, 2$ using the reasoning above. Similar methods are used for the terms involving O_i. The $M_{\nu_0}^{(i)}(z, h)$ are analytic at $z = 1$, as are the $N_{\nu_0}^{(i)}(z, h)$ modulo terms of order $(z-1)^{1/2}$ which is $O(((z-1)^{1/2})^2)/(z-1)^{1/2}$, and, hence, may be estimated for $r = 0, 1, 2$.

Finally, we consider the case when $\alpha_2 \neq 0$. Consider first α_1/α_2 with negative real part. Then the terms in (5.34):

$$\frac{\dfrac{\alpha_1}{\alpha_2} h + \left(\dfrac{z-1}{\lambda a(x_0)}\right)^{1/2}}{\dfrac{\alpha_1}{\alpha_2} h - \left(\dfrac{z-1}{\lambda a(0)}\right)^{1/2}}, \qquad \frac{O(((|z-1|)^{1/2} + h)^2)}{\dfrac{\alpha_1}{\alpha_2} h - \left(\dfrac{z-1}{\lambda a(0)}\right)^{1/2}} (\det B(z, h))^{-1}, \qquad \frac{(z-1)^{1/2}}{\det B(z, h)}$$

are bounded on the path of integration. Thus, we may estimate the terms involving

τ_1', as above. We may combine this argument with the analogous one used in the Dirichlet case to estimate the terms involving the coefficients of the $p_i(\nu, z)$.

If α_1/α_2 has positive real part, we must consider the influence of certain new singularities in the integrand. If $\beta \geqq 4\,|\lambda a(0)\alpha_1/\alpha_2|\max(1, 1/(C_1')^{1/2}) = \beta_0$, then we may use exactly the same reasoning as above to obtain the bound. The estimate for $\beta = \gamma < \beta_0$ follows, since

$$K_0 \exp\,[C\gamma^2(q - p + 1)h^2 - (x + x_0)\gamma] \geqq \exp[C\beta_0^2(q - p + 1)h^2 - (x + x_0)\beta_0],$$

with $K_0 = \exp[C\beta_0^2 T/\lambda]$. If α_1/α_2 is purely imaginary, we merely require first that $\beta \geqq 1$. Then the estimates follow as before for all such β. For $0 \leqq \beta < 1$, we merely use the same trick as in the previous case.

Now we write the Green's function $\Gamma(x, qk, x_0, pk)$ in the following way, (imitating Widlund [8]):

$$\begin{aligned}\Gamma(x, qk, x_0, pk) &= G(x, qk, x_0, pk)\\ &\quad + \sum_{\nu=p}^{q-1}\sum_{y=ah}^{\infty} G(x, qk, y, (\nu+1)k)\psi(y, \nu k, x_0, pk),\quad x_0 \geqq ah.\end{aligned}\tag{6.10}$$

We have, denoting by E_k an operator such that $E_k v(qk) = v((q+1)k)$,

$$\begin{aligned}\psi(x, qk, x_0, pk) &= (\tilde{Q}(x, qk, hD, h) - E_k)G(x, qk, x_0, pk)\\ &\quad + \sum_{\nu=p}^{q-1}\sum_{y=ah}^{\infty}(\tilde{Q}(x, qk, hD, h) - E_k)\\ &\quad \cdot G(x, qk, y, (\nu+1)k)\psi(y, \nu k, x_0, pk).\end{aligned}\tag{6.11}$$

Let $\psi^0(x, qk, x_0, pk) = (\tilde{Q}(x, qk, hD, pk) - E_k)G(x, qk, x_0, pk)$, and, for $m \geqq 1$:

$$\psi^{(m)}(x, qk, x_0, pk) = \sum_{\nu=p}^{q-1}\sum_{y}\psi^{(0)}(x, qk, y, (\nu+1)k)\psi^{(m-1)}(y, \nu k, x_0, pk).\tag{6.12}$$

We shall show that $\sum \psi^{(m)}$ converges absolutely and uniformly. It is then easy to show that $\psi = \sum \psi^{(m)}$ solves (6.10).

LEMMA 6.3. *Assume that the conditions of the main theorem are fulfilled. Then, $\sum \psi^{(m)}$ converges uniformly and absolutely and solves* (6.10). *Furthermore, there exists a constant C such that*

$$\begin{aligned}&|\psi(x, qk, x_0, pk)|\\ &\quad\leqq \frac{Ch^\gamma}{(q - p + 1)^{(3-\gamma)/2}}\exp\,[2C\beta^2(q - p + 1)h^2 - |x - x_0|\,\beta]\end{aligned}\tag{6.13}$$

for all β such that $0 \leqq h\beta \leqq C$.

Proof. In order to calculate $\psi^{(0)}$, we need

$$\begin{aligned}&-\frac{1}{2\pi i}\oint z^{q-p}(\tilde{Q}(x, qk, hD, h) - z)\\ &\qquad\cdot[\Phi(x, x_0, pk, h, z) + \eta(x, x_0, pk, h, z)]\,dz.\end{aligned}\tag{6.14}$$

We may multiply through by the operator

$$\begin{bmatrix} (I+Q_{-1}) & 0 & \cdots & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & \cdots & 1 \end{bmatrix}$$

without affecting the estimate, because of its uniform invertibility on the space of functions which obey the homogeneous boundary conditions. We know that

$$0 = \oint z^{(q-p)}(\tilde{Q}^{(1)}(x_0, pk, hD) - z)\eta(x, x_0, pk, h, z)\, dz.$$

Thus, we need only consider

(6.15)

$$\oint z^{q-p} \begin{bmatrix} +z(Q_{-1} - Q_{-1}^{(1)}) & -(Q_0 - Q_0^{(1)}) & \cdots & -(Q_R - Q_R^{(1)}) \\ 0 & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & & \cdots & 0 \end{bmatrix} \eta(x, x_0, pk, h, z)\, dz.$$

All these terms may be estimated as in Widlund [8], except for those corresponding to $\sigma = 0$. Thus, we must consider terms of the form:

$$\begin{aligned} &[B_{\mu,0,j}(x, qk) - B_{\mu,0,j}(x_0, pk)](hD)^2 G_i(x, qk, x_0, pk) \\ (6.16) \qquad &\leqq C\,\frac{[|x - x_0|^\gamma + [(q-p)k]^{\gamma/2}]}{(q-p+1)^{3/2}} \exp\,[C\beta^2(q-p+1)h^2 - (x - x_0)\beta]. \end{aligned}$$

The $(x - x_0)^\gamma$ term is estimated as in Widlund's paper. The $[(q-p)k]^{\gamma/2}$ gives no difficulty.

Thus, for this part of $\psi^{(0)}$, which we call $\psi_i^{(0)}$, we have

$$(6.17) \quad |\psi_i^{(0)}| \leqq \frac{C(\epsilon)h^\gamma}{(q-p+1)^{(3-\gamma)/2}} \exp\,[(1+2\epsilon)C\beta^2(q-p+1)h^2 - |x - x_0|\beta]$$

for all β, $0 \leqq \beta \leqq C$.

Next, we recall that

$$0 = (\tilde{Q}^{(1)}(0, pk, hD) - z)\Phi(x, x_0, pk, h, z) \quad \text{for } x \geqq ah.$$

Thus, we consider

$$(6.18) \qquad \frac{1}{2\pi i}\oint z^{q-p} \begin{bmatrix} z[Q_{-1} - Q_{-1}^{(1)}] & -[Q_0 - Q_0^{(1)}] & \cdots & -[Q_R - Q_R^{(1)}] \\ 0 & & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & & \cdots & 0 \end{bmatrix} \Phi\, dz.$$

Again, we need only consider the terms of the form:

(6.19)

$$[B_{\mu,0,j}(x, qk) - B_{\mu,0,j}(0, pk)](hD)^2 G_R$$
$$\leqq c[x^\gamma + (qk - pk)^{\gamma/2}] \frac{1}{(q-p+1)^{3/2}} \exp[C\beta^2(q-p+1)h^2 - (x+x_0)\beta].$$

The $(q - p)k$ term is again easily estimated. We notice

$$\frac{x^\gamma}{((q-p+1)h^2)^{\gamma/2}} \exp\left[-\frac{\epsilon x}{(q-p+1)^{\gamma/2}h^\gamma}\right] \leqq C(\epsilon).$$

The remainder of the argument follows Widlund's. Thus, we have

$$|\psi_R^{(0)}| \leqq \frac{C(\epsilon)h^\gamma}{(q-p+1)^{(3-\gamma)/2}} \exp[(1+\epsilon)C\beta^2(q-p+1)h^2 - |x+x_0|\beta] \tag{6.20}$$

for all β, $0 \leqq \beta \leqq C$.

We now have Widlund's estimate for $\psi^{(0)}$, since $x + x_0 \geqq |x - x_0|$. Thus, we may prove Lemma 6.3 in the same manner as in [8].

The following theorem is proved using the argument Widlund used to prove his Theorem 4.1.

THEOREM 6.4. *Assume the conditions of the main theorem concerning the right-half problem are fulfilled. Then, there exist universal positive constants C, such that*

$$|D^\tau\Gamma(x, qk, x_0, pk)|$$
$$\leqq \frac{C}{h^\tau(q-p+1)^{(1+\tau)/2}} \exp[3C\beta^2(q-p+1)h^2 - |x-x_0|\beta] \tag{6.21}$$

for all β, $0 \leqq \beta \leqq C/h$, $\tau = 0, 1$.

7. Construction of the Green's Function for the Two-Point Problem. We now consider the two-point boundary value problem, and the difference approximation to it. We construct the right-half and left-half plane functions which obey

$$\begin{aligned} (\tilde{Q}^{(1)}(0, pk, hD) - zI)\Phi_0(x, pk, z) &= 0, \qquad x \geqq a_0 h, \\ (\tilde{Q}^{(1)}(1, pk, hD) - zI)\Phi_1(x, pk, z) &= 0, \qquad x \leqq 1 - a_1 h. \end{aligned} \tag{7.1}$$

(a_0, a_1, are 0 or 1 depending on the problem.)

We perform the same reductions as before, obtaining scalar homogeneous equations for $\chi_0(\nu h)$ and $\chi_1(\nu h)$. We may write the general solution as in (5.14), (5.15), this time obtaining $(l + s)$ unknowns, l of which correspond to χ_1. Finally, we demand that $\chi_0(\cdot, pk, z) + \chi_1(\cdot, pk, z) + \zeta(\cdot, x_0, pk, z)$ obey the boundary conditions, or

$$\begin{aligned} &\chi_0(\mu h) - \sum_{j=a_0}^{a_0} b_{\mu j}^{(0)}(h)\chi_0(jh) + \chi_1(\mu h) - \sum_{j=a_0}^{a_0} b_{\mu j}^{(0)}(h)\chi_1(jh) \\ &\qquad = -\zeta(\mu h) + \sum_{j=a_0}^{a_0} b_{\mu j}^{(0)}(h)\zeta(jh), \qquad \mu = a_0, a_0 - 1, \cdots, a_0 - s, \\ &\chi_0(1+ph) - \sum_{j=a_1}^{a_1} b_{pj}^{(1)}(h)\chi_0(1-jh) + \chi_1(1+ph) - \sum_{j=a_1}^{a_1} b_{ph}^{(1)}(h)\chi_1(1-jh) \\ &\qquad = -\zeta(1+ph) + \sum_{j=a_1}^{a_0} b_{pj}^{(1)}(h)\zeta(1-jh), \ p = a_1, a_1 + 1, \cdots, a_1 + l. \end{aligned} \tag{7.2}$$

We may get an estimate of the type (6.4) on the path Γ_0, since the χ_1 terms in the first equation and the χ_0 terms in the second equation decay like $C\rho^{1/h}$, $0 \leqq \rho < 1$. Thus, we need examine (7.2) carefully only on the paths Γ_j, in fact, only on Γ_1, near $z = 1$.

In the neighborhood of $z = 1$, we have

$$\chi_1(\nu h) = c_1\tau_1^\nu + \sum_{j=2}^{s} c_j p_j^{(0)}(\nu, z), \tag{7.3}$$
$$\chi_0(\nu h) = c_{s+1}\tau_{s+1}^{N-\nu} + \sum_{j=s+2}^{l+s} c_j p_j^{(1)}(N-\nu, z).$$

The analogue of Eq. (5.24) is now

$$\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} W_1 \\ W_2 \end{bmatrix}, \qquad \begin{aligned} V_1 &= [c_1, \cdots, c_s], \\ V_2 &= [c_{s+1}, \cdots, c_{l+s}], \end{aligned} \tag{7.4}$$

B_{11} and B_{22} are the matrices we obtained in the right- and left-half plane cases, respectively; W_1 and W_2 are the right- and left-half W's, respectively. We multiply both sides by

$$\begin{bmatrix} B_{11}^{-1} & 0 \\ 0 & B_{22}^{-1} \end{bmatrix}$$

and obtain

$$\begin{bmatrix} I & B_{11}^{-1}B_{12} \\ B_{22}^{-1}B_{21} & I \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} B_{11}^{-1} & W_1 \\ B_{22}^{-1} & W_2 \end{bmatrix}. \tag{7.4'}$$

The first column of B_{12} is

$$B_{12}^{(1,j)} = \tau_{s+1}^{-N}[\tau_{s+1}^{-j} - \sum b_{-j,k}^{(0)}(h)\tau_{s+1}^{k}], \tag{7.5}$$

while its remaining columns are analytic and decay like $C\rho^{1/h}$. Also,

$$B_{21}^{(1,j)} = \tau_1^N[\tau_1^j - \sum b_{jk}^{(1)}(h)\tau_1^{-k}], \tag{7.6}$$

while its remaining columns are analytic and decay like $C\rho^{1/h}$.

$B_{11}^{-1}\ W_1$ and $B_{22}^{-1}\ W_2$ have been obtained by expressions like (5.31), (5.33), or (5.31′), (5.32′) and their analogues for the right-half plane problem. We now use similar arguments to those in Section 5 in order to estimate $B_{11}^{-1}B_{12}^{-1}$ and $B_{22}B_{21}$.

Assume first $\alpha_2 \neq 0$. Then

$$(B_{11}^{-1}B_{12})_{(1,j)}$$
$$= \tau_{s+1}^{-N}\left[\frac{\dfrac{\alpha_1}{\alpha_2}h + \left(\dfrac{z-1}{\lambda a(1)}\right)^{1/2}}{\dfrac{\alpha_1}{\alpha_2}h - \left(\dfrac{z-1}{\lambda a(0)}\right)^{1/2}}\delta_{1j} + \frac{(z-1)^{-1/2}O(((|z-1|)^{1/2}+h)^3)}{\left(\dfrac{\alpha_1}{\alpha_2}h - \left(\dfrac{z-1}{\lambda a(0)}\right)^{1/2}\right)\det B_{11}(z,h)} \right. \tag{7.7}$$
$$\left. + C_{1j}\rho_{1j}^N(\det B_{11}(z,h))^{-1}\right],$$

while, for $k > 1$,

$$(B_{11}^{-1}B_{12})^{ki} = \rho_{ki}^{N}\left(1 + \frac{O((h + (|z-1|)^{1/2})^2))}{\det B_{11}(z, h)}\right)$$

$$+ \tau_{s+1}^{-N}\left[\frac{O(((|z-1|)^{1/2} + h)^3)}{\dfrac{\alpha_1}{\alpha_2}h - \left(\dfrac{z-1}{\lambda a(0)}\right)^{1/2}}\left(1 + \frac{O((h + (|z-1|)^{1/2})^2))}{\det B_{11}(z, h)}\right)\right].$$

Next, suppose $\alpha_2 = 0$. We have

$$(B_{11}^{-1}B_{12})^{(1,i)} = \tau_{s+1}^{-N}[\delta_{1i} + O(((|z-1|)^{1/2} + h)^2) + \rho_{1i}^{N}], \tag{7.7'}$$

while for $k > 1$,

$$(B_{11}^{-1}B_{12})^{(k,i)} = \rho_{ki}^{N}C_{ki} + O(((|z-1|)^{1/2} + h)^2)\tau_{s+1}^{-N},$$

each ρ_{ki} and C_{ki} is an analytic function of $(z-1)^{1/2}$, each $|\rho_{ki}| < 1 - c$. We may obtain analogous results for $B_{22}^{-1}B_{21}$. Thus, if we wish to invert this matrix and obtain the estimates for the contour integral as we did in Theorem 6.1, we need only worry about the element in the (1, 1) position in both matrices. In all cases, we merely multiply the first row by

$$-\tau_1^{N}\frac{\left[\beta_1 h - \beta_2\left(\dfrac{z-1}{\lambda a(0)}\right)^{1/2}\right]}{\left[\beta_1 h + \beta_2\left(\dfrac{z-1}{\lambda a(1)}\right)^{1/2}\right]}$$

and add it to row number $s + 1$. The only troublesome equation becomes, modulo harmless terms:

$$\left[1 - (\tau_{s+1}^{-N}(1)\tau_1^{N}(0))\left\{\frac{\beta_1 h - \beta_2\left(\dfrac{z-1}{\lambda a(0)}\right)^{1/2}}{\beta_1 h + \beta_2\left(\dfrac{z-1}{\lambda a(1)}\right)^{1/2}}\right\}\left\{\frac{\alpha_1 h + \alpha_2\left(\dfrac{z-1}{\lambda a(0)}\right)^{1/2}}{\alpha_1 h - \alpha_2\left(\dfrac{z-1}{\lambda a(1)}\right)^{1/2}}\right\}\right]c_{s+1}$$

$$\begin{aligned}&= \frac{\tau_1^{N-\nu_0}(x_0)A_1(z)}{(\tau_1(x_0) - \tau_{s+1}(x_0))}\,\tau_{s+1}^{-\nu_0}(x_0)\left(\frac{\tau_1(1)}{\tau_1(x_0)}\right)^{N-\nu_0}\tau_1^{\nu_0}(1)\\ &\quad\cdot\frac{\alpha_1 h + \alpha_2\left(\dfrac{z-1}{\lambda a(x_0)}\right)^{1/2}}{\alpha_1 h - \alpha_2\left(\dfrac{z-1}{\lambda a(0)}\right)^{1/2}}\;\frac{\beta_1 h - \beta_2\left(\dfrac{z-1}{\lambda a(0)}\right)^{1/2}}{\beta_1 h + \beta_2\left(\dfrac{z-1}{\lambda a(1)}\right)^{1/2}}\\ &\quad+\frac{\beta_1 h - \beta_2\left(\dfrac{z-1}{\lambda a(x_0)}\right)^{1/2}}{\beta_1 h + \beta_2\left(\dfrac{z-1}{\lambda a(1)}\right)^{1/2}}\left[\frac{-\tau_{s+1}^{N-\nu_0}(x_0)A_2(z)}{\tau_1(x_0) - \tau_{s+1}(x_0)}\right].\end{aligned} \tag{7.8}$$

Now, we merely require that $\beta \geqq c > 0$, c fixed independent of h, on the path of integration Γ_1. Then the coefficient of c_{s+1} is uniformly bounded away from zero and hence, we may invert this and obtain the appropriate estimates.

We may now obtain the parametrix for this problem, which will be divided into three parts instead of two, and follow the procedure of the last section, in order to obtain:

THEOREM 7.1. *Assume that the conditions of the main theorem are fulfilled. Then, there exist universal positive constants C such that*

$$\tag{7.9} \begin{aligned} &|D^{\tau}\Gamma(x, qk, x_0, pk)| \\ &\quad \leqq Ch^{-\tau}(q-p+1)^{-((1+\tau)/2)} \exp(3C\beta^2(q-p+1)h^2 - |x-x_0|\beta), \end{aligned}$$

for all β, $0 \leqq \beta \leqq C/h$, $\tau = 0, 1$.

8. Proof of Main Theorem II. If the functions $g_0(t)$ and $g_1(t)$ are identically zero, it then becomes easy to prove the second main theorem, (3.5).

We merely write:

$$\tag{8.1} \begin{aligned} \tilde{v}(x, t, h) = &\sum_{y \in \text{mesh}} \Gamma(x, t, y, Rk)\tilde{v}(y, Rk, k) \\ &+ k \sum_{\nu=R}^{(t-k)/k} \sum_{y \in \text{mesh}} \Gamma(x, t, y, (\nu+1)k)(I + Q_{-1}(y, \nu k, hD, h))^{-1} \\ &\cdot (f(y, \nu k), 0, \cdots, 0)^T, \qquad (R+1)k \leqq t \leqq T. \end{aligned}$$

Then, we may use exactly the same proof as in Widlund [8].

Next, we consider the general case as follows. We write $v = v_1 + v_2$, where v_1 satisfies the problem with homogeneous boundary conditions and v_2 satisfies the problem with inhomogeneous boundary conditions and with everything else homogeneous.

We consider first the function $v_2(x, t, h)$ for the right-half problem when $\alpha_2 \neq 0$. (For simplicity of notation, we call it $v(x, t, h)$.) It satisfies

$$\tag{8.2} \begin{aligned} &v(jh, 0, h) = 0, \quad \text{if } j \geqq 0, \\ &v(ph, 0, h) = \frac{h}{\alpha_2} g(0)\left[p - \sum_{j=0}^{a_0} jb_{pj}^{(0)}(h)\right], \\ &\qquad\qquad p = -1, -2, -3, \cdots, -s, \\ &v(ph, nk, h) - \sum_{j=0}^{a_0} b_{pj}^{(0)}(h)v(jh, nk, h) = \frac{h}{\alpha_2} g_0(nk)\left[p - \sum_{j=0}^{a_0} jb_{pj}^{(0)}(h)\right], \\ &\qquad\qquad p = -1, -2, \cdots, -s, \; 0 \leqq n \leqq T/K, \end{aligned}$$

and the difference equation (2.7) in the right-half plane for $f \equiv 0$. We may solve this as follows. Let

$$\tag{8.3} \begin{aligned} &\tilde{w}(ph, nk, h) = 0, \quad \text{if } p \geqq 0, \\ &\tilde{w}(ph, nk, h) = \left[\frac{h}{\alpha_2} g_0(n - Rk)\left(p - \sum_{j=0}^{a_0} jb_{pj}^{(0)}(h)\right), 0, \cdots, 0\right]^T, \\ &\qquad\qquad \text{if } p = -1, -2, \cdots, -s. \end{aligned}$$

Then,

$$\tilde{v}(x, t, h) = \sum_{\nu=R}^{(t-k)/R} \sum_{\nu \geq 0} \Gamma(x, t, y, (\nu + 1)k)\tilde{Q}(y, \nu k, hD, h)\tilde{w}(y, \nu k, h) + \tilde{w}(x, t, h). \tag{8.4}$$

However, for $\tau = 0, 1$,

$$\begin{aligned} &\sum_{\nu \geq 0} D^\tau \Gamma(x, t, y, (\nu + 1)k)\tilde{Q}(y, \nu h, hD, h)\tilde{w}(y, \nu k, h) \\ &\leqq |g_0(\nu - Rk)| \frac{Ch^2}{((q - \nu)h^2)^{(\tau+1)/2}} \exp[3C\beta^2(q - \nu)h^2 - x\beta], \qquad t = qk, \end{aligned} \tag{8.5}$$

for all β, $0 \leqq \beta \leqq C/h$.

This follows easily from the fact that

$$|[I + Q_{-1}(y, t, hD, h)]^{-1} \delta(x, 0)| \leqq Ce^{-cx/h}. \tag{8.6}$$

Thus, we have

$$\begin{aligned} |D^\tau v(x, t, h)| \leqq \sup_{0 \leq s \leq t} |g_0(s)|\, Ch^2 \sum_{\nu=R}^{q-1} \frac{1}{((q - \nu)h^2)^{(\tau+1)/2}} \\ \cdot \exp[3C\beta_{x,\nu}^2(q - \nu)h^2 - x\beta_{x,\nu}]. \end{aligned} \tag{8.7}$$

For $\tau = 0$, we choose $\beta_{x,\nu} = \min(C/h, X/4C(q - \nu)h^2)$, and we may easily show

$$|v(x, t, h)| \leqq \sup_{0 \leq s \leq t} |g_0(s)| \sqrt{t}\, C \max\left[\exp\left[\frac{-cx}{h}\right], \exp\left[\frac{-cx^2}{t}\right]\right]. \tag{8.8}$$

Next, let us suppose $\tau = 1$. We use the same definition for $\beta_{x,\nu}$ as above. The first part of the sum is bounded by

$$C \ln\left(\frac{(x - h)}{Ch}\right) \exp\left[\frac{-Cx}{h}\right], \tag{8.9}$$

and this is all there is if $(q - R) < xh^{-1}/C$. In general, we have this term, plus

$$\left|\sum_{[\nu > x/Ch]} \frac{1}{\nu} \frac{\lambda h^2}{\lambda h^2} \exp\left(-\frac{cx^2}{\nu k}\right)\right| \leqq \int_{xh/C}^{t} \exp\left[\frac{-cx^2}{s}\right] \frac{1}{s}\, ds, \tag{8.10}$$

suppose $x^2/t \geqq C''$, C'' to be chosen later. We have $\exp[-x^2/t] \leqq C \exp[-x/\sqrt{t}]$. We may multiply the integrand above by $x/(sC'')^{1/2}$, we then have

$$\leqq C \exp[-Cx/\sqrt{t}]. \tag{8.11}$$

If $x^2/t < C''$, then in the integrand above, we integrate first from $s = xh/C$ to $s = x^2/C''$, multiply by $x/(sC'')^{1/2}$, and obtain a constant for a bound. Next, we consider

$$\int_{x^2/C}^{t} \exp\left[-C\frac{x^2}{s}\right] \frac{1}{s}\, ds = \int_{Cx^2/t}^{CC''} \frac{1}{y} e^{-y}\, dy. \tag{8.12}$$

Integrating by parts, we get

$$\int_{Cx^2/t}^{CC''} \frac{1}{y}(1 - y \ln y)e^{-y} = -\ln\frac{Cx^2}{t} \exp\left[-\frac{Cx^2}{t}\right] + \ln(CC'')e^{-CC''} \tag{8.13}$$

if $C'' = 1/C$. Then $y \ln y \leqq 0$ and we have the estimate

$$\int_{x^2 C}^{t} \exp\left[-\frac{Cx^2}{s}\right]\frac{1}{s}\,ds \leqq -\ln\frac{Cx^2}{t}\exp\left[-\frac{Cx^2}{t}\right]. \tag{8.14}$$

Thus, we have estimate (3.6) in view of (8.9), (8.11), and (8.14). Next, we consider the Dirichlet case for the right-half problem. Unfortunately, our estimates for the Green's function are not strong enough, in general, to obtain estimates for the solution in terms of the maximum form of the boundary data. Thus, we must use the usual nefarious trick which follows. Let

$$w(jh, nk, h) = v(jh, nk, h) - g_0(nk). \tag{8.15}$$

Then w satisfies the homogeneous boundary conditions and

$$\begin{aligned} w(jh, 0, h) &= -g_0(0), && \text{if } j \geqq 1, \\ w(jh, 0, h) &= \left(-\sum_{\nu=1}^{q_0} b_{j\nu}^{(0)}(h)\right)g_0(0), && \text{if } j = -s+1, \cdots, 0. \end{aligned} \tag{8.16}$$

Also

$$\begin{aligned} &[I - Q_{-1}(x, t, hD, h)]w(x, t+k, h) \\ &\quad - \sum_{\mu=0}^{R}(a_\mu + Q_\mu(x, t, hD, h))w(x, t-\mu k, h) \\ &= \left(g_0(t+k) - \sum_{\mu=1}^{R} a_\mu g_0(t-\mu k)\right) + \sum_{\sigma\geqq 2}\sum_{\mu=-1}^{R} B_{\mu,\sigma,0}(x, t)h^\sigma g_0(t-\mu k) \\ &= k\tilde{f}(x, t). \end{aligned} \tag{8.17}$$

We may then use Eq. (8.1), with a modification necessary to estimate the second term. We know $\sum_{\nu\geqq h}|\Gamma(x, t, y, (\nu+1)k)|$ is bounded. Thus, it follows

$$\begin{aligned} |v(x, t, h)| \leqq{}& C_0'\,|g_0(0)| + |g_0(t)| + C_0 t\,|||g_0||| \\ &+ C_0''\sup_k \sum_{s=Rk}^{(t-k)/k}\left|g_0(s+k) - \sum_{\mu=0}^{R} a_\mu g_0(s-\mu k)\right|. \end{aligned} \tag{8.18}$$

Also

$$\begin{aligned} |(t+k)^{1/2}Dv(x, t, h)| \leqq{}& C_1 t\,|||g_0||| + C_1'\,|g_0(0)| \\ &+ C_1''\sup_k \sum_{s=Rk}^{(t-k)/k}\left|\frac{g_0(s+k) - \sum_{\mu=0}^{R} a_\mu g_0(s-\mu k)}{(t-s)^{1/2}}\right|(t+k)^{1/2}. \end{aligned} \tag{8.19}$$

Finally, we consider the general two-point case. We may merely decompose it into the sum of two problems where one side of each has homogeneous boundary conditions.

9. On the Necessity of Certain Conditions. Three-Point Schemes. We now give an example to demonstrate that the lack of sufficient accuracy at the boundary and assumption (i) at $z = 1$ can lead to a stable scheme for which the first divided difference is not estimable in terms of the data.

This example indicates a difficulty that will arise in multi-dimensional problems where the space boundary is curved. Accurate boundary conditions will be difficult to maintain in such a case.

Consider a parabolic difference approximation to

$$u_t = u_{xx}, \qquad u(x, 0) = u_0(x), \qquad t, x \geqq 0, \; u(0, t) = 0,$$

(9.1)
$$v_j^{n+1} = v_j^n + \lambda(v_{j+1}^n - 2v_j^n + v_{j-1}^n) + \lambda h^2 \frac{v_{j+2}^n - 2v_j^n + v_{j-2}^n}{4},$$

$$j = 1, 2, \cdots, 0 < \lambda < \tfrac{1}{2},$$

with boundary conditions:

$$v_{-1}^n = 0, \qquad v_0^n = \frac{1}{1 + z(h)} v_1^n, \qquad z(h) = O(h^2)$$

chosen below. Elementary maximum principle analysis assures us that the problem is stable. The equation

(9.2)
$$z - 2 + \frac{1}{z} + h^2\left(z - \frac{1}{z}\right)^2 = 0$$

has the root $z = 1$ and $z(h) = O(h^2)$. Consider the solution

(9.3)
$$v_j^n = 1 - (z(h))^{j+1}.$$

This obeys the equation and the boundary conditions. Now consider

(9.4)
$$\left|\frac{v_0^{n+1} - v_{-1}^{n+1}}{h}\right| = \frac{1 - z(h)}{h} \to \infty \quad \text{as } h \to 0.$$

(However, if $x > 0$, $Dv(x, t, h) \to 0$ as $h \to 0$.)

Consider condition (i). We may view it as a restriction on the difference approximation $Q^{(1)}(0, t_0, hD)$ to $u_t = a(0, t_0)u_{xx}$, in the region $0 \leqq x, t$ with boundary conditions $u_x(0) = 0$ or $u(0) = 0$, respectively.

Condition (i) requires, among other things, that $v(x, t) = x \exp[i\varphi_j t/k]$ and $v(x, t) = \exp[i\varphi_j t/k]$, respectively, are not solutions to the difference approximations to the above problems. This condition can easily be shown to be necessary in order that the estimates in the main theorem are true for the constant coefficient problem.

Next, we consider the important special case $s = 1 = l$ in (2.9), a "three-point scheme". We may weaken our accuracy assumptions at the boundary in this case for the Dirichlet problem. If $\alpha_2 = 0$, we do not need the condition:

$$\mu - \sum_{j=1}^{a_0} b_{\mu j}^{(0)}(h) j = 0.$$

We, of course, still need the other assumptions, in particular (i) and (j). We may then modify our proof in a simple way to obtain the estimates (6.1) and (6.2), but this time we have then for all $\tau = 0, 1, 2, \cdots$, as we do in this case with $\alpha_2 \neq 0$. The main results then follow under these weaker hypotheses.

We notice that our estimates on the Green's function would enable us to obtain L_p estimates, $1 \leqq p \leqq \infty$, instead of the L_∞ estimates, with no extra difficulty.

Last, we notice that even for the Cauchy problem for $u_t = u_{xx}$, without the hypothesis of parabolicity, stable consistent difference schemes do not necessarily yield estimates on derivatives. For example,

$$v_j^{n+1} = \frac{v_{j-1}^n + v_{j+1}^n}{2}, \qquad j = 0, \pm 1, \pm 2, \cdots, \tag{9.5}$$

has the property that v_{n+p}^n is independent of v_{n+q}^n if $p - q$ is odd, hence, an estimate on $(v_j^n - v_{j-1}^n)/h$ in terms of $\sup|v_j^0|$ is impossible.

A similar statement can be made for the well-known DuFort-Frankel scheme.

Mathematics Department
State University of New York at Stony Brook
Stony Brook, New York 11790

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