

# A Note Concerning the Two-Step Lax-Wendroff Method in Three Dimensions

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**Abstract.** The two-step Lax-Wendroff method in three spatial dimensions is discussed and, dealing with its linear stability in the hydrodynamic case, the sufficiency of the von Neumann condition is proved.

In their paper [1], Rubin and Preiser suggest a difference scheme for the conservation law:

$$(1) \quad W_t + \partial f_i / \partial x_i = 0.$$

Their scheme is an extension of Richtmyer's two-step method to three spatial dimensions. In order to deal with the linear stability they take the linearized equation:  $W_t + A_i \cdot \partial w / \partial x_i$  (where  $A_i = \partial f_i / \partial W$  are taken locally constant), and, in order to get a stability criterion, they compute the amplification matrix:

$$(2) \quad G = I - \frac{2}{3} i \lambda [\cos \xi_1 + \cos \xi_2 + \cos \xi_3] M - 2 \lambda^2 M^2.$$

Here  $\lambda = \Delta t / \Delta x_1 = \Delta t / \Delta x_2 = \Delta t / \Delta x_3$ ,  $\xi_i = k_i \Delta x_i$  (where  $k_i$  are the dual variables) and  $M = A_1 \sin \xi_1 + A_2 \sin \xi_2 + A_3 \sin \xi_3$ .

To prove sufficiency of the von Neumann condition, Rubin and Preiser use a theorem due to Kreiss [3] where the dissipativity of the scheme is assumed. However, it is easy to verify that their scheme is not dissipative because for  $\xi = (\pi, 0, 0)$  (so that  $|\xi| = \pi$ ), for example,  $M$  is the null matrix and  $G = I$  so that its eigenvalues are on the unit circle, as is the case for two dimensions (see [4]).

We give a different proof for the sufficiency of the von Neumann condition but only for the hydrodynamic case. (This is an extension of Richtmyer's proof in [2] to three spatial dimensions.)

In this case,  $W = (\rho, \rho u, \rho v, \rho w, E)$ , where  $\rho$ ,  $E$  and  $V = (u, v, w)$  are the density, total energy per unit volume and the velocity vector, respectively. We shall make use of the following sufficiency theorem (see [2]): "If  $G$  has a complete set of eigenvectors and there exists a constant  $\delta$  such that  $\Delta \geq \delta > 0$ , where  $\Delta^2$  is the Gram determinant of the normalized eigenvectors, then the von Neumann condition is sufficient as well as necessary for stability".

Instead of calculating the eigenvectors of  $G$ , we shall consider another matrix  $G'$  obtained from  $G$  by a similarity transformation. We introduce  $W' = (\rho, u, v, w, p)$ , where  $p$  is the pressure, and the transformation is

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$$(2.a) \quad dW = P \cdot dW',$$

$$(2.b) \quad A_i = PA'_i P^{-1},$$

and so  $M = PM'P^{-1}$  and  $G = PG'P^{-1}$ . This is done because the original  $A_i$  are too complicated.

If we compute  $P$  from (2.a), we get that

$$(3) \quad \det P = \rho^4 \partial e / \partial p,$$

where  $e$  is the internal energy per unit mass. It turns out that  $P$  is triangular so that  $\det(P^{-1}) = (\det P)^{-1}$ .

Let  $y_i$  be the normalized eigenvectors of  $G$  (and  $M$ ), then  $\alpha_i P^{-1} y_i$  are the normalized eigenvectors of  $G'$  (and  $M'$ ), where  $\alpha_i > 0$  are the normalizing factors.

If we define

$$(4) \quad \Delta_1 = |\det(y_1, y_2, \dots, y_n)|,$$

$$(5) \quad \Delta_2 = |\det(\alpha_1 P^{-1} y_1, \alpha_2 P^{-1} y_2, \dots, \alpha_n P^{-1} y_n)|,$$

and  $\alpha = \alpha_1 \cdot \alpha_2 \cdot \dots \cdot \alpha_n > 0$ , then

$$\begin{aligned} \Delta_2 &= |\det[\alpha P^{-1}(y_1, y_2, \dots, y_n)]| \\ &= \alpha |\det P^{-1}| \cdot |\det(y_1, y_2, \dots, y_n)| \\ &= \alpha |\det P^{-1}| \Delta_1 = \frac{\alpha}{\rho^4 \partial e / \partial p} \Delta_1. \end{aligned}$$

This is the case because  $\rho$  is always bounded away from zero and in the usual fluids the same holds for  $\partial e / \partial p$  and consequently for  $\det P^{-1}$ . Therefore,  $\Delta_1$  is bounded away from zero if and only if  $\Delta_2$  is.

Rubin and Preiser found  $M'$  to be

$$(6) \quad M' = L \begin{bmatrix} u' & \rho \cos r & \rho \cos s & \rho \cos t & 0 \\ 0 & u' & 0 & 0 & 1/\rho \cos r \\ 0 & 0 & u' & 0 & 1/\rho \cos s \\ 0 & 0 & 0 & u' & 1/\rho \cos t \\ 0 & \rho C^2 \cos r & \rho C^2 \cos s & \rho C^2 \cos t & u' \end{bmatrix}$$

where  $L = (\sin^2 \xi_1 + \sin^2 \xi_2 + \sin^2 \xi_3)^{1/2}$ ,  $\cos r = (\sin \xi_1)/L$ ,  $\cos s = (\sin \xi_2)/L$ ,  $\cos t = (\sin \xi_3)/L$  and  $u' = u \cos r + v \cos s + w \cos t$ .

A direct computation of its eigenvectors shows

$$(7) \quad \Delta_2 = \left| \det \begin{bmatrix} 1 & 0 & 0 & -k\rho/C \cos r & k\rho/C \cos r \\ 0 & -\cos r \cdot \text{Ctg } s & -\cos t / \sin s & k & k \\ 0 & \sin s & 0 & k \cos s / \cos r & k \cos s / \cos r \\ 0 & -\cos t \cdot \text{Ctg } s & \cos r / \sin s & k \cos t / \cos r & k \cos t / \cos r \\ 0 & 0 & 0 & -k\rho C / \cos r & k\rho C / \cos r \end{bmatrix} \right|$$

$$= \frac{2\rho C k^2}{\cos^2 r}.$$

$k^2 = (\cos^2 r)/(\rho^2 C^2 + 1 + \rho^2/C^2)$  is a normalizing factor so that finally

$$(8) \quad \Delta_2 = 2\rho C/(\rho^2 C^2 + 1 + \rho^2/C^2).$$

We see that  $\Delta_2$ , and so  $\Delta_1$ , is bounded away from zero. Hence, by the sufficiency theorem quoted above, the von Neumann condition, namely  $\Delta t/\Delta x \leq 1/\sqrt{3}(|V| + C)$ , implies linear stability.

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1. E. L. RUBIN & S. PREISER, "Three-dimensional second-order accurate difference schemes for discontinuous hydrodynamic flows," *Math. Comp.*, v. 24, 1970, pp. 57-63.
2. R. D. RICHTMYER & K. W. MORTON, *Difference Methods for Initial-Value Problems*, 2nd ed., Interscience Tracts in Pure and Appl. Math., no. 4, Interscience, New York, 1967. MR 36 #3515.
3. H.-O. KREISS, "On difference approximations of the dissipative type for hyperbolic differential equations," *Comm. Pure Appl. Math.*, v. 17, 1964, pp. 335-353. MR 29 #4210.
4. S. Z. BURSTEIN, "High order accurate difference methods in hydrodynamics," in *Nonlinear Partial Differential Equations*, W. F. Ames (Editor), Academic Press, New York, 1967, pp. 279-290. MR 36 #510.