On the Nonexistence of Simplex Integration Rules for Infinite Integrals

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Abstract. It is shown that there do not exist integration rules of the form

$$\int_0^\infty f(x) dx = \sum_{i=1}^n w_i f(x_i) + C_n f^{(m)}(\xi), \quad 0 < \xi < \infty.$$

Almost all classical integration rules over a finite interval are simplex, that is, they have the form

$$\int_a^b f(x) dx = \sum_{i=1}^n w_i f(x_i) + C f^{(k)}(\xi), \quad a < \xi < b, \xi = \xi(f),$$

where C is a constant depending on the rule and interval, but independent of f , and k is some integer which is characteristic for the rule. Some special rules, for example Weddle's rule, which are not simplex are multiplex, that is, the error has the form $\sum_{i=1}^m C_i f^{(k_i)}(\xi_i)$. It is the aim of this note to show that there can exist no simplex or multiplex rule for the infinite integral $\int_0^\infty f(x) dx$. Although the Gauss-Laguerre rule

$$\int_0^\infty e^{-x} f(x) dx = \sum_{i=1}^n w_i f(x_i) + C_n f^{(2n)}(\xi)$$

appears to have the form of a simplex rule, this is not so, since we are concerned with unweighted integrals and if we write $f(x) = e^{-x} e^x f(x)$, we have that

$$\int_0^\infty f(x) dx = \sum_{i=1}^n w_i e^{x_i} f(x_i) + C_n (e^x f(x))_{x=\xi}^{(2n)}$$

which is neither simplex nor multiplex in form.

We now show that it is impossible to have an integration rule of the form

$$(1) \quad If \equiv \int_0^\infty f(x) dx = \sum_{i=1}^n w_i f(x_i) + C f^{(k)}(\xi), \quad 0 < \xi < \infty,$$

valid for all $f \in L[0, \infty) \cap C^k(0, \infty)$, or for that matter, one where there are a finite number of terms of the form $C_i f^{(k_i)}(\xi_i)$. The proof is based on the simple fact that, for any $r > 0$,

$$(2) \quad \int_0^\infty f(x) dx = r \int_0^\infty f(rx) dx.$$

Received February 7, 1972.

AMS 1970 subject classifications. Primary 41A55; Secondary 65D30.

Key words and phrases. Numerical integration, infinite interval, simplex rule, integration error.

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If (1) were true, then (2) would imply that

$$\begin{aligned}
 (3) \quad If &= r \int_0^\infty f(rx) \, dx = r \sum_{i=1}^n w_i f(rx_i) + r C f^{(k)}(r\xi) \\
 &= r \sum_{i=1}^n w_i f(rx_i) + r^{k+1} C f^{(k)}(\xi), \quad 0 < \xi < \infty,
 \end{aligned}$$

which must hold for all $f \in L[0, \infty) \cap C^k(0, \infty)$ and any real r . If we now choose such a function which is bounded together with its k th derivative on $[0, \infty)$, say $f(x) = 1/(1 + x^2)$, and let r approach zero, we see that the right-hand side of (3) approaches zero while the left-hand side has a constant value. This contradiction proves our result.

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