Numerical Investigation of Certain Asymptotic Results in the Theory of Partitions

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Abstract. A numerical investigation of some of the asymptotic formulas in partitions is made. Comparisons with actual computed values show that in certain cases only the relative error tends to zero and the errors are significant. Only in the case of $p_k(n)$ is it found that only a few terms of the asymptotic series are sufficient to obtain the exact value.

1. **Introduction.** Let $p_k(n)$ be defined by

$$\prod_{i=1}^{\infty} (1 - x^i)^{-k} = \sum_{n=0}^{\infty} p_k(n)x^n.$$

In an earlier paper, Cheema and Haskell [2] obtained asymptotic results for $p_k(n)$. These results are useful in obtaining similar results for $u_k(n)$, the number of k -line partitions of n . More recently, Gordon and Houten [5] derived the generating functions for $b_k(n)$, the number of k -line partitions of n , whose nonzero parts decrease strictly along rows, and extended their results to obtain the generating function for $b(n)$, the number of plane partitions of n , whose nonzero parts decrease strictly along rows. They also derived generating functions for the number of k -line and plane partitions under various other restrictions. MacMahon [7] had obtained the generating function for $u_k(n)$ and $u(n)$, the number of k -line and plane partitions of n , and Wright [12] obtained an asymptotic formula for $u(n)$. Following a similar technique, Gordon and Houten [4] have obtained an asymptotic formula for $b(n)$. In this article, we make a numerical investigation of some of these asymptotic results. The asymptotic results for $b(n)$ or $u(n)$, comparable in accuracy to those for $p(n)$ obtained by Hardy-Ramanujan-Rademacher, probably do not exist. Lehmer [8] investigated the asymptotic results for $p(n)$ and showed that $\alpha n^{1/2}$ terms of the series are sufficient so that the error will be less than $\frac{1}{2}$ in absolute value. The generating functions for $b(n)$ or $u(n)$ do not belong to the class of modular functions, thus making it impossible to obtain asymptotic formulas of the same type as that for $p(n)$. Our comparison with actual values of $b(n)$, $b_4(n)$, $p_2(n)$, $p_3(n)$ show certain interesting features. Results indicate that the error terms in the asymptotic formulas for $b_k(n)$ and $b(n)$ are quite significant, and only the relative errors tend to zero as n tends to infinity. Our computations confirm that results similar to those of Lehmer [8] also hold for the series for $p_k(n)$. Lehmer obtained estimates like $A_q(n) = O(q^{1/2+\epsilon})$ for

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$A_q(n)$. Similar estimates have been obtained for $A_q(k, n)$ by Agarwal and Gandhi [1], with the extra condition that $(k, q) = 1$. If one could extend these estimates for all q , then the proof that $\alpha n^{1/2}$ terms of the series for $p_k(n)$ are sufficient so that the error will be less than $\frac{1}{2}$ in absolute value will follow. Actual computations show the number of terms required is even much smaller.

2. Generating Functions. The generating functions for $p_k(n)$, $u(n)$, $u_k(n)$, $b(n)$, and $b_k(n)$ are given by the following:

$$(2.1) \quad \prod_{i=1}^{\infty} (1 - x^i)^{-k} = \sum_{n=0}^{\infty} p_k(n) x^n,$$

$$(2.2) \quad \prod_{i=1}^{\infty} (1 - x^i)^{-1} = \sum_{n=0}^{\infty} u(n) x^n,$$

$$(2.3) \quad \prod_{i=1}^{\infty} (1 - x^i)^{-\min(i, k)} = \sum_{n=0}^{\infty} u_k(n) x^n,$$

$$(2.4) \quad \prod_{i=1}^{\infty} (1 - x^i)^{-\{(i+1)/2\}} = \sum_{n=0}^{\infty} b(n) x^n,$$

$$(2.5) \quad P(x)^{[k/2]} Q(x)^{2[k/2]} \prod_{i=1}^{k-2} (1 - x^i)^{-\{(k-i)/2\}} = \sum_{n=0}^{\infty} b_k(n) x^n,$$

where

$$P(x) = \prod_{i=1}^{\infty} (1 - x^i)^{-1}, \quad Q(x) = \prod_{i=1}^{\infty} (1 - x^{2i-1})^{-1},$$

$[k]$ and $\{k\}$ denote the integer and fractional part of k .

3. Asymptotic Results. These are stated by the following:

For $r \leq 12$,

$$(3.1) \quad p_{2r}(n) = \frac{(r/3)^{1/2}}{2^r \pi^{r-1} i} \sum_{q=1}^{\infty} A_q(2r, n) q^{r-1} \frac{d^r}{dn^r} \left[\frac{J_1 \left[\frac{2\pi i \left(\frac{r}{3} \left(n - \frac{r}{12} \right) \right)^{1/2}}{q} \right]}{(n - r/12)^{1/2}} \right].$$

For $2r + 1 < 24$,

$$(3.2) \quad p_{2r+1}(n) = \frac{1}{2^{r+1/2} \pi^{r+1} i} \sum_{q=1}^{\infty} A_q(2r+1, n) q^{r+1/2} \frac{d^{r+1}}{dn^{r+1}} \left[\frac{\sinh \left[\frac{\pi \left(\frac{4r+2}{3} \left(n - \frac{2r+1}{24} \right) \right)^{1/2}}{q} \right]}{\left(n - \frac{2r+1}{24} \right)^{1/2}} \right],$$

where

$$A_q(k, n) = \sum_p (\omega_{p,q})^k \exp(-2\pi i p n / q),$$

$\omega_{p,q}$ being certain $24q$ th roots of unity, summation extends over all p , $1 \leq p \leq q$, such that $(p, q) = 1$.

$$(3.3) \quad b(n) \sim 2^{-3/4} (3\pi\zeta(3))^{-1/2} N^{-49/24} \exp\left\{\frac{3}{2}\zeta(3)N^2 + \frac{\pi^2}{24}N + C - \frac{\pi^4}{3456\zeta(3)}\right\}$$

where

$$N = \left(\frac{n}{\zeta(3)}\right)^{1/3}, \quad C = \int_0^\infty \frac{y \log y}{e^{2\pi y} - 1} dy.$$

A similar result for $u(n)$ is given in Wright [12]. Asymptotic results for $u_k(n)$ and $b_k(n)$ are derived from those of $p_k(n)$. In particular,

$$(3.4) \quad u_k(n) \sim (k-1)!(k-2)! \cdots 2! \frac{d^{k(k-1)/2}}{dn^{k(k-1)/2}} (p_k(n)),$$

$$b_4(n) \sim \frac{1}{i\sqrt{3}} \frac{d^3}{dn^3} \left[\frac{J_1\left(2\pi i \left(\frac{1}{3}\left(n - \frac{1}{12}\right)\right)\right)^{1/2}}{\left(n - \frac{1}{12}\right)^{1/2}} \right],$$

which reduces to

$$(3.5) \quad b_4(n) \sim \frac{\pi^7}{81} \sum_{m=0}^{\infty} \frac{\pi^{2m} \left(\frac{1}{3}\left(n - \frac{1}{12}\right)\right)^m}{m! (m+4)!}.$$

Similarly, the asymptotic series for $p_2(n)$ and $p_3(n)$ reduce to

$$(3.6) \quad p_2(n) = \frac{1}{18} \sum_{q=1}^{\infty} A_q(2, n) \sum_{m=0}^{\infty} \frac{\pi^{2m+3} \left(\frac{1}{3}\left(n - \frac{1}{12}\right)\right)^m}{m! (m+2)! q^{2m+3}},$$

$$(3.7) \quad p_3(n) = \frac{1}{2^{3/2} \pi^2} \sum_{q=1}^{\infty} A_q(3, n) q^{3/2} \cdot \left(\left(\frac{A-B}{2} \right) \exp\left(\frac{\pi}{q} \left(2n - \frac{1}{4}\right)^{1/2}\right) - \left(\frac{A+B}{2} \right) \exp\left(-\frac{\pi}{q} \left(2n - \frac{1}{4}\right)^{1/2}\right) \right)$$

where

$$A = \left(\frac{3}{2} + \frac{\pi^2}{q^2} \left(n - \frac{1}{8}\right)\right) / 2 \left(n - \frac{1}{8}\right)^{5/2}, \quad B = \frac{3\pi}{2\sqrt{2} q \left(n - \frac{1}{8}\right)^2}.$$

The coefficients $A_q(n) = A_q(1, n)$ first occurred in Hardy-Ramanujan's asymptotic series for $p(n)$ and were studied by Lehmer [8], Whiteman [10], [11], Rademacher [9] and others. Selberg discovered the formula

$$(3.8) \quad A_q(n) = \frac{1}{4} \left(\frac{k}{3}\right)^{1/2} \sum (-1)^{|l/6|} \exp\left(\frac{\pi i l}{6k}\right).$$

Summation is over integers l such that $(l, 6) = 1$ and $l^2 \equiv 1 - 24n \pmod{24q}$ and $|x|$ denotes the integer nearest to x . Such a formula should also exist for $A_q(k, n)$. Gandhi [3] has studied $A_q(k, n)$ and has obtained some of their properties.

A table of $A_q(n)$ for $q \leq 18$ was given by Hardy and Ramanujan [6]. It is easy to obtain a similar table for $A_q(k, n)$. For example,

$$\begin{aligned}
 A_1(k, n) &= 1, \\
 A_2(k, n) &= (-1)^n, \\
 A_3(k, n) &= 2 \cos\left(\frac{2n\pi}{3} - \frac{k\pi}{18}\right), \\
 A_4(k, n) &= 2 \cos\left(\frac{n\pi}{2} - \frac{k\pi}{8}\right), \\
 A_5(k, n) &= 2 \cos\left(\frac{2n\pi}{5} - \frac{k\pi}{5}\right) + 2 \cos\left(\frac{4n\pi}{5}\right).
 \end{aligned}
 \tag{3.9}$$

For $k = 1$, these reduce to the formulas obtained by Hardy and Ramanujan.

The value of C in (3.3) was obtained by numerical integration of

$$\int_0^\infty \frac{y \log(y)}{e^{2\pi y} - 1} dy = \int_0^1 \frac{y \log y}{e^{2\pi y} - 1} dy + \int_1^\infty \frac{y \log(y)}{e^{2\pi y} - 1} dy.$$

The second integral is easy to evaluate. The substitution $2\pi y = u$ reduces the first integral to $(1/4\pi^2) \int_0^{2\pi} (u \log(u/2\pi)/(e^u - 1)) du$ which is written as

$$\frac{1}{4\pi^2} \int_0^{2\pi} \left(\frac{u}{e^u - 1} - 1 \right) \log\left(\frac{u}{2\pi}\right) du - \frac{1}{2\pi}$$

or as

$$\frac{1}{4\pi^2} \int_0^\epsilon \sum_{k=1}^\infty \frac{u^k}{k!} B_k \log\left(\frac{u}{2\pi}\right) du + \frac{1}{4\pi^2} \int_\epsilon^{2\pi} \left(\frac{u}{e^u - 1} - 1 \right) \log\left(\frac{u}{2\pi}\right) du - \frac{1}{2\pi}$$

where B_k are the well-known Bernoulli numbers. Taking $\epsilon = .1$ in these integrals and adding the second integral, we obtain $C = -.08271057185022965322$.

In the asymptotic calculations for $b(n)$, $\zeta(3) = 1.20205690315959428540$ and $\pi = 3.14159265358979323846$ were used.

NUMERICAL RESULTS

$b_4(n)$

n	Exact Values	Asymptotic values
50	7861151	10769606
100	81121690641	102674670000
150	122723468142192	149502323956804
200	65197255804176049	77556919390336123
250	17410617101196225420	20368778539157277707
299	2563183245773772069521	2961963804368820491881

$b(n)$		
n	Exact Values	Asymptotic values
50	134035641	131450402
100	40271460800459	39515125499612
150	1695070342263449292	1664761688894903178
200	23318651621796353657014	22920072294220827550007
250	150270591063287164944741750	147801858831116594095994117
299	470334179683225117053159476103	462868009280107428736730772768

4. Comments on Numerical Results. Comparisons of exact values with those given by asymptotic formulas for $b_4(n)$ and $b(n)$ indicate that the error in the asymptotic results are quite significant. However, in the case of $p_2(n)$ and $p_3(n)$, the asymptotic results are excellent. We were able to calculate the exact values of $p_2(50)$, $p_2(100)$, $p_3(100)$, and $p_3(200)$ by the asymptotic formula. In fact, three terms of the asymptotic series gave the exact value of $p_2(50)$, while four terms were needed for $p_2(100)$. For $p_3(100)$, four terms of the series were needed, while for $p_3(200)$ it took six terms. The double precision accuracy of twenty eight digits for the CDC 6400 was exceeded by $p_3(299)$. However, the asymptotic series after four terms gave $p_3(299) = 37464893330090346069484200521$ which is larger than the exact value by only 14. This value was not improved by using additional terms up to and including eighteen, which is not surprising as we think the error is in the first four terms and is due to the limit on the accuracy of double precision arithmetic on the CDC 6400.

The first six terms of the series for $p_3(200)$ are listed as are the first four terms of the series for $p_2(100)$, together with their exact values in the following table. In the appendix, we give the exact values of $b_4(n)$, $b(n)$, $p_2(n)$ and $p_3(n)$ for n ranging from one to fifty only. The complete table up to two hundred and ninety nine has been published on the microfiche card that accompanies this issue of the journal. All computations were made on the CDC 6400 at the University of Arizona Computing Center.

First six terms of series for $p_3(200)$

56669336086583184772691.88617

875179402.76032

— 33461.76862

64.90130

4.30801

.57670

sum = 56669336086584059918702.66388

exact value = 56669336086584059918703

First four terms of series for $p_2(100)$

1843645804262.16325

16512.32889

−10.51372

1.70633

sum = 1843645820765.68475

exact value = 1843645820766

APPENDIX

n	$b_4(n)$	$b(n)$	$p_2(n)$	$p_3(n)$
1	1	1	2	3
2	2	2	5	9
3	4	4	10	22
4	7	7	20	51
5	11	12	36	108
6	19	21	65	221
7	29	34	110	429
8	46	56	185	810
9	70	90	300	1479
10	106	143	481	2640
11	156	223	752	4599
12	232	348	1165	7868
13	334	532	1770	13209
14	482	811	2665	21843
15	686	1224	3956	35581
16	971	1834	5822	57222
17	1357	2725	8470	90882
18	1894	4031	12230	1 42769
19	2612	5914	17490	2 21910
20	3592	8638	24842	3 41649
21	4900	12540	35002	5 21196
22	6656	18116	49010	7 88460
23	8980	26035	68150	11 83221
24	12077	37262	94235	17 62462
25	16137	53070	1 29512	26 06604
26	21490	75292	1 77087	38 29437
27	28476	1 06377	2 40840	55 90110
28	37600	1 49738	3 26015	81 11346
29	49422	2 09980	4 39190	117 01998
30	64763	2 93473	5 89128	167 90136
31	84511	4 08734	7 86814	239 64594
32	1 09953	5 67484	10 46705	340 34391
33	1 42539	7 85409	13 86930	481 04069
34	1 84244	10 83817	18 31065	676 79109
35	2 37368	14 91247	24 08658	948 00537
36	3 04996	20 46233	31 57789	1322 30021
37	3 90688	28 00125	41 26070	1836 86994
38	4 99189	38 21959	53 74390	2541 70332
39	6 36059	52 03515	69 78730	3503 71088
40	8 08489	70 67373	90 35539	4812 25800
41	10 25017	95 76142	116 64896	6586 26822
42	12 96595	129 46065	150 18300	8983 65204
43	16 36173	174 62920	192 83830	12213 40197
44	20 60246	235 05298	246 97480	16551 51489
45	25 88440	315 72120	315 51450	22361 38826
46	32 45381	423 21495	402 10481	30120 47298
47	40 60519	566 18358	511 24970	40454 35515
48	50 70574	755 99640	648 54575	54181 00581
49	63 19336	1007 54843	820 88400	72367 23897
50	78 61151	1340 35641	1036 79156	96401 59893

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