On the General Hermite Cardinal Interpolation

By R. Kress

Abstract. A sequence of interpolation series is given which generalizes Whittaker's cardinal function to the case of Hermite interpolation. By integrating the interpolation series, a sequence of new quadrature formulae for $\int_{-\infty}^{\infty} f(x) dx$ is obtained. Derivative-free remainders are stated for these interpolation and quadrature formulae.

Given a function $f: \mathbb{R} \rightarrow \mathbb{C}$ and a real number $h > 0$, the series

$$\begin{aligned} T_h(f)(z) &:= \frac{h}{\pi} \sum_{m=-\infty}^{\infty} \frac{(-1)^m f(mh)}{z - mh} \sin \frac{\pi}{h} z \\ &= \frac{h}{\pi} \sum_{m=-\infty}^{\infty} \frac{f(mh)}{z - mh} \sin \frac{\pi}{h} (z - mh), \quad z \in \mathbb{C}, \end{aligned}$$

is called the *cardinal series* of the function f with respect to the interval h . If the series converges, its sum $T_h(f)$ is called the *cardinal function* or *cardinal interpolation* of the function f . Obviously,

$$T_h(f)(mh) = f(mh), \quad m = 0, \pm 1, \pm 2, \dots,$$

holds. In the case when $f: B \rightarrow \mathbb{C}$ is analytic in a strip $B: \mathbb{R} \times [-a, a] \subset \mathbb{C}$, $a > 0$, and satisfies certain conditions at infinity, a derivative-free remainder for this cardinal interpolation was independently found by Kress [2] and McNamee, Stenger and Whitney [6].

In the present paper, we generalize the cardinal interpolation and give a sequence of *Hermite cardinal interpolations* $T_{p,h}(f)$, $p = 0, 1, 2, \dots$, with

$$T_{p,h}^{(q)}(f)(mh) = f^{(q)}(mh), \quad q = 0, 1, \dots, p, \quad m = 0, \pm 1, \pm 2, \dots.$$

The usual cardinal interpolation is included as the particular case $p = 0$.

In Section 1, we give the explicit form of $T_{p,h}(f)$ and state a derivative-free remainder. Making use of this remainder, we describe a class of functions for which $T_{p,h}(f) = f$.

In Section 2, we apply the general cardinal functions to derive a sequence $I_{p,h}(f)$, $p = 0, 2, 4, \dots$, of integration formulae for infinite integrals involving the derivatives $f^{(q)}(mh)$, $q = 0, 2, \dots, p$, $m = 0, \pm 1, \pm 2, \dots$, which may be regarded as generalizations of the trapezoidal rule. The remainder, given by Goodwin [1], Martensen [4] and McNamee [5] for the trapezoidal rule, is extended to the quadrature formulae $I_{p,h}(f)$.

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The general cardinal interpolation developed in this paper is closely related to the general Hermite trigonometric interpolation of periodic functions [3].

1. Interpolation. Let $p \geq 0$ be an integer and let $h > 0$ be real. Define $p + 1$ entire functions t_q , $q = 0, 1, \dots, p$, by

$$(1.1) \quad t_q(z) := \frac{z^q}{q!} \left(\frac{\sin(\pi/h)z}{(\pi/h)z} \right)^{p+1} \sum_{r=0; r \text{ even}}^{2[(p-q)/2]} a_r(p) \left(\frac{\pi}{h} z \right)^r, \quad z \in \mathbb{C},$$

where the $a_r(p)$ are the coefficients of the Laurent expansion

$$(1.2) \quad \frac{1}{\sin^{p+1} z} = \sum_{r=0; r \text{ even}}^{\infty} \frac{a_r(p)}{z^{1+p-r}}, \quad 0 < |z| < \pi.$$

To avoid indexing difficulties, we do not indicate the dependence of the t_q on p and h .

LEMMA 1.1. For every $r = 0, 1, \dots, p$, the functions t_q , $q = 0, 1, \dots, p$, satisfy

$$(1.3) \quad t_q^{(r)}(0) = \delta_q^r,$$

$$(1.4) \quad t_q^{(r)}(mh) = 0, \quad m = \pm 1, \pm 2, \dots.$$

Proof. From (1.1) and (1.2), we obtain

$$t_q(z) = z^q/q! + z^{p+1}u_q(z), \quad q = 0, 1, \dots, p,$$

with certain entire functions u_q , and (1.3) immediately follows. The relation (1.4) trivially holds.

Definition 1.1. Let $p \geq 0$ be an integer and let $h > 0$ be real. Given a function $f: \mathbb{R} \rightarrow \mathbb{C}$, $f \in C^p(\mathbb{R})$, the p th cardinal series of f with respect to the interval h is defined by

$$(1.5) \quad T_{p,h}(f)(z) := \sum_{m=-\infty}^{\infty} \sum_{q=0}^p f^{(q)}(mh) t_q(z - mh), \quad z \in \mathbb{C}.$$

If the series converges, its sum $T_{p,h}(f)$ is called the p th cardinal function of f .

Lemma 1.1 implies

THEOREM 1.1. The p th cardinal function $T_{p,h}(f)$ is a Hermite interpolation of the function f with equidistant interpolation points

$$(1.6) \quad T_{p,h}^{(q)}(f)(mh) = f^{(q)}(mh), \quad q = 0, 1, \dots, p, \quad m = 0, \pm 1, \dots.$$

The first cardinal series is listed below.

$$\begin{aligned} T_{0,h}(f)(z) &= \frac{h}{\pi} \sum_{m=-\infty}^{\infty} (-1)^m \frac{f(mh)}{z - mh} \sin \frac{\pi}{h} z, \\ T_{1,h}(f)(z) &= \left(\frac{h}{\pi} \right)^2 \sum_{m=-\infty}^{\infty} \left\{ \frac{f(mh)}{(z - mh)^2} + \frac{f'(mh)}{z - mh} \right\} \sin^2 \frac{\pi}{h} z, \\ T_{2,h}(f)(z) &= \left(\frac{h}{\pi} \right)^3 \sum_{m=-\infty}^{\infty} (-1)^m \left\{ \frac{f(mh)}{(z - mh)^3} - \frac{1}{2} \left(\frac{\pi}{h} \right)^2 \frac{f(mh)}{z - mh} \right. \\ &\quad \left. + \frac{f'(mh)}{(z - mh)^2} + \frac{f''(mh)}{z - mh} \right\} \sin^3 \frac{\pi}{h} z. \end{aligned}$$

In the case when the function f is analytic in a strip $B := \mathbb{R} \times [-a, a] \subset \mathbb{C}$, $a > 0$, we shall give a sufficient condition on the convergence of the p th cardinal

series of f and shall obtain a representation of the remainder

$$(1.7) \quad R_{p,h}(f) := f - T_{p,h}(f).$$

LEMMA 1.2. *Let the function f be analytic in the strip $B := \mathbb{R} \times [-a, a] \subset \mathbb{C}$, $a > 0$. Then*

$$(1.8) \quad \begin{aligned} \chi_n(z)f(z) - \sum_{m=-n}^n \sum_{q=0}^p f^{(q)}(mh)t_q(z-mh) \\ = \frac{1}{2\pi i} \sin^{p+1} \frac{\pi}{h} z \int_{C_n} \frac{f(\zeta) d\zeta}{(\zeta - z) \sin^{p+1}(\pi/h)\zeta}, \quad z \notin C_n, \end{aligned}$$

where C_n denotes the boundary of a rectangle $B_n := [-(n + \frac{1}{2})h, (n + \frac{1}{2})h] \times [-a, a] \subset B$ and where χ_n denotes the characteristic function of B_n with $\chi_n(z) = 1$, $z \in B_n$ and $\chi_n(z) = 0$, $z \notin B_n$.

Proof. The function $F: B_n \rightarrow \mathbb{C}$, defined by

$$(1.9) \quad F(z) := \frac{1}{\sin^{p+1}(\pi/h)z} \left(f(z) - \sum_{m=-n}^n \sum_{q=0}^p f^{(q)}(mh)t_q(z-mh) \right), \quad z \in B_n,$$

is analytic. Hence, by Cauchy's theorem,

$$(1.10) \quad \chi_n(z)F(z) = \frac{1}{2\pi i} \int_{C_n} \frac{F(\zeta)}{\zeta - z} d\zeta, \quad z \notin C_n.$$

Using the identities

$$\int_{C_n} \frac{d\zeta}{(\zeta - z)(\zeta - mh)^{q+1}} = \frac{-2\pi i}{(z - mh)^{q+1}} (1 - \chi_n(z)), \quad z \notin C_n,$$

$q = 0, 1, \dots, p$, $m = 0, \pm 1, \dots, \pm n$, we substitute (1.9) into (1.10) and obtain (1.8).

THEOREM 1.2. *Let the function f be analytic and bounded in the strip $B := \mathbb{R} \times [-a, a] \subset \mathbb{C}$, $a > 0$, and let*

$$(1.11) \quad \int_{-\infty-ia}^{\infty-ia} |f(z)|^2 ds < \infty, \quad \int_{-\infty+ia}^{\infty+ia} |f(z)|^2 ds < \infty.$$

Then, for arbitrary $p \geq 0$ and $h > 0$, the p th cardinal series of f with respect to the interval h is locally uniformly convergent for all $x \in \mathbb{R}$ and the remainder (1.7) is given by

$$(1.12) \quad \begin{aligned} R_{p,h}(f)(x) = \frac{1}{2\pi i} \sin^{p+1} \frac{\pi}{h} x \left\{ \int_{-\infty-ia}^{\infty-ia} \frac{f(\zeta) d\zeta}{(\zeta - x) \sin^{p+1}(\pi/h)\zeta} \right. \\ \left. - \int_{-\infty+ia}^{\infty+ia} \frac{f(\zeta) d\zeta}{(\zeta - x) \sin^{p+1}(\pi/h)\zeta} \right\}, \quad x \in \mathbb{R}, \end{aligned}$$

and bounded by

$$(1.13) \quad \begin{aligned} |R_{p,h}(f)(x)| \leq \frac{1}{2(\pi a)^{1/2}} \left(\frac{\sin(\pi/h)x}{\sinh(\pi/h)a} \right)^{p+1} \left\{ \left(\int_{-\infty-ia}^{\infty-ia} |f(\zeta)|^2 ds \right)^{1/2} \right. \\ \left. + \left(\int_{-\infty+ia}^{\infty+ia} |f(\zeta)|^2 ds \right)^{1/2} \right\}, \quad x \in \mathbb{R}. \end{aligned}$$

Proof. Let f be bounded by M . Then we estimate the integrals

$$\left| \int_{\pm(n+1/2)h-ia}^{\pm(n+1/2)h+ia} \frac{f(\zeta) d\zeta}{(\zeta - x) \sin^{p+1}(\pi/h)\zeta} \right| \leq \frac{2Ma}{(n + \frac{1}{2})h - |x|}.$$

Thus, by Lemma 1.2,

$$(1.14) \quad f(x) = \lim_{n \rightarrow \infty} \left[\sum_{m=-n}^n f^{(q)}(mh) t_q(x - mh) + \frac{1}{2\pi i} \sin^{p+1} \frac{\pi}{h} x \left\{ \int_{-(n+1/2)h-ia}^{+(n+1/2)h-ia} \frac{f(\zeta) d\zeta}{(\zeta - x) \sin^{p+1}(\pi/h)\zeta} - \int_{-(n+1/2)h+ia}^{+(n+1/2)h+ia} \frac{f(\zeta) d\zeta}{(\zeta - x) \sin^{p+1}(\pi/h)\zeta} \right\} \right], \quad x \in \mathbf{R},$$

where convergence is locally uniform for all $x \in \mathbf{R}$. Upon noting that

$$\left| \sin \frac{\pi}{h} \zeta \right| \geq \sinh \frac{\pi}{h} a, \quad \zeta = \xi \pm ia,$$

and

$$\int_{-\infty \pm ia}^{\infty \pm ia} ds / |\zeta - x|^2 = \pi/a,$$

Schwarz's inequality yields

$$\left| \int_{-\infty \pm ia}^{\infty \pm ia} \frac{f(\zeta) d\zeta}{(\zeta - x) \sin^{p+1}(\pi/h)\zeta} \right| \leq \frac{\sqrt{\pi}}{\sqrt{a} \sinh^{p+1}(\pi/h)a} \left(\int_{-\infty \pm ia}^{\infty \pm ia} |f(\zeta)|^2 ds \right)^{1/2}, \quad x \in \mathbf{R}.$$

Hence, letting $n \rightarrow \infty$ in (1.14) completes the proof.

Remark. From the bound (1.13), we easily see that

$$\lim_{h \rightarrow 0} T_{p,h}(f)(x) = f(x), \quad p = 0, 1, \dots,$$

and

$$\lim_{p \rightarrow \infty} T_{p,h}(f)(x) = f(x), \quad h > 0, \quad \sinh \frac{\pi}{h} a > 1,$$

where convergence is uniform for all $x \in \mathbf{R}$. In both cases $h \rightarrow 0$, p fixed and $p \rightarrow \infty$, h fixed, the convergence is exponential.

The following theorem describes a class of functions for which $T_{p,h}(f) = f$ is true.

THEOREM 1.3. Let f be an entire function, such that

$$(1.15) \quad |f(z)| \leq ce^{\rho|y|}, \quad z = x + iy \in \mathbf{C},$$

with real numbers $c \geq 0$ and $0 \leq \rho < (p+1)\pi/h$. Then the p th cardinal series for f with respect to h is locally uniformly convergent for all $z \in \mathbf{C}$, and the identity

$$(1.16) \quad T_{p,h}(f)(z) = f(z), \quad z \in \mathbf{C},$$

holds.

Proof. By (1.15) we have

$$\left| \frac{f(\zeta)}{\sin^{p+1}(\pi/h)\zeta} \right| \leq \frac{ce^{\rho|\eta|}}{\sinh^{p+1}(\pi/h)|\eta|} \\ = O\left(\exp\left[-\left[(p+1)\frac{\pi}{h} - \rho\right]|\eta|\right]\right), \quad \zeta = \xi + i\eta \in \mathbb{C},$$

as $|\eta| \rightarrow \infty$, and therefore

$$\lim_{a \rightarrow \infty} \int_{-(n+1/2)h \pm ia}^{(n+1/2)h \pm ia} \frac{f(\zeta) d\zeta}{(\zeta - x) \sin^{p+1}(\pi/h)\zeta} = 0,$$

that is, by Lemma 1.2,

$$(1.17) \quad f(z) = \sum_{m=-n}^n \sum_{q=0}^p f^{(q)}(mh) t_q(z - mh) \\ + \frac{1}{2\pi i} \sin^{p+1} \frac{\pi}{h} z \left\{ \int_{(n+1/2)h - i\infty}^{(n+1/2)h + i\infty} \frac{f(\zeta)}{(\zeta - z) \sin^{p+1}(\pi/h)\zeta} d\zeta \right. \\ \left. - \int_{-(n+1/2)h - i\infty}^{-(n+1/2)h + i\infty} \frac{f(\zeta)}{(\zeta - z) \sin^{p+1}(\pi/h)\zeta} d\zeta \right\}, \\ z = x + iy \in \mathbb{C}$$

for all n with $(n + \frac{1}{2})h > |x|$. Making use of

$$\left| \sin \frac{\pi}{h} \zeta \right| = \cosh \frac{\pi}{h} \eta \geq \frac{1}{2} \exp\left[\frac{\pi}{h} |\eta|\right], \quad \zeta = \pm(n + \frac{1}{2})h + i\eta,$$

we conclude that

$$\left| \int_{\pm(n+1/2)h - i\infty}^{\pm(n+1/2)h + i\infty} \frac{f(\zeta) d\zeta}{(\zeta - z) \sin^{p+1}(\pi/h)\zeta} \right| \leq \frac{2^{p+2}c}{(p+1)(\pi/h) - \rho} \frac{1}{(n + \frac{1}{2})h - |x|}.$$

Letting $n \rightarrow \infty$ in (1.17), the assertion of the theorem follows.

Example. If we choose $f(z) := e^{i\rho z}$, $z \in \mathbb{C}$, $0 \leq \rho < (p+1)\pi/h$, we obtain the local uniform convergent expansion

$$(1.18) \quad e^{i\rho z} = \sum_{m=-\infty}^{\infty} \sum_{q=0}^p (i\rho)^q e^{i\rho mh} t_q(z - mh), \quad z \in \mathbb{C}.$$

Setting $\rho := r\pi/h$, $r = 0, 1, \dots, p$, we derive

$$(1.19) \quad \exp\left[ir \frac{\pi}{h} z\right] = \sum_{m=-\infty}^{\infty} \sum_{q=0}^p \left(ir \frac{\pi}{h}\right)^q (-1)^m t_q(z - mh), \quad z \in \mathbb{C}.$$

2. Numerical Integration. We integrate the p th cardinal series of f termwise and obtain the series

$$(2.1) \quad I_{p,h}(f) := h \sum_{m=-\infty}^{\infty} \sum_{q=0; q \text{ even}}^p \left(\frac{h}{2\pi}\right)^q a_{q,p} f^{(q)}(mh)$$

with the weights

$$(2.2) \quad a_{q,p} := \frac{1}{2\pi} \left(\frac{2\pi}{h}\right)^{q+1} \int_{-\infty}^{\infty} t_q(x) dx, \quad q = 0, 2, \dots, p.$$

If q is odd, the integral (2.2) vanishes, since in this case the function t_q is odd. The series (2.1) may be regarded as generalizations of the trapezoidal rule approximation for the integral $\int_{-\infty}^{\infty} f(x) dx$.

In order to derive simple recurrence formulae for the weights $a_{q,p}$, we state

THEOREM 2.1. *Let p be even. Then the weights $a_{q,p}$ are uniquely determined by the identity*

$$(2.3) \quad \sum_{q=0; q \text{ even}}^p a_{q,p} z^q = \prod_{q=1}^{p/2} (1 + (z/q)^2),^* \quad z \in \mathbb{C}.$$

Proof. Integrating (1.19), we have

$$\int_0^h \exp \left[ir \frac{\pi}{h} x \right] dx = \sum_{q=0; q \text{ even}}^p \left(ir \frac{\pi}{h} \right)^q \int_{-\infty}^{\infty} t_q(x) dx, \quad r = 0, 2, \dots, p,$$

thus, we are led to the system of $p/2 + 1$ linear equations

$$(2.4) \quad \begin{aligned} a_{0,p} &= 1, \\ \sum_{q=0; q \text{ even}}^p (ir)^q a_{q,p} &= 0, \quad r = 1, \dots, p/2. \end{aligned}$$

Since the determinant D_p of (2.4) is a Vandermonde determinant with

$$D_p = i^{(p/2)(p/2+1)} \prod_{q>r=0}^{p/2} (q^2 - r^2) \neq 0,$$

the weights $a_{q,p}$ are uniquely determined by the system (2.4).

Define a polynomial P_p of degree p by

$$P_p(z) := \sum_{q=0; q \text{ even}}^p a_{q,p} z^q, \quad z \in \mathbb{C}.$$

Then (2.4) reads

$$\begin{aligned} P_p(0) &= 1, \\ P_p(ri) &= 0, \quad r = \pm 1, \dots, \pm p/2. \end{aligned}$$

Hence $P_p(z) = \prod_{q=1}^{p/2} (1 + (z/q)^2)$, and (2.3) is established.

From (2.3) it follows that

$$(1 + (2z/p)^2) \sum_{q=0; q \text{ even}}^{p-2} a_{q,p-2} z^q = \sum_{q=0; q \text{ even}}^p a_{q,p} z^q, \quad p = 2, 4, \dots.$$

Comparing the coefficients, we find the desired recursion formulae

$$(2.5) \quad \begin{aligned} a_{0,p} &= 1, \quad p = 0, 2, \dots, \\ a_{q,p} &= a_{q,p-2} + (2/p)^2 a_{q-2,p-2}, \quad q = 2, 4, \dots, p-2, \quad p = 2, 4, \dots, \\ a_{p,p} &= \frac{1}{((p/2)!)^2}, \quad p = 0, 2, \dots. \end{aligned}$$

Using (2.5), we obtain

* $\prod_{q=1}^{p/2}$ is to be interpreted as unity when $p = 0$.

$$I_{0,h}(f) = h \sum_{m=-\infty}^{\infty} f(mh),$$

$$I_{2,h}(f) = h \sum_{m=-\infty}^{\infty} f(mh) + \frac{h^3}{4\pi^2} \sum_{m=-\infty}^{\infty} f''(mh),$$

$$I_{4,h}(f) = h \sum_{m=-\infty}^{\infty} f(mh) + \frac{5h^3}{16\pi^2} \sum_{m=-\infty}^{\infty} f''(mh) + \frac{h^5}{64\pi^4} \sum_{m=-\infty}^{\infty} f''''(mh).$$

In the case of odd p , we integrate (1.19) for $r = 0, 2, \dots, p-1$ and get the system of $(p+1)/2$ linear equations

$$(2.6) \quad \begin{aligned} a_{0,p} &= 1, \\ \sum_{q=0; q \text{ even}}^{p-1} (ir)^q a_{q,p} &= 0, \quad r = 1, \dots, (p-1)/2. \end{aligned}$$

Comparing (2.6) with (2.4), we see that $a_{q,p} = a_{q,p-1}$, $q = 0, 2, \dots, p-1$. Thus, if p is odd, $I_{p,h}(f) = I_{p-1,h}(f)$ is valid. Therefore, we may restrict ourselves to even p .

In the case when the function f is analytic, we state a sufficient condition on the convergence of the series (2.1) and give a remainder in the following:

THEOREM 2.2. *Let the function f be analytic in the strip $B := \mathbf{R} \times [-a, a] \subset \mathbf{C}$, $a > 0$, let $f(z) \rightarrow 0$, $z = x + iy$ as $x \rightarrow \pm \infty$ uniformly for all $-a \leq y \leq a$ and let*

$$(2.7) \quad \int_{-\infty - ia}^{\infty - ia} |f(z)| ds < \infty, \quad \int_{-\infty + ia}^{\infty + ia} |f(z)| ds < \infty.$$

Then $\int_{-\infty}^{\infty} f(x) dx$ exists, and the series (2.1) is convergent for even $p \geq 0$ and $h > 0$. The remainder

$$(2.8) \quad E_{p,h}(f) := \int_{-\infty}^{\infty} f(x) dx - I_{p,h}(f)$$

is given by

$$(2.9) \quad \begin{aligned} E_{p,h}(f) &= \frac{1}{(2i)^{p+1}} \sum_{q=0}^{p/2} (-1)^q \binom{p+1}{q} \left\{ \int_{-\infty + ia}^{\infty + ia} \frac{\exp[i(p+1-2q)(\pi/h)\xi]}{\sin^{p+1}(\pi/h)\xi} f(\xi) d\xi \right. \\ &\quad \left. - \int_{-\infty - ia}^{\infty - ia} \frac{\exp[-i(p+1-2q)(\pi/h)\xi]}{\sin^{p+1}(\pi/h)\xi} f(\xi) d\xi \right\} \end{aligned}$$

with the bound

$$(2.10) \quad |E_{p,h}(f)| \leq \frac{\exp[-(\pi/h)a]}{2 \sinh^{p+1}(\pi/h)a} \left(\int_{-\infty + ia}^{\infty + ia} |f(z)| ds + \int_{-\infty - ia}^{\infty - ia} |f(z)| ds \right).$$

Proof. From the assumption (2.7), we see by Cauchy's theorem that $\int_{-\infty}^{\infty} f(x) dx$ exists.

Using the identity

$$\sin^{p+1} \frac{\pi}{h} x = \frac{\exp[i(p+1)(\pi/h)x]}{(2i)^{p+1}} \sum_{q=0}^{p+1} (-1)^q \binom{p+1}{q} \exp[-2iq(\pi/h)x],$$

we deduce from the residue theorem that

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin^{p+1}(\pi/h)x}{x - \zeta} dx = \frac{\exp[i(p+1)(\pi/h)\zeta]}{(2i)^p} \sum_{q=0}^{p/2} (-1)^q \binom{p+1}{q} \exp[-2iq(\pi/h)\zeta],$$

$$\zeta = \xi + ia,$$

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin^{p+1}(\pi/h)x}{x - \zeta} dx = \frac{\exp[-i(p+1)(\pi/h)\zeta]}{(2i)^p} \sum_{q=0}^{p/2} (-1)^q \binom{p+1}{q} \exp[2iq(\pi/h)\zeta],$$

$$\zeta = \xi - ia.$$

From this the estimates

$$(2.11) \quad \left| \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin^{p+1}(\pi/h)x}{x - \zeta} dx \right| \leq \frac{\exp[-(p+1)(\pi/h)a]}{2^p} \sum_{q=0}^{p/2} \binom{p+1}{q} \exp[2q(\pi/h)a]$$

$$\leq \exp[-(\pi/h)a], \quad \zeta = \xi \pm ia,$$

follow.

Integrating (1.8) over $(-\infty, \infty)$ and interchanging the order of integration, we obtain

$$\int_{-(n+1/2)h}^{(n+1/2)h} f(x) dx = h \sum_{m=-n}^n \sum_{q=0}^p \left(\frac{h}{2\pi} \right)^q a_{q,p} f^{(q)}(mh)$$

$$+ \frac{1}{2\pi i} \int_{c_n} \frac{f(\zeta)}{\sin^{p+1}(\pi/h)\zeta} \left(\int_{-\infty}^{\infty} \frac{\sin^{p+1}(\pi/h)x}{\zeta - x} dx \right) d\zeta.$$

With the aid of (2.11), we can estimate

$$\left| \frac{1}{\pi} \int_{\pm(n+1/2)h-ia}^{\pm(n+1/2)h+ia} \frac{f(\zeta)}{\sin^{p+1}(\pi/h)\zeta} \left(\int_{-\infty}^{\infty} \frac{\sin^{p+1}(\pi/h)x}{\zeta - x} dx \right) d\zeta \right|$$

$$\leq 2a \max_{\eta \in [-a, a]} |f(\pm(n + \frac{1}{2})h + i\eta)|,$$

and

$$\left| \frac{1}{\pi} \int_{-(n+1/2)h \pm ia}^{(n+1/2)h \pm ia} \frac{f(\zeta)}{\sin^{p+1}(\pi/h)\zeta} \left(\int_{-\infty}^{\infty} \frac{\sin^{p+1}(\pi/h)x}{\zeta - x} dx \right) d\zeta \right|$$

$$\leq \frac{\exp[-(\pi/h)a]}{\sinh^{p+1}(\pi/h)a} \int_{-(n+1/2)h \pm ia}^{(n+1/2)h \pm ia} |f(\zeta)| ds.$$

Thus, by the assumptions on f , letting $n \rightarrow \infty$ completes the proof.

Remark. From the bound (2.10), we have

$$\lim_{h \rightarrow 0} I_{p,h}(f) = \int_{-\infty}^{\infty} f(x) dx, \quad p = 0, 2, \dots,$$

and

$$\lim_{p \rightarrow \infty} I_{p,h}(f) = \int_{-\infty}^{\infty} f(x) dx, \quad h > 0, \sinh \frac{\pi}{h} a > 1,$$

where convergence is exponential in both cases $h \rightarrow 0$, fixed and $p \rightarrow \infty$, h fixed.

Universität Göttingen
Institut für Numerische und Angewandte Mathematik
D-34, Göttingen, Germany

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