## **Chebyshev Approximations for the Psi Function**\*

By W. J. Cody, Anthony J. Strecok and Henry C. Thacher, Jr.

Abstract. Rational Chebyshev approximations to the psi (digamma) function are presented for  $.5 \le x \le 3.0$ , and  $3.0 \le x$ . Maximum relative errors range down to the order of  $10^{-20}$ .

1. Introduction. The principal mathematical properties of the psi (digamma) function

(1) 
$$\psi(z) = d[\ln \Gamma(z)]/dz = \Gamma'(z)/\Gamma(z)$$

are summarized by Davis [2] and Luke [3]. For real arguments, the function is traditionally computed using either the classical power series expansion

(2) 
$$\psi(1+z) = -\gamma + \sum_{n=2}^{\infty} (-1)^n \zeta(n) z^{n-1}, \quad |z| < 1,$$

or the asymptotic expansion

(3) 
$$\psi(z) \sim \ln(z) - \frac{1}{2z} - \sum_{n=1}^{\infty} \frac{B_{2n}}{2nz^{2n}}$$

with the recurrence relation

(4) 
$$\psi(z+1) = \psi(z) + 1/z$$
.

The reflection formula

(5) 
$$\psi(1-z) = \psi(z) + \pi \cot(\pi z)$$

allows computation for negative arguments. (For complex arguments, see Luke [4].)

Recently, Luke [3] presented an expansion of  $\psi(x + 3)$ ,  $0 \le x \le 1$ , in Chebyshev polynomials, 17 coefficients being required to compute the function with an absolute error on the order of  $10^{-20}$ . For computations outside of the primary range, it is still necessary to use one or more of the relations (3), (4) and (5) in addition to Luke's expansion. In this note, we present rational Chebyshev approximations which allow direct computation of  $\psi(x)$  for any  $x \ge .5$  with various choices of maximum *relative* error, including some of the order of  $10^{-20}$ . For x < .5, either (4) or (5) is still required in conjunction with our approximations.

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TABLE I  $\varepsilon_{ik} = -100 \log_{10} \max \left| \frac{\psi(x) - \psi_{ik}(x)}{\psi(x)} \right|$ .5 < X < 3. \*\*\*\*\* ø 214\* 435\* 674\* 56Ø 930\* 1199\* 148Ø\* 1771\* 2071\* 3. < X ø 114Ø 1257 1366 717\* 1060\* 1353\* 1Ø18 1617\* 174Ø 1860\* 2088\* 

\*Coefficients for these approximations only are given in Tables II and III.

2. Generation of the Approximations. The approximation forms used are

 $\psi_{ik}(x) = (x - x_0)R_{ik}(x), \quad .5 \leq x \leq 3.0,$ 

and

 $\psi_{ik}(x) = \ln(x) - 1/2x + R_{ik}(1/x^2), \quad 3.0 \leq x,$ 

where  $x_0$  is the positive zero of  $\psi(x)$ ,

 $x_0 = 1.46163\ 21449\ 68362\ 34126\ 26595\ 42325\ 72132\ 5\ \cdots$ 

and the  $R_{jk}$  are rational functions of degree j in the numerator and k in the denominator. Our value of  $x_0$ , determined in 40S arithmetic by applying the secant method to a Taylor series expansion of  $\psi(x)$  about x = 1.5, agrees with the 33D value given by Wrench [5].

The approximations were generated with standard versions of the Remes algorithm [1] in 25S arithmetic on a CDC 3600 computer, using values computed from variations of the methods described in Section 1. A Taylor series expansion about  $x_0$  was used to compute  $\psi(x)/(x - x_0)$  for arguments close to  $x_0$ . For other small arguments, the computation was based upon Eq. (2), using the form

$$\psi(1+z) = -\gamma - \sum_{n=2}^{\infty} (\zeta(n) - 1)(-z)^{n-1} + z/(1+z), \qquad |z| \leq \frac{1}{2},$$

## TABLE II

x       y       y       y       y         1       B       1.2456       (60)       1.6546       (-61)         1       B       1.2456       (60)       1.6546       (-61)         2       B       1.701576       (60)       2.667376       (-62)         2       B       1.701576       (60)       2.667376       (-62)         2       B       1.29563       (20)       2.291607       (-62)         3       H       9.18265       (60)       3.93668       191       (-62)         1       1.02639       2255       (60)       3.935686       191       (-62)         2       3.1318       HB       (00)       3.91656       1.640       (00)         3       3.94931       823       (-62)       1.060666       666       (01)         3       3.1318       HB       (01)       3.4119       271636       (01)         3       3.31393       31179       661405       (01)       1.34119       271636       (01)         3       3.8139       3179       66214       (20)       2.2529       71718       (62)         3       3.13363117976				$\psi(\mathbf{x}) = (\mathbf{x} - \mathbf{x}_0)$	$\sum_{j=0}^{n} p_{j} x^{j} / j$	$\sum_{j=0}^{n} q_{j} x^{j},  .5 \leq x \leq 3.0$
n       1 $P_1$ $q_1$ 1       1.2456       (20)       1.5946       (-61)         1       2.2307       (-01)       1.0000       (20)         2       6       1.78157       6       (00)       2.6877       (-22)         2       8.23517       9       (00)       2.6877       (-22)       (200)       (200)         2       8.24324       7       (-22)       1.00000       0       (200)       (200)         3       9.491896       925       (00)       7.12335       364       (00)         2       3.1318       460       (00)       7.12335       364       (01)         3       3.9431       823       (01)       5.23146       54092       7       (-64)         3       3.9431       823       (01)       3.4119       27163       (01)         3       3.81336       3177       6       (02)       1.000000       0       (02)         4       6       2.3421       30617       6       (02)       1.000000       0       (02)         3       3.81336       31179       66       (02)       2.25257       74317 <th>**</th> <th>ŦŦ</th> <th>****</th> <th>****</th> <th>*******</th> <th><u>⋠</u>⋇⋇⋇⋇⋩⋩⋩⋩⋩⋩⋩⋩⋩⋩⋩⋩⋩⋩⋩⋩⋩⋩⋩⋩⋩⋩⋩⋩⋩</th>	**	ŦŦ	****	****	*******	<u>⋠</u> ⋇⋇⋇⋇⋩⋩⋩⋩⋩⋩⋩⋩⋩⋩⋩⋩⋩⋩⋩⋩⋩⋩⋩⋩⋩⋩⋩⋩⋩
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5						
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23.32425 $0.6815$ $0.6421$ 3.80425 $0.6421$ $0.6421$ $0.6421$ $0.6421$ $0.6421$ 37.5456896431969 $0.011$ 1.82176 $0.2814$ $2.66$ $0.021$ 44.3387592564 $704$ $0.601$ 1.82176 $0.2814$ $2.66$ $0.021$ 51.3559474028651 $(-62)$ 1.00000 $0.0000$ $0.0000$ $0.0000$ $0.0000$ 601.3056026982789694 $0.031$ 1.90831 $0.7659$ $630602$ 2 $0.033$ 23.6335184680649872 $0.033$ 1.90831 $0.7659$ $630602$ 2 $0.033$ 31.1864520071342521 $0.033$ 2.21000079924782975 $0.033$ 41.4244158508402850 $0.023$ 5.2075277146716184 $0.023$ 54.777628286426274 $0.000$ 4.484525734298264 $0.0000$ $0.0000$ $0.00000$ 61.3524999667726346383 $0.041$ 1.976857463 $0.4673$ $0.00000$ $0.00000$ $0.00000$ $0.00000$ 701.3524999667726346383 $0.041$ 1.976857463 $0.421$ $0.00000$ $0.00000$ $0.00000$ $0.00000$ 71.3524999667726346383 $0.041$ 1.976857463 $0.421$ $0.041$ <t< td=""><td></td><td>1</td><td>4.54652 99037</td><td>3Ø1</td><td>(02)</td><td></td></t<>		1	4.54652 99037	3Ø1	(02)	
3       7.54565 96431 989       (90)       1.7675 92514 731       (91)         4       4.33875 92564 704       (90)       2.77171 52851 731       (91)         5       1.325594 74028 651       (-02)       1.000000 00000 0000       (00)       (90)         6       0       1.30560 26982 78969 4       (03)       1.90831 07659 63000 2       (03)         2       3.63351 84680 64987 2       (03)       3.64127 34907 93806 0       (03)         3       1.18645 20071 34252 3       (03)       2.21000 79924 78297 5       (03)         4       4.77762 82804 26274 0       (00)       4.48452 57342 98264 0       (01)         5       4.77762 82804 26274 0       (00)       4.48452 57342 98264 0       (01)         6       8.95385 02298 19699 9       (-03)       1.000000 00000 0000 0       00000 0       (00)         7       0       1.35249 99667 72634 6383       (04)       6.93891 11753 76344 4376       (-07)         1       4.52856 01699 54728 9655       (04)       1.97685 74263 04673 6421       (04)         2       4.53915 25149 40693 3532       (03)       9.00196 66074 85517 0271       (03)         5       2.40680 32474 35720 1831       (02)       1.24474 77785 67085 6439       (03)		2	3.32425 06881	581	( 64 )	
5       1.35594       74028       651       (-02)       1.000000       000000       0000       0000       0000       0000       0000       0000       0000       000000       0000000       000000       000000		3	1,22022 00431	969 געקר	( 66)	2 77171 52851 731 (Ø1)
6       1.30560 26982 78969 4       (03)       6.91091 68271 45328 9       (-06)         1       4.13810 16126 90130 0       (03)       1.90831 07659 63060 2       (03)         2       3.63351 84680 64987 2       (03)       3.64127 34907 93806 0       (03)         3       1.18645 20071 34252 3       (03)       2.21000 79924 78297 5       (03)         4       1.42441 58508 40285 0       (02)       5.20752 77146 71618 4       (02)         5       4.77762 82804 26274 0       (00)       4.46452 57342 98264 0       (01)         6       8.95385 02298 19699 9       (-03)       1.00000 000000 00000 00000 0       (00000 0         7       0       1.35249 99667 72634 6383       (04)       6.93891 11753 76344 4376       (-07)         1       4.52856 01699 54728 9655       (04)       1.97685 74263 04673 6421       (04)         2       4.51351 68469 73666 2555       (04)       4.12551 60835 35383 2333       (04)         3       1.65290 11318 58261 0168       (04)       2.93002 87119 93268 1918       (04)         4       3.32915 25149 40693 5532       (03)       9.08196 66074 85517 0271       (03)         5       2.40680 32474 35720 1831       (02)       1.24474 77785 67085 6039       (03) <t< td=""><td></td><td>5</td><td>1.35594 74028</td><td>651</td><td>(-02)</td><td></td></t<>		5	1.35594 74028	651	(-02)	
$ \begin{array}{c c c c c c c c c c c c c c c c c c c $		-				
$ \begin{array}{c ccccccccccccccccccccccccccccccccccc$	6	ø	1.30560 26982	78969 4	(Ø3)	6.91091 68271 45328 9 (-06)
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31.1864520071 $34252$ 1 $037$ $221000$ $7924$ $78297$ $5$ $0037$ 41.4244158508 $40285$ 0 $027$ $5.20752$ $77146$ $71618$ $4$ $(02)$ 54.7776282804 $26274$ 0 $007$ $007$ $4.48452$ $57342$ $98264$ 0 $011$ 68.95385 $02298$ $19699$ 9 $(-03)$ $1.00000$ $000000$ $00000$ $00000$ $0000000$ $0000000$ <t< td=""><td></td><td>12</td><td>3.63351 84680</td><td>64987 2</td><td>· (Ø3)</td><td>3.64127 34967 93866 6 [63]</td></t<>		12	3.63351 84680	64987 2	· (Ø3)	3.64127 34967 93866 6 [63]
1.42441 $5503$ $40235$ $0$ $(02)$ $4.48452$ $57342$ $98264$ $0$ $(01)$ $5$ $4.77762$ $82894$ $426274$ $0$ $0$ $0$ $4.48452$ $57342$ $98264$ $0$ $0$ $7$ $0$ $1.35249$ $99667$ $72634$ $6383$ $041$ $6.93891$ $11753$ $76344$ $4376$ $(-07)$ $1$ $4.52856$ $01699$ $54728$ $9655$ $041$ $1.97685$ $74263$ $04673$ $6421$ $041$ $2$ $4.51351$ $68469$ $73666$ $2555$ $041$ $4.12551$ $60835$ $35382$ $2333$ $044$ $3$ $3.2915$ $25149$ $40693$ $5532$ $031$ $9.08196$ $66074$ $85517$ $0271$ $033$ $5$ $2.40680$ $32474$ $35720$ $1831$ $023$ $0.24474$ $77785$ $67085$ $6039$ $033$ $5$ $5.15778$ $92000$ $13968$ $4710$ $000$ $6.74291$ $29516$ $37635$ $033773$ $031$ $7$ $6.22835$ $06918$ $98474$ $5826$ $(-03)$ $1.000000$ $00000$ $00000$ $00000$ $00000$ $00000$ $8$ $0$ $1.65856$ $95029$ $76102$ $23207$ $66$ $055$ $6.41552$ $23783$ $57622$ $59962$ $50$ $-685$ $1$ $5.80413$ $12783$ $53756$ $99927$ $83$ $055$ $2.42421$ $85002$ $01798$ $52519$ $81$		3	1.18645 20071	34252 3	( ( ( ( ( ( ( ( ( ( ( ( ( ( ( ( ( ( ( (	$5 20752 771 \mu 6 71618 \mu$ ( $02)$
6       8.95385       02298       19699       9       (-03)       1.00000       00000       0       (00)         7       0       1.35249       99667       72634       6383       (04)       6.93891       11753       76344       4376       (-07)         1       4.52856       01699       54728       9655       (04)       1.97685       74263       04673       6421       (04)         2       4.51351       68469       73666       2555       (04)       4.12551       60835       35382       2333       (04)         3       1.85290       11818       58261       0168       (04)       2.93902       87119       93268       1918       (04)         4       3.32915       25149       40693       5532       (03)       9.08196       66074       85517       0271       (03)         5       2.40680       32474       35720       1831       (02)       1.24474       77785       67085       6039       (03)         6       5.15778       92000       13908       4710       (05)       6.74291       29516       3773       (01)         7       6.22835       05918       9474		5	1.42441 50500 4 77762 82804	98274 B	( 88)	4,48452 57342 98264 Ø (Ø1)
7       Ø       1.35249       99667       72634       6383       (Ø4)       6.93891       11753       76344       4376       (-Ø7)         1       4.52856       Ø1699       54728       9655       (Ø4)       1.97685       74263       Ø4673       6421       (Ø4)         2       4.51351       68469       73666       2555       (Ø4)       4.12551       6Ø835       35382       2333       (Ø4)         3       1.8529Ø       11818       58261       Ø168       (Ø4)       2.939Ø2       87119       93268       1918       (Ø4)         4       3.32915       25149       40693       5532       (Ø3)       9.Ø8196       66Ø74       85517       Ø271       (Ø3)         5       2.4Ø68Ø       32474       3572Ø       1831       (Ø2)       1.24474       77785       67085       6339       (Ø3)         6       5.15778       92ØØØ       139Ø8       471Ø       (Ø0)       6.74291       29516       37859       3773       (Ø1)         7       6.22835       Ø69927       83       (Ø5)       2.44241       85Ø2Ø       Ø1798       52519       81       (Ø5)         1       5.8Ø413		a	8,95385 02298	19699 9	(-Ø3)	
$ \begin{array}{c ccccccccccccccccccccccccccccccccccc$		<b> </b>				
1       4.52856 Ø1699 54728 9655       (Ø4)       1.97685 74263 Ø4673 6421       (Ø4)         2       4.51351 68469 73666 2555       (Ø4)       4.12551 6Ø835 35383 2333       (Ø4)         3       1.8529Ø 11818 58261 Ø168       (Ø4)       2.93902 87119 93268 1918       (Ø4)         4       3.32915 25149 4Ø693 5532       (Ø3)       9.Ø8196 66074 85517 Ø271       (Ø3)         5       2.4Ø68Ø 32474 3572Ø 1831       (Ø2)       1.24474 77785 67085 6Ø39       (Ø3)         6       5.15778 92ØØØ 139Ø8 471Ø       (ØØ)       6.74291 29516 37859 3773       (Ø1)         7       6.22835 Ø6918 98474 5826       (-Ø3)       1.ØØØØØ ØØØØØ ØØØØØ       (ØØ)         8       Ø       1.65856 95Ø29 761Ø2 232Ø7 66 (Ø5)       6.41552 23783 57622 59962 5Ø (-Ø8)       (Ø6)         1       5.8Ø413 12783 53756 99927 83 (Ø5)       2.42421 85ØØ2 Ø1798 52519 81 (Ø5)       2.42421 85ØØ2 Ø1798 52519 81 (Ø5)         2       6.36Ø69 97788 96445 87965 52 (Ø5)       5.42563 84537 26999 37332 49 (Ø5)       5.42563 84537 26929 3732 49 (Ø5)         3       3.Ø6559 763Ø1 98736 56738 Ø4 (Ø5)       4.34878 8Ø712 76832 9Ø368 16 (Ø5)       4.34878 8Ø712 76832 9Ø368 16 (Ø5)         4       7.14515 95818 95193 32102 93 (Ø4)       1.62Ø65 66Ø91 53367 16388 42 (Ø5)       5.99254 9Ø849 15199 8Ø654 ØØ (Ø3)       2.98624 97022 25£Ø27 79195 66 (Ø4)	7	ø	1.35249 99667	72634 6383	( Ø4)	6.93891 11753 76344 4376 (-07)
2       4.51351       68469       73656       2555       (64)       4.12551       68483       5383       2333       (64)         3       1.85290       11818       58261       0168       (64)       2.93902       87119       93268       1918       (64)         4       3.32915       25149       40693       5532       (63)       9.08196       66074       85517       0271       (63)         5       2.40680       32474       35720       1831       (62)       1.24474       77785       67085       6039       (63)         6       5.15778       92000       13908       4710       (60)       6.74291       29516       37859       3773       (61)         7       6.22835       06918       98474       5826       (-03)       1.00000       000000       00000       00000       (000       (00)         8       0       1.65856       95029       76102       23207       66       (05)       6.41552       23783       57622       59962       50       (-06)         1       5.80413       12783       53756       99927       83       (05)       2.42421       85002       01798       5219		1	4.52856 Ø1699	54728 9655	( Ø4)	1.97685 74263 Ø4673 6421 (Ø4)
3       1.85290       11318       52281       0168       (047)       2.93500       67119       52283       1515       (047)         4       3.32915       25149       40693       5532       (031)       9.08196       66074       85517       0271       (03)         5       2.40680       32474       35720       1831       (02)       1.24474       77785       67085       6039       (03)         6       5.15778       92000       13908       4710       (00)       6.74291       29516       37859       3773       (01)         7       6.22835       06918       98474       5826       (-03)       1.000000       000000       00000       (000       (05)         8       0       1.65856       95029       76102       23207       66       (05)       6.41552       23783       57622       59962       50       (-06)         1       5.80413       12783       53756       99927       83       (05)       2.42421       85002       01798       52519       81       (05)         2       6.36069       97788       96445       87965       52       (05)       5.42563       84537       26999 <td></td> <td>12</td> <td>4.51351 68469</td> <td>73666 2555</td> <td>(84)</td> <td></td>		12	4.51351 68469	73666 2555	(84)	
5       2.42688       32474       435728       1831       (62)       1.24474       77785       67865       6639       (63)         6       5.15778       92000       13908       4710       (60)       6.74291       29516       37859       3773       (61)         7       6.22835       66918       98474       5826       (-03)       1.00000       60000       60000       (60)         8       0       1.65856       95029       76102       23207       66       (05)       6.41552       23783       57622       59962       50       (-08)         1       5.80413       12783       53756       99927       83       (05)       2.42421       85002       01000       (0000		3	1.85290 11818	10261 0168 10263 5532	(83)	9 88196 66874 85517 8271 ( 83)
6       5.15778       92000       13908       4710       (00)       6.74291       29516       37859       3773       (01)         7       6.22835       06918       98474       5826       (-03)       1.00000       00000       00000       00000       (00)       (00)         8       0       1.65856       95029       76102       23207       66       (05)       6.41552       23783       57622       59962       50       (-08)         1       5.80413       12783       53756       99927       83       (05)       2.42421       85002       01098       52519       81       (05)         2       6.36069       97788       96445       87965       52       (05)       5.42563       84537       26999       3732       49       (05)         3       3.06559       76301       98736       56738       64       (05)       4.34878       80712       76832       90368       16       (05)         4       7.14515       95818       95193       32102       93       (04)       1.62065       66091       53367       16388       42       (05)         5       7.95254       90849       151		5	2.40680 32474	35720 1831	( 02)	1.24474 77785 67085 6039 (03)
7       6.22835 06918 98474 5826       (-03)       1.00000 00000 00000 00000 0000       (00)         8       0       1.65856 95029 76102 23207 66       (05)       6.41552 23783 57622 59962 50       (-08)         1       5.80413 12783 53756 99927 83       (05)       2.42421 85002 01798 52519 81       (05)         2       6.36069 97788 96445 87965 52       (05)       5.42563 84537 26999 3732 49       (05)         3       3.06559 76301 98736 56738 04       (05)       4.34878 80712 76832 90368 16       (05)         4       7.14515 95818 95193 32102 93       (04)       1.62065 66091 53367 16388 42       (05)         5       7.95254 90849 15199 80654 00       (03)       2.98624 97022 25027 79195 06       (04)         6       3.76466 93175 92927 68559 71       (02)       2.62877 15790 58119 33301 23       (03)         7       5.49328 55833 00038 53561 68       (00)       9.61416 54774 22235 85246 14       (01)		ā	5.15778 92000	13908 4710	( 00)	6.74291 29516 37859 3773 ( Ø1)
8         Ø         1.65856         95Ø29         761Ø2         232Ø7         66         (Ø5)         6.41552         23783         57622         59962         5Ø         (-Ø8)           1         5.8Ø413         12783         53756         99927         83         (Ø5)         2.42421         85Ø22         Ø1798         52519         81         (Ø5)           2         6.36Ø69         97788         96445         87965         52         (Ø5)         5.42563         84537         26999         3732         49         (Ø5)           3         3.06559         763Ø1         98736         56738         Ø4         (Ø5)         4.34878         8Ø712         76832         9Ø368         16         (Ø5)           4         7.14515         95818         95133         32102         93         (Ø4)         1.62065         66091         53367         16388         42         (Ø5)           5         7.95254         9Ø849         15199         8Ø654         ØØ         (Ø3)         2.62877         1579Ø         58119         33301         23         (Ø3)           7         5.49328         55833         ØØ038         53561         68         (ØØ)		7	6.22835 Ø6918	98474 5826	(-Ø3)	1.00000 00000 00000 0000 ( 00)
8       0       1.65856       95029       76102       23207       66       (057)       2.41552       23783       57622       59962       50       (-063)         1       5.80413       12783       53756       99927       83       (051)       2.42421       85002       01798       52519       81       (05)         2       6.36069       97788       96445       87965       52       (05)       5.42563       84537       26999       3732       49       (05)         3       3.06559       76301       98736       56738       04       (05)       4.34878       80712       76832       90368       16       (05)         4       7.14515       95818       95133       32102       93       (04)       1.62065       66091       53367       16388       42       (05)         5       7.95254       90849       15199       80654       00       (03)       2.98624       97022       25027       79195       66       (04)         6       3.76466       93175       92927       68559       71       (22)       2.62877       15790       58129       33301       23       (03)         7	-	1~		20400 03003	66 ( <del>6</del> 5)	C 114550 03703 57000 50000 50 ( 00)
$\begin{array}{cccccccccccccccccccccccccccccccccccc$	8	10		53756 00097	83 ( 75)	0.41332 23/83 3/622 39962 50 (-08)
3       3.06559       76301       98736       56738       04       (05)       4.34878       80712       76832       90368       16       (05)         4       7.14515       95818       95193       32102       93       (04)       1.62065       66091       53367       16388       42       (05)         5       7.95254       90849       15199       80654       00       (03)       2.98624       97022       25027       79195       06       (04)         6       3.76466       93175       92927       68559       71       (02)       2.62877       15790       58119       33301       23       (03)         7       5.49328       55833       00038       53561       68       (00)       9.61416       54774       22235       85246       14       (01)		12	6.36069 97788	96445 87965	52 ( 05)	5.42563 84537 26999 37332 49 ( 05)
4       7.14515       95818       95193       32102       93       (04)       1.62065       66091       53367       16388       42       (05)         5       7.95254       90849       15199       80654       00       (03)       2.98624       97022       25027       79195       06       (04)         6       3.76466       93175       92927       68559       71       (02)       2.62877       15790       58119       33301       23       (03)         7       5.49328       55833       00038       53561       68       (00)       9.61416       54774       22235       85246       14       (01)		3	3.06559 76301	98736 56738	84 ( 85)	4.34878 80712 76832 90368 16 ( 05)
5       7.95254       90849       15199       80654       00       03)       2.98624       97022       25027       79195       06       04)         6       3.76466       93175       92927       68559       71       02)       2.62877       15790       58119       33301       23       03)         7       5.49328       55833       00038       53561       68       00)       9.61416       54774       22235       85246       14       01)		4	7.14515 95818	95193 32102	93 ( Ø4)	1.62065 66091 53367 16388 42 ( 05)
6         3.76466         93175         92927         68559         71         ( Ø2 )         2.62877         1579Ø         58119         333Ø1         23         ( Ø3 )           7         5.49328         55833         ØØØ38         53561         68         ( Ø0 )         9.61416         54774         22235         85246         14         ( Ø1 )		5	7.95254 90849	15199 80654	00 ( 03)	2.98624 97022 25027 79195 06 ( 04)
11 2.4328 22833 00038 23201 08 ( 00) 9.61416 24/14 22232 82246 14 ( 01)		6	3,76466 93175	92927 68559	/1 ( Ø2)	2.628// 15/90 58119 33301 23 ( 03)
IRI U 51846 81245 76293 41596 89 (-83)   1,88888 88888 88888 88888 88 ( 88)			1 2.49328 22833 4 51846 81945	76293 41596	89 (-83)	1 BARAR BARAR BARAR BARAR RA ( BA)

and upon a Taylor series expansion about 2.5, applying Eq. (4) when necessary. For arguments greater than 15.0, the asymptotic expansion was used.

3. Results. Table I lists the values of

$$\epsilon_{ik} = -100 \log_{10} \max \left| \frac{\psi(x) - \psi_{ik}(x)}{\psi(x)} \right|,$$

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$\psi(\mathbf{x}) = \ln(\mathbf{x}) - \frac{1}{2\mathbf{x}} + \sum_{j=0}^{n} p_j \mathbf{x}^{-2j} / \sum_{j=0}^{n} q_j \mathbf{x}^{-2j},  3.0 \le \mathbf{x}$												
i i n	j	******************** P	- - 	****	************* q <sub>:</sub>	****	****					
96 1	***	******	\$********************* 7	****	\$ \$ \$ U	****	****					
1	Ø 1	-2.71580 1589 -8.87212 8684	(-06) (-01)	1.06496 6945 1.00000 0000		í í	01) 00)					
2	Ø 1 2	-2.00288 09639 95 -1.52827 61729 27 -1.39916 28425 82	5 (-09) 7 (00) 2 (00)	1.83393 20868 1.86234 52532 1.00000 00000	04 39 00	( ( (	01) 01) 00)					
ÿ	19 1 2 3	-2.10638 77134 38 -1.24670 03283 19 -3.91846 20126 40 -1.79307 10243 80	5026 (-12) 9607 (00) 3745 (00) 3592 (00)	1.49604 03955 4.85175 82510 2.56563 23856 1.00000 00000	01592 26104 68056 00000	( ( (	01) 01) 01) 00)					
4	Ø 1234	-2.72817 57513 15 -6.48157 12376 61 -4.48616 54391 80 -7.01677 22776 61 -2.12940 44513 10	5296 783 (-15) 1965 099 (-01) 2193 579 (00) 7586 642 (00) 2105 168 (00)	7.77788 54852 5.46117 7381Ø 8.92920 70048 3.22703 49379 1.00000 00000	29616 Ø42 32150 702 18613 702 11433 614 ØØØØØ 800	( ( ( (	00) 01) 01) 01) 01) 00)					
5	Ø 12345	-4.03243 06017 35 -2.46151 39673 45 -3.05024 76808 03 -1.04226 83363 86 -1.07724 05634 64 -2.43139 31584 34	5749 11804 (-18) 5628 90390 (-01) 3867 49109 (00) 8352 86361 (01) 4792 99398 (01) 4655 50347 (00)	2.95381 676Ø8 3.68983 53845 1.28621 37781 1.4Ø521 63132 3.868Ø4 66Ø83 1.00000 00000	14838 86052 69604 30939 52642 53627 63703 12714 54867 03234 00000 00000	[ [ [ [ [	00) 01) 02) 02) 01) 00)					
6	8 1 2 3 4 5 6	-6.51353 87732 7 -7.36896 00332 3 -1.44796 14616 8 -8.81009 58828 3 -1.97845 54148 7 -1.51662 71776 8 -2.71032 28277 7	1817 13058 11 (-21) 9454 99107 26 (-02) 9984 29858 77 (00) 1221 98214 36 (00) 1921 86672 38 (01) 9612 13830 24 (01) 9783 41916 47 (00)	8.84275 20398 1.74639 65060 1.07425 43875 2.47369 79003 2.02409 55312 4.49927 60373 1.00000 60300	87348 Ø3422 67856 99Ø61 7Ø227 83259 31529 ØØ565 67993 11593 78936 58461 ØØØØ BØØØØ	02 (- 23 ( 79 ( 08 ( 17 ( 73 ( 00 (	-01) 01) 02) 02) 02) 02) 01) 00)					

TABLE III

where the maximum is taken over the appropriate interval, for the initial segments of the  $L_{\infty}$  Walsh arrays. Tables II and III present coefficients for the approximations along the main diagonals of these arrays.

All coefficients are given to an accuracy greater than that justified by the maximal errors to allow precise determination of the corresponding octal or hexadecimal representations. Each approximation listed, with the coefficients just as they appear here, was tested against the master function routines with 5000 pseudorandom arguments. In all cases, the maximal error agreed in magnitude and location with the values given by the Remes algorithm.

4. Use of the Coefficients. The rational approximations all appear to be well conditioned. With a little care, they can be used to generate function values close to working machine precisions up to 20S.

To maintain machine precision in  $\psi(x)$  for x close to  $x_0$ , the computation of  $(x - x_0)$  must be carried out in higher than machine precision to preserve the low order bits of  $x_0$ . This can be achieved by breaking  $x_0$  into two parts,  $x_1$  and  $x_2$ , such that  $x_0 \equiv x_1 + x_2$  to the precision desired, and such that the floating-point exponent on  $x_2$  is much less than that on  $x_1$ . Then  $(x - x_0)$  is computed as  $(x - x_0) =$  $(x - x_1) - x_2$ . This breakup of  $x_0$  is most easily accomplished by examining the octal or hexadecimal representation

 $x_0 = 1.35426\ 60615\ 26574\ 37556\ 06516\ 21031\ 36024\ 47402_8$ 

 $= 1.762D8 6356B E3F6E 1A9C8 865E0 A4F02_{16}$ .

One remaining avoidable source of error is in the use of the reflection formula (5) for negative arguments. We suggest that z be broken into  $z = z_i + z_f$ , where  $z_i$ is the integer part of z, and  $z_1$  is the fractional part. Then Eq. (5) should be reformulated as

$$\psi(1-z) = \psi(z) + \pi \cot(\pi z_f).$$

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