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On Integral Groups. III: Normalizers

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Abstract. Methods for determining a generating set for the normalizer of a finite group of $n \times n$ integral matrices, i.e., an n-dimensional crystallographic point group, are discussed. Necessary and sufficient conditions for the finiteness of such a normalizer are derived, and several examples of the application of the methods to cases when the normalizer is infinite are presented.

This is the third in a series of papers dealing with the finite subgroups of $\mathrm{GL}(n, \mathbf{Z})$. In the first two papers [3], [4], we discussed the integral classification of these groups, and, in this paper, we consider their normalizers in $\mathrm{GL}(n, \mathbf{Z})$, which are needed, e.g., for the determination of the n-dimensional space groups [2], [17]. As in the previous two papers, the discussion results in a complete determination for the case $n = 4$. We refer to the first paper for basic definitions and notation.

1. Introduction. For any finite subgroup G of $\mathrm{GL}(n, \mathbf{Z})$, i.e., an f.u. group, it is known that the normalizer $N(G)$ of G in $\mathrm{GL}(n, \mathbf{Z})$ is finitely generated [15].

We assume that a representative set of the integral equivalence classes of the f.u. groups of dimension n is given. For the case $n = 4$, such a representative set has been determined [7], [8]. As the normalizers of integrally equivalent groups are also integrally equivalent under the same transformation, we only need determine the normalizers of such a representative set. In Section 2 of this paper, we show that a representative set can be so chosen that in fact it suffices to determine only the normalizers of the so-called Bravais groups in this set, as the normalizer of any other member of this set can be obtained by a finite algorithmic process which we describe.

In Section 3 we determine fairly easily applied necessary and sufficient conditions for the normalizer of an f.u. group to be finite. For these groups, the normalizer can be read off from the subgroup lattices of the maximal n-dimensional f.u. groups. For $n = 4$, these lattices have been computed [7].

In Section 4, we consider "block triangular," i.e., reduced f.u. groups G, and we give some sufficient conditions for their normalizers $N(G)$ to be of the same block triangular form.

In Section 5, we give examples of some methods for finding generating sets of the normalizers when they are infinite. For the case $n = 4$, these methods suffice to determine all infinite normalizers.

For dimensions 2, 3 and 4, representative sets of the Bravais groups together with generating sets for their normalizers will be listed in a subsequent paper.

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2. Bravais Groups.

(2.1) Let G be a subgroup of $\mathrm{GL}(n, \mathbf{Z})$. The set of all symmetric rational $n \times n$-matrices X (or, equivalently, n-dimensional quadratic forms) satisfying

$$g^t X g = X \quad \text{for all } g \in G \tag{2.11}$$

forms a $\mathbf{Q}$-vectorspace $S(G)$. Note that (2.11) is valid if and only if it is valid for a set of generators of G.

The set of all unimodular matrices h such that

$$h^t X h = X \quad \text{for all } X \in S(G) \tag{2.12}$$

forms a subgroup $B(G) \leqq \mathrm{GL}(n, \mathbf{Z})$, which we call the *Bravais group* of G. Note that $G \leqq B(G)$, and that (2.12) is valid for all $X \in S(G)$ if and only if it is valid for a $\mathbf{Q}$-basis of $S(G)$. Note also that $G \leqq H$ implies $S(H) \leqq S(G)$ and that $S(B(G)) = S(G)$. Hence $B(B(G)) = B(G)$. Therefore, we can call a subgroup $B \leqq \mathrm{GL}(n, \mathbf{Z})$ a Bravais group if $B(B) = B$.

Let $G \leqq \mathrm{GL}(n, \mathbf{Z})$, $y \in N(G)$ and $X \in S(G)$. Then $y^{-1} \in N(G)$ and $(ygy^{-1})^t Xygy^{-1} = X$ for all $g \in G$. Hence, $g^t(y^tXy)g = y^tXy$, and $y^tS(G)y = S(G)$. For a Bravais group B, also the converse holds. For let $y^tS(B)y = S(B)$. Then, for any $b \in B$ and $X \in S(B)$, since $y^{-1t}Xy^{-1} \in S(B)$, we have $(y^{-1}by)^tX(y^{-1}by) = y^tb^t(y^{-1t}Xy^{-1})by = X$, i.e., $y^{-1}by \in B(B) = B$. Hence $y \in N(B)$. From these two remarks we have

(2.13) Lemma. *Let $G \leqq \mathrm{GL}(n, \mathbf{Z})$; then $N(G) \leqq N(B(G))$.*

From now on, we shall confine our consideration to finite subgroups G of $\mathrm{GL}(n, \mathbf{Z})$. As is well known, a subgroup G of $\mathrm{GL}(n, \mathbf{Z})$ is finite if and only if $S(G)$ contains a positive definite symmetric matrix [16]. Hence, in particular, if G is finite, so is $B(G)$. From this we have

(2.14) Lemma. *Let G be a finite subgroup of $\mathrm{GL}(n, \mathbf{Z})$. Then the index $N(B(G))$: $N(G)$ is finite.*

Proof. As $B(G)$ is finite, there are only finitely many subgroups of $B(G)$ which are $\mathbf{Z}$-equivalent to G. Let Ω be the orbit of G under transformation by elements from $N(B(G))$. Ω is finite and $N(B(G))$ is represented as a group of permutations on Ω. $N(G)$ is the stabilizer of G in this permutation representation; hence $N(B(G))$: $N(G)$ is finite.

(2.2) Let $\mathfrak{U}_n$ be a representative set of all $\mathbf{Z}$-equivalence classes of finite subgroups of $\mathrm{GL}(n, \mathbf{Z})$.

Let B' be $\mathbf{Z}$-equivalent to a Bravais group $B(G)$ of $G \leqq \mathrm{GL}(n, \mathbf{Z})$. Then, there exists a group $G' \leqq \mathrm{GL}(n, \mathbf{Z})$ such that $G' \sim_{\mathbf{Z}} G$ and $B(G') = B'$. Hence, we can choose the set $\mathfrak{U}_n$ in such a way that it satisfies the following property:

(2.21) If $G \in \mathfrak{U}_n$, then $B(G) \in \mathfrak{U}_n$.

From now on, we assume that we have a fixed set $\mathfrak{U}_n$ with property (2.21). For a Bravais group B, $B \in \mathfrak{U}_n$, we define its family to consist of all $G \in \mathfrak{U}_n$ with $B(G) = B$. We shall describe in Section (2.3) how we obtained such a $\mathfrak{U}_n$ for $n = 4$. Using property (2.21), we can obtain the normalizers of all groups in $\mathfrak{U}_n$ from the normalizers of the Bravais groups in $\mathfrak{U}_n$. Let $G \in \mathfrak{U}_n$. By Lemma (2.14), $N(B(G))$: $N(G)$ is finite. Also, the normalizers of finite unimodular groups are finitely generated [15]. There are efficient procedures [6] to determine from a finite generating set $\{g_1, \cdots, g_n\}$ of $N(B(G))$ a set of coset representatives $\mathfrak{R} = \{r_1, \cdots, r_k\}$ of the cosets of $N(G)$ in $N(B(G))$. By Schreier's theorem [14],

$$N(G) = \langle r_i g_j \cdot \overline{r_i g_j}^{-1} \mid i = 1, \cdots, n, j = 1, \cdots, k \rangle$$

where $\overline{r_i g_j}$ denotes the coset representative of the element $r_i g_j$ in the set $\mathfrak{R}$. As G is finite, there is an effective method to decide if a given element is in $N(G)$ and, hence, since $N(B(G))$:$N(G)$ is finite, an effective method to determine $\overline{r_i g_j}$ from $r_i g_j$. All these procedures have been implemented on a computer.

(2.3) In dimension 4, from the application of existing computer programs, a listing of the lattices of subgroups of all Dade groups [12] with all **Z**-equivalence relations between these subgroups was available [7]. From this, a set $\mathfrak{U}_4$ of representatives of all **Z**-equivalence classes of finite subgroups of GL(4, **Z**) satisfying property (2.21) was determined [8]. For $n = 4$, among the 710 groups in $\mathfrak{U}_4$, there are 64 Bravais groups.

3. Finite Normalizers.

(3.1) In this section, we shall find necessary and sufficient conditions for a finite subgroup $G \leqq \mathrm{GL}(n, \mathbf{Z})$ to have a finite normalizer in GL(n, **Z**).

(3.11) LEMMA. *$N(G)$ is finite if and only if $Z(G)$, the centralizer of G in* GL(n, **Z**), *is finite.*

Proof. For $y \in N(G)$, the correspondence φ: $y \to y\varphi \in \mathrm{Aut}(G)$ defined by $g(y\varphi) = y^{-1}gy$ is a homomorphism from $N(G)$ into Aut(G). Since G is finite, so is Aut(G), and thus $N(G)$:ker φ is finite. The kernel of φ is precisely $Z(G)$, and the result follows.

(3.2) We now consider $Z(G)$ for a f.u. group G.

Let $C(G) = \{Y \in M_{n\times n}(\mathbf{Z}) \mid Yg = gY \text{ for all } g \in G\}$, i.e., the commuting ring of G in $M_{n\times n}(\mathbf{Z})$. Then $Z(G)$ is the unit group of $C(G)$. Since $C(G)$ is a **Z**-submodule of the finitely generated free **Z**-module $M_{n\times n}(\mathbf{Z})$, $C(G)$ has a finite **Z**-basis which is also a **Q**-basis for $C_{\mathbf{Q}}(G) = \{Y \in M_{n\times n}(\mathbf{Q}) \mid Yg = gY \text{ for all } g \in G\}$. Moreover, $C_{\mathbf{Q}}(G) = \mathbf{Q}C(G)$ and $C(G)$ is a subring with identity of the **Q**-algebra $\mathbf{Q}C(G)$. Thus, $C(G)$ is a **Z**-order in the classical sense, and $\mathbf{Q}C(G)$ is a semisimple **Q**-algebra [1].

Since $\mathbf{Q}C(G)$ is semisimple, by Wedderburn's structure theorems,

$$\mathbf{Q}C(G) = \bigoplus_1^s A_i$$

where each A_i is **Q**-isomorphic to a full matrix ring $M_{f_i \times f_i}(D_i)$ for some finite-dimensional division algebra D_i over **Q**. Also, there exists a maximal **Z**-order, $\mathfrak{O}_{\max}$, of $\mathbf{Q}C(G)$ which contains $C(G)$, and $\mathfrak{O}_{\max}$ can be decomposed as a ring theoretic direct sum

$$\mathfrak{O}_{\max} \cong \bigoplus_1^s \mathfrak{O}_i \tag{3.21}$$

where each $\mathfrak{O}_i$ is a maximal **Z**-order in $M_{f_i \times f_i}(D_i)$ [9].

In general, if $\mathfrak{O}$ and $\mathfrak{O}'$ are two orders over **Z** of equal (finite) rank such that $\mathfrak{O} \supset \mathfrak{O}'$, then, trivially, the unit group $U(\mathfrak{O}')$ of $\mathfrak{O}'$ is contained in the unit group $U(\mathfrak{O})$ of $\mathfrak{O}$. Also, we have in this case:

(3.22) LEMMA. *$U(\mathfrak{O})$:$U(\mathfrak{O}')$ is finite.*

Proof. Since $\mathfrak{O}$ and $\mathfrak{O}'$ are of equal rank, $\mathfrak{O}$:$\mathfrak{O}'$ is finite. More precisely, if $\mathfrak{O} = \bigoplus_1^k \mathbf{Z}a_i$ and $\mathfrak{O}' = \bigoplus_1^k \mathbf{Z}b_i$ where $b_i = \sum_1^k c_{ij}a_j$, then $\mathfrak{O}$:$\mathfrak{O}' = |\det(c_{ij})|$. Let

$A_1, \cdots, A_t$ be representatives of the isomorphism classes of **Z**-modules of order $\mathfrak{O}: \mathfrak{O}'$. Each submodule of $\mathfrak{O}$ of index $\mathfrak{O}: \mathfrak{O}'$ occurs as a kernel of a **Z**-epimorphism from $\mathfrak{O}$ to one of the A_i. Since $\mathfrak{O}$ is free of finite rank, there are only finitely many **Z**-homomorphisms from $\mathfrak{O}$ to each A_i. Thus, there are only finitely many submodules of $\mathfrak{O}$ of index $\mathfrak{O}: \mathfrak{O}'$. Left multiplication of these finitely many submodules by elements of $U(\mathfrak{O})$ induces a finite permutation representation of $U(\mathfrak{O})$. Since $\mathfrak{O}'$ is unital, the stabilizer of $\mathfrak{O}'$ in $U(\mathfrak{O})$ is precisely $U(\mathfrak{O}')$, and hence $U(\mathfrak{O}):U(\mathfrak{O}')$ is finite.

By this lemma, we have in our case $Z(G) = U(C(G))$ is finite if and only if $U(\mathfrak{O}_{\max})$ is finite. As a consequence of (3.21),

$$U(\mathfrak{O}_{\max}) \cong \bigoplus_1^s U(\mathfrak{O}_i),$$

and thus $Z(G)$ is finite if and only if each $U(\mathfrak{O}_i)$ is finite. There exists a maximal **Z**-order $\mathfrak{O}_i'$ of $M_{f_i \times f_i}(D_i)$ such that $M_{f_i \times f_i}(Z) \subseteq \mathfrak{O}_i'$, and thus $\mathrm{GL}(f_i, \mathbf{Z}) \subseteq U(\mathfrak{O}_i')$ [9], [10]. Since $\mathfrak{O}_i$ and $\mathfrak{O}_i'$ are both maximal **Z**-orders in $M_{f_i \times f_i}(D_i)$, $\mathfrak{O}_i': \mathfrak{O}_i' \cap \mathfrak{O}_i$ is finite. Also, if $f_i > 1$, $\mathrm{GL}(f_i, \mathbf{Z})$ contains elements of infinite order. Hence, we have as a necessary condition for the finiteness of $Z(G)$ that $f_i = 1$, $i = 1, \cdots, s$. Note that in this case each $\mathfrak{O}_i$ is a maximal order in D_i.

(3.3) We now seek conditions for the unit group $U(\mathfrak{O}_i)$ of the maximal order $\mathfrak{O}_i$ in the finite-dimensional division algebra D_i over Q to be finite.

(3.31) THEOREM [5]. *Let D be a finite-dimensional division algebra over* Q, *and let $\mathfrak{O}$ be a maximal order in D. Then $U(\mathfrak{O})$, the unit group of $\mathfrak{O}$, is finite if and only if D is* Q*-isomorphic to* Q, *an imaginary quadratic extension of* Q *or a positive definite quaternion algebra over* Q.

It follows from this theorem that $U(\mathfrak{O}_i)$ is finite if and only if $\mathrm{Q}\mathfrak{O}_i$ is one of the permissible types.

(3.4) In order to apply these criteria to the unit group of $C(G)$, we consider the behaviour of the natural representation

$$\Delta : g \to g$$

of G with respect to its reduction over Q. Let

$$\Delta = \bigoplus_1^s f_i \Delta_i$$

be a full reduction of Δ over Q where the Δ_i are inequivalent irreducible representations of G over Q with multiplicities $f_i > 0$. Such a reduction can be obtained using character theory. It follows from a generalization of Schur's lemma [11] that, in the Wedderburn decomposition of $\mathrm{Q}C(G)$,

$$\mathrm{Q}C(G) \cong \bigoplus_1^s M_{f_i \times f_i}(D_i),$$

the division algebra D_i can be chosen as the commuting algebra of Δ_i, $i = 1, \cdots, s$. Thus, the previous results yield

(3.41) THEOREM. *The unit group of $C(G)$, and thus the normalizer of G in* $\mathrm{GL}(n, \mathbf{Z})$, *is finite if and only if*

(a) $f_i = 1$, $i = 1, \cdots, s$.

(b) *Each D_i is one of the following three types*:

I. Q.

II. *An imaginary quadratic extension of* Q.

III. *A positive definite quaternion algebra over* Q.

Let n_i be the degree of the irreducible representation Δ_i of G over Q. The algebra $\mathbf{Q}\Delta_i(G)$ is a simple subalgebra of $M_{n_i \times n_i}(\mathbf{Q})$ [1]. The centre of $\mathbf{Q}\Delta_i(G)$, Z_i, is a finite extension of Q, say of degree z_i, and $\mathbf{Q}\Delta_i(G)$ is isomorphic to a full ring of matrices of finite degree r_i over a division algebra B_i of dimension m_i^2 over Z_i. Here, m_i is the Schur index of Δ_i. The numerical relation

$$n_i = z_i m_i^2 r_i$$

holds [4]. From the theory of algebras, it is known that the dimension of the commuting algebra, D_i, over Q is equal to $z_i m_i^2$. In fact, D_i is anti-isomorphic to B_i [1]. Thus, condition (b) above is equivalent to

(b′) I. $m_i = 1$, $z_i = 1$.

II. $m_i = 1$, $z_i = 2$, Z_i imaginary quadratic.

III. $m_i = 2$, $z_i = 1$, B_i positive definite quaternion algebra.

In the special case $n = 4$, the above criterion is very easy to apply. Trivially, for $n_i = 1$, (b′) is always satisfied; and using the methods of [4], one has

$n_i = 2$: (b′) is always satisfied.

$n_i = 3$: (b′) is satisfied if and only if $\Delta_i G$ is not a cyclic group.

$n_i = 4$: (b′) is satisfied if and only if $\Delta_i G$ is not a cyclic or a dihedral group.

For $n = 4$, of the 64 Bravais groups in the list $\mathfrak{U}_4$ of representatives of all integral classes, 38 have finite normalizers in GL(4, Z).

4. Block Triangular Normalizers. Let G be a reduced f.u. group and $\Delta\colon G \to G$ the natural representation of G. Say

$$\Delta g = \begin{bmatrix} \Delta_1 g & * \\ 0 & \Delta_2 g \end{bmatrix}.$$

For $u \in N(G)$, let α be the automorphism of G induced by u, i.e., $g\alpha = u^{-1}gu$. Then $u^{-1}\Delta g u^{-1} = \Delta(g\alpha)$ for every g in G. Thus, Δ is Z-equivalent as a representation to $\bar{\Delta}$ where $\bar{\Delta} g = \Delta(g\alpha)$.* Let $\Delta_i(g\alpha) = \bar{\Delta}_i(g)$, $i = 1, 2$, and

$$u = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}, \qquad u^{-1} = \begin{bmatrix} u'_{11} & u'_{12} \\ u'_{21} & u'_{22} \end{bmatrix}$$

be block decompositions of u and u^{-1} corresponding to the block pattern of Δ.

It follows directly that $\Delta_2(g)u_{21} = u_{21}\bar{\Delta}_1(g)$ and $\bar{\Delta}_2(g)u'_{21} = u'_{21}\Delta_1(g)$ for every g in G. Hence, from a generalization of Schur's lemma, we have that if Δ_1 and $\bar{\Delta}_2$ or $\bar{\Delta}_1$ and Δ_2 have no Q-constituents (as representations) in common, then $u_{21} = u'_{21} = 0$, i.e., u is block triangular.

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* Note that the equivalence of two faithful representations Δ and $\bar{\Delta}$ of a group G is a more restrictive condition than the equivalence as groups of the images ΔG and $\bar{\Delta} G$. If Δ and $\bar{\Delta}$ are equivalent representations, then, clearly, ΔG and $\bar{\Delta} G$ are equivalent groups; but if ΔG and $\bar{\Delta} G$ are equivalent groups, all that can be said is that for some $\alpha \in \mathrm{Aut}(G)$, Δ^{α} and $\bar{\Delta}$ are equivalent representations. Here $\Delta^{\alpha}(g) = \Delta(g\alpha)$.

For $i = 1, 2$, let χ_{Δ_i}, $\chi_{\bar{\Delta}_i}$ be the Q-characters of Δ_i and $\bar{\Delta}_i$, respectively, and let

$$\chi_{\Delta_i} = \bigoplus_{j=1}^{k_i} n_{ij}\chi_{\Delta_{ij}}, \qquad \chi_{\bar{\Delta}_i} = \bigoplus_{j=1}^{\bar{k}_i} \bar{n}_{ij}\chi_{\bar{\Delta}_{ij}}$$

be their reductions into distinct irreducible Q-characters with positive multiplicities. By the definition of $\bar{\Delta}_i$, χ_{Δ_i} and $\chi_{\bar{\Delta}_i}$ must have the same reduction pattern, i.e., $k_i = \bar{k}_i$ and the irreducible characters can be ordered such that $n_{ij} = \bar{n}_{ij}$ and $\dim \chi_{\Delta_{ij}} = \dim \chi_{\bar{\Delta}_{ij}}$.

For α in Aut(G), let $\Delta_i^\alpha(g) = \Delta_i(g\alpha)$. From the above comments, we have

(4.11) THEOREM. *If, for each α in the subgroup of* Aut(G) *induced by $N(G)$,*

$$\chi_{\Delta_{1s}} \neq \chi_{\Delta_{2t}\alpha}, \qquad 1 \leqq s \leqq k_1,\ 1 \leqq t \leqq k_2,$$

then $N(G)$ is block triangular.

The hypothesis of (4.11) is satisfied if Δ fulfills the following condition:

(4.12) (i) Δ_1 and Δ_2 have no Q-constituents in common.

(ii) Whenever $\dim \Delta_{1s} = \dim \Delta_{2t}$, then $n_{1s} \neq n_{2t}$, $1 \leqq s \leqq k_1$, $1 \leqq t \leqq k_2$.

To show this, assume, e.g., that $\Delta_{1s} \sim_{\mathrm{Q}} \Delta_{2t}^\alpha = \bar{\Delta}_{2t}$ and $n_{1s} > n_{2t}$. Since Δ and $\Delta^\alpha = \bar{\Delta}$ are Z-equivalent representations, the Q-constituents of Δ and $\bar{\Delta}$ must be Q-equivalent in some order. Also, $\bar{\Delta}_1$ and $\bar{\Delta}_2$ have no Q-constituents in common. From $n_{1s} > n_{2t}$, it follows that $\Delta_{1s} \sim_{\mathrm{Q}} \bar{\Delta}_{1r}$, some $1 \leqq r \leqq k_1$, or $\Delta_{1s} \sim_{\mathrm{Q}} \bar{\Delta}_{2v}$, some $v \neq t$. Hence, $\bar{\Delta}_{2t} \sim_{\mathrm{Q}} \bar{\Delta}_{1r}$ or $\bar{\Delta}_{2t} \sim_{\mathrm{Q}} \bar{\Delta}_{2v}$, $v \neq t$. This is a contradiction.

The above condition is easily applied, particularly in the case $n = 4$. Of the 26 classes of Bravais groups with infinite normalizers, 12 have block triangular normalizers and, in fact, satisfy this condition.

5. Computational Methods for Determining Generators for Infinite Normalizers. Some Examples.

(5.1) There are 26 classes of Bravais groups in 4 dimensions which have infinite normalizers. Of these, one is the group $\langle -I_4 \rangle$ which has normalizer GL(4, **Z**); three of these classes consist of irreducible groups which do not satisfy (3.41) and hence have infinite normalizers; 12 have reduced representatives which satisfy (4.11) and hence have block triangular normalizers; and 10 classes consist of reducible groups which do not satisfy (4.11) and do not have block triangular normalizers.

In three cases $Z(G)$, the centralizer of G, is naturally isomorphic to GL(2, R) where R is the ring of integers in the 4th or 6th cyclotomic field. For these cases, the following lemma is useful:

(5.11) LEMMA. *Let $\alpha \in \mathbf{C}$ be an algebraic integer of degree m such that* $\mathbf{Z}[\alpha]$ *is a Euclidean domain. Let U be the unit group of* $\mathbf{Z}[\alpha]$. *Then* GL(2, $\mathbf{Z}[\alpha]$) *is generated by the elements*

$$g = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}; \qquad h_i = \begin{bmatrix} 0 & 1 \\ -1 & \alpha^i \end{bmatrix}, \qquad i = 0, \cdots, m-1;$$

$$t_u = \begin{bmatrix} u^{-1} & 0 \\ 0 & u \end{bmatrix}, \qquad u \in U; \quad \textit{and} \quad v_u = \begin{bmatrix} 1 & 0 \\ 0 & u \end{bmatrix}, \qquad u \in U.$$

Proof. GL(2, $\mathbf{Z}[\alpha]$) $= \{A \in M_{2\times 2}(\mathbf{Z}[\alpha]) \mid \det A \in U\}$. The map ϕ: GL(2, $\mathbf{Z}[\alpha]$) $\to U$ given by $A\phi = \det A$ is an epimorphism.

Let $H = \ker \phi$. Then the elements g, h_i and t_u are in H, and $\mathrm{GL}(2, \mathbf{Z}[\alpha])\colon H = |U|$. For any $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in H$,

$$\prod_{i=0}^{m-1} (g^3 h_i)^{k_i} A = \begin{bmatrix} a - c \sum_{i=0}^{m-1} k_i \alpha^i & * \\ c & * \end{bmatrix}$$

and

$$\prod_{i=0}^{m-1} (h_i g^3)^{k_i} A = \begin{bmatrix} a & * \\ c + a \sum_{i=0}^{m-1} k_i \alpha^i & * \end{bmatrix}.$$

Now $\{\alpha^0, \cdots, \alpha^{m-1}\}$ forms a $\mathbf{Z}$-basis for $\mathbf{Z}[\alpha]$. Thus, by premultiplication of A with matrices of the above type which are determined by the quotients in a Euclid's algorithm scheme for a and c, we can transform A into a matrix of the form $\left[\begin{smallmatrix} x & y \\ 0 & z \end{smallmatrix}\right]$. Since the premultiplication matrices and A are in H, $\left[\begin{smallmatrix} x & y \\ 0 & z \end{smallmatrix}\right]$ must also be in H, i.e., $xz = 1$. Hence, $x \in U$, say $x = u$, and $z = u^{-1}$. Premultiplication of $\left[\begin{smallmatrix} x & y \\ 0 & z \end{smallmatrix}\right]$ by t_u transforms it into a matrix of the form $\left[\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right]$. Since the elements $h_i^{-1} g$, $i = 0, \cdots, m - 1$, generate all matrices of this latter form, g, and h_i and the t_u generate H. The set $\{v_u \mid u \in U\}$ is clearly a set of coset representatives for H in $\mathrm{GL}(2, \mathbf{Z}[\alpha])$, and the result follows.

In several cases, $Z(G)$ is naturally isomorphic to a subgroup T of $\mathrm{GL}(2, \mathbf{Z})$ where the entries of the matrices in T must satisfy certain congruence conditions. For these cases, the following two lemmas are useful:

(5.12) Lemma. *Let $a, b \in \mathbf{Z}$, $b \neq 0$. If $a - b \equiv 1(2)$, then there is a Euclid's algorithm scheme for a and b in which all the quotients are even.*

This lemma is easily proved by noting that for two integers c, d with $d \neq 0$ and $c - d \equiv 1(2)$, if from the division algorithm we have $c = dq + r$, $0 < r < |d|$ with $q \equiv 1(2)$, then $c = d(q + d/|d|) + (r - |d|)$ with $0 < |r - |d|\,| < |d|$, and $d - (r - |d|) \equiv 1(2)$.

(5.13) Lemma. *Let $a, b \in \mathbf{Z}$, $b \neq 0$. Then there is a terminating Euclid's algorithm type scheme for a and b in which the quotient in each odd numbered step is a multiple of 3.*

Proof. Choose $q, r \in \mathbf{Z}$ such that $a = bq + r$, $0 \leqq r < |b|$, say $q = 3t + k$, $k \in \{-1, 0, 1\}$. Set $q_1 = 3t$ and $r_1 = r + kb$. Then $a = bq_1 + r_1$, $0 \leqq |r_1| < |2b|$. If $r_1 \neq 0$, then

$$b = kr_1 + r_2 \quad \text{with } 0 \leqq |r_2| < |b| \text{ if } k \neq 0$$

and

$$b = (b/|b|)r_1 + r_2 \quad \text{with } 0 \leqq |r_2| < |b| \text{ if } k = 0.$$

Continuing in this manner, since $|b| > |r_2| > |r_4| > \cdots$, we get the desired result.

(5.2) Let G be a representative of one of the three irreducible 4-dimensional Bravais classes not satisfying (3.14). The reader is referred to [4] for the verification of the following observations about G.

The centre of the $\mathbf{Q}$-enveloping algebra of G is a real quadratic extension of $\mathbf{Q}$, and G is a dihedral group of order 16, 20 or 24, say $|G| = 2m$. The cyclic subgroup of order m, S_G, of G is also an irreducible f.u. group. Since there is only one $\mathbf{Z}$-

equivalence class of the irreducible cyclic groups of order m, we may assume $S_G = \langle A \rangle$ where A is the companion matrix of the mth-cyclotomic polynomial.

Let $K = \mathbf{Q}(A)$. Then K is isomorphic to the mth-cyclotomic field. The ring of integers in K is $\mathbf{Z}[A] \subseteq M_{4\times 4}(\mathbf{Z})$. Let $U(A)$ be the unit group of $\mathbf{Z}[A]$. From the Cayley-Hamilton theorem it follows that $U(A) = K \cap \mathrm{GL}(4, \mathbf{Z})$. Also, K is its own commuting algebra in $M_{4\times 4}(\mathbf{Q})$. Thus, $U(A) = \{X \in \mathrm{GL}(4, \mathbf{Z}) \mid XA = AX\}$.

The Galois group $G_{K/\mathbf{Q}}$ of K over $\mathbf{Q}$ can be faithfully represented in $\mathrm{GL}(4, \mathbf{Z})$ in such a way that for any $\delta \in G_{K/\mathbf{Q}}$, the action of δ on K is the same as conjugation by the matrix corresponding to δ. Moreover, this representation can be determined constructively. Let H be the group of matrices corresponding to $G_{K/\mathbf{Q}}$, and let $N \in H$ be the matrix corresponding to the element of $G_{K/\mathbf{Q}}$ induced by $A \to A^{-1}$. Since $N^2 = I_4$ and $N^{-1}AN = A^{-1}$, we may assume that G has concrete representation $G = \langle A, N \rangle$.

If $X \in N(S_G)$, then $X^{-1}AX = \delta A$ for some $\delta \in G_{K/\mathbf{Q}}$. Hence $X^{-1}AX = T^{-1}AT$ for some $T \in H$, and $XT^{-1} \in U(A)$. Thus $N(S_G) = U(A)\cdot H$.

(5.21) (a) $|G| = 16$.

$U(A) = \langle A, I_4 + A + A^{-1} \rangle$ [13].

$H = \langle N, N_1 \rangle \cong C_2 \times C_2$ where N_1 is the matrix corresponding to $A \to A^3$. Thus $N(S_G) = \langle G, N_1, I_4 + A + A^{-1} \rangle$. Now $NN_1 = N_1N$ and $N(I_4 + A + A^{-1}) = (I_4 + A + A^{-1})N$. Hence $N(G) = N(S_G)$.

(b) $|G| = 20$.

$U(A) = \langle A, A + A^{-1} \rangle$ [13].

$H = \langle N_1 \rangle \cong C_4$ where N_1 is the matrix corresponding to $A \to A^7$.

$N(G) = N(S_G) = \langle A, A + A^{-1}, N_1 \rangle$.

(c) $|G| = 24$.

$U(A) = \langle A, I_4 + A \rangle$ [13].

$H = \langle N, N_1 \rangle \cong C_2 \times C_2$ where N_1 is the matrix corresponding to $A \to A^5$.

$N(G) = N(S_G) = \langle G, N_1, I_4 + A \rangle$.

(5.3) Let $G \neq \langle -I_4 \rangle$ be a representative of one of the ten classes of reducible Bravais groups not satisfying (4.11). From direct inspection of the list of 4-dimensional Bravais groups, G is a cyclic group of order 4 or 6, a dihedral group of order 8 or 12 or a Klein 4-group.**

(5.31) Let G be cyclic or dihedral. By inspection of the Bravais group list, the natural representation Δ of G may be assumed to have the form

$$\Delta g = \begin{bmatrix} \Delta_1 g & * \\ 0 & \Delta_2 g \end{bmatrix},$$

where Δ_1 and Δ_2 are $\mathbf{Q}$-equivalent irreducible 2-dimensional representations of G.

$\mathbf{Q}\Delta_1 G$ and $\mathbf{Q}\Delta_2 G$ are $\mathbf{Q}$-isomorphic simple algebras over $\mathbf{Q}$. By Wedderburn, $\mathbf{Q}\Delta_i G \cong M_{r\times r}(B)$ where B is a finite-dimensional division algebra over $\mathbf{Q}$. Let F be the centre of B, say $B{:}F = f$, and let s be the Schur index of B. Then the relation $2 = fs^2r$ holds [1].

If G is cyclic, $\mathbf{Q}\Delta_i G$ is commutative, and we have $f = 2$, $s = r = 1$. Thus $\mathbf{Q}\Delta_i G \cong F$, where F is an mth-cyclotomic field, $m = 4$ or $m = 6$. If G is dihedral, then

** A list of the Bravais groups in dimensions 2, 3 and 4, as well as their normalizers and other related information will be published in the near future.

$Q\Delta_i G$ is noncommutative, and we have $f = s = 1$, $r = 2$. Thus $Q\Delta_i G \cong M_{2\times 2}(Q)$.

For any subgroup $H < GL(n, \mathbf{Z})$, let $C(H)$ denote the integral commuting algebra of H. The centralizer of H in $GL(n, \mathbf{Z})$, $Z(H)$, is the unit group of $C(H)$. From the general theory of algebras, we have $QC(\Delta_i G)$ is anti-isomorphic to B and from a generalization of Schur's lemma $QC(G) \cong M_{2\times 2}(B)$.

In the case G is cyclic, $QC(G) \cong M_{2\times 2}(F)$ where $F{:}Q = 2$. Hence $QC(G){:}Q = 8$. In the case G is dihedral, $C(G) \cong M_{2\times 2}(Q)$, and $QC(G){:}Q = 4$.

(5.32) *The Cyclic Case.* Let $|G| = m$ and let α be a primitive mth-root of unity. Let $G = \langle g \rangle$. We may choose G such that $\Delta_1 g = \Delta_2 g = A$ where A is the companion matrix of the mth-cyclotomic polynomial.

By direct computation,

$$\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & I_2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ I_2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & I_2 \end{bmatrix},$$
$$\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ A & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix}$$

are in $C(G)$, and since $QC(G){:}Q = 8$, they form an integral basis for $QC(G)$. Thus,

$$C(G) = \{(x_{ij}I_2 + y_{ij}A) \mid 1 \leqq i, j \leqq 2, x_{ij}, y_{ij} \in \mathbf{Z}\}.$$

$C(G)$ is isomorphic to $M_{2\times 2}(\mathbf{Z}[\alpha])$ under $I_2 \to 1$, $A \to \alpha$; and $Z(G) \cong GL(2, \mathbf{Z}[\alpha])$.

Now $\mathbf{Z}[\alpha]$ is precisely the ring of algebraic integers in $Q(\alpha)$, and for $m = 4$ or $m = 6$, $\mathbf{Z}[\alpha]$ is a Euclidean domain [13]. Applying Lemma (5.11), we obtain a set of generators for $\mathbf{Z}(G)$.

The natural homomorphism from $N(G)$ to $\mathrm{Aut}(G)$ has kernel $Z(G)$. Also $|\mathrm{Aut}(G)| = 2$. Thus $N(G){:}Z(G) \leqq 2$.

For $m = 4$, the matrix

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & T_1 \end{bmatrix}, \qquad T_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

is in $N(G)\backslash Z(G)$, and $N(G) = \langle Z(G), T \rangle$. For $m = 6$, the matrix

$$N = \begin{bmatrix} N_1 & 0 \\ 0 & N_1 \end{bmatrix}, \qquad N_1 = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix},$$

is in $N(G)\backslash Z(G)$, and $N(G) = \langle Z(G), N \rangle$.

(5.33) *Examples of the Dihedral Case.* Let $G = \langle A, B \mid A^m = B^2 = (AB)^2 = I_4 \rangle$.

(a) By inspection of the list of Bravais groups, in one dihedral case of each of the orders 8 and 12, we may choose G such that

$$\Delta g = \begin{bmatrix} \Delta_1 g & 0 \\ 0 & \Delta_2 g \end{bmatrix}$$

with $\Delta_1 = \Delta_2$.

$$\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & I_2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ I_2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & I_2 \end{bmatrix}$$

are in $C(G)$, and since $QC(G){:}Q = 4$, they form an integral basis for $QC(G)$. Thus

$$C(G) = \{(x_{ij}I_2) \mid 1 \leqq i, j \leqq 2, x_{ij} \in \mathbf{Z}\}.$$

$C(G)$ is isomorphic to $M_{2\times 2}(\mathbf{Z})$ under $I_2 \to 1$, and $Z(G) \cong \mathrm{GL}(2, \mathbf{Z})$. Generating sets for $\mathrm{GL}(2, \mathbf{Z})$ are well known.

Let $S < \mathrm{Aut}(G)$ be the subgroup of $\mathrm{Aut}(G)$ induced by $N(G)$. From $N(G)/Z(G) \cong S$ it follows that $Z(G)$ together with a set of elements from $N(G)$ inducing S generate $N(G)$. Since S must contain the inner automorphism group, $\mathrm{Inn}(G)$, of G and $\mathrm{Aut}(G){:}\mathrm{Inn}(G) = 2$, we must have $S = \mathrm{Aut}(G)$ or $S = \mathrm{Inn}(G)$. In both of the two above cases, we have by direct computation that the outer automorphism

$$\Psi : \begin{cases} A \to A \\ B \to A^3B \end{cases} \text{ is not induced by } N(G).$$

Hence $N(G) = \langle Z(G), G\rangle$.

(b) $m = 6$ and G can be so chosen that

$$\Delta g = \begin{bmatrix} \Delta_1 g & 0 \\ 0 & \Delta_2 g \end{bmatrix}$$

with

$$\Delta_1 A = \Delta_2 A = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix},$$

$$\Delta_1 B = -\Delta_2 B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Using the technique of case (a) above, we have

$$C(G) = \left\{ \left[\begin{array}{c|c} xI_2 & yT \\ \hline zT & \omega I_2 \end{array}\right] \;\middle|\; T = \begin{bmatrix} 1 & 2 \\ -2 & -1 \end{bmatrix}, x, y, z, \omega \in \mathbf{Z} \right\}.$$

$C(G)$ is isomorphic to $R = \{[\begin{smallmatrix} x & -3y \\ z & \omega \end{smallmatrix}] \mid x, y, z, \omega \in \mathbf{Z}\}$ under the map indicated by the notation. Thus, to determine $Z(G)$, we need only determine the unit group $U(R)$ of R. Note that $U(R) = R \cap \mathrm{GL}(2, \mathbf{Z})$.

Using Lemma (5.13) and the same technique as in the proof of Lemma (5.11), we can show that

$$U(R) = \left\langle -I_2, \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\rangle,$$

and hence obtain a generating set for $Z(G)$. As in (a), to determine $N(G)$ from $Z(G)$ it suffices to see if

$$\Psi : \begin{cases} A \to A \\ B \to A^3B \end{cases} \text{ is induced by } N(G).$$

In this case, Ψ is induced by $T = [\begin{smallmatrix} 0 & I_2 \\ I_2 & 0 \end{smallmatrix}]$. Thus $N(G) = \langle Z(G), G, T\rangle$.

(c) $m = 4$ and G can be so chosen that

$$\Delta A = \left[\begin{array}{cc|cc} 0 & -1 & & \\ 1 & 0 & \multicolumn{2}{c}{0} \\ \hline & & 0 & -1 \\ \multicolumn{2}{c|}{0} & 1 & 0 \end{array}\right],$$

$$\Delta B = \left[\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ \hline & & 0 & 1 \\ \multicolumn{2}{c|}{0} & 1 & 0 \end{array}\right].$$

Using the technique of case (a), we have

$$C(G) = \left\{ \left[\begin{array}{cc|cc} x & y & z & k \\ -y & x & -k & z \\ \hline 2y & 0 & 2k+x & -y \\ 0 & 2y & y & 2k+x \end{array}\right] \,\middle|\, x, y, k, z \in \mathbf{Z} \right\}.$$

$C(G)$ is isomorphic to

$$R = \left\{ \begin{bmatrix} x & 2z-y \\ y & 2k+x \end{bmatrix} \,\middle|\, x, y, k, z \in \mathbf{Z} \right\}$$

under the map indicated by the notation. $U(R) = R \cap \mathrm{GL}(2, \mathbf{Z})$, and if we let $t = 2z - y$, $\omega = x + 2k$, then

$$U(R) = \left\{ \begin{bmatrix} x & t \\ y & \omega \end{bmatrix} \in \mathrm{GL}(2, \mathbf{Z}) \,\middle|\, \omega \equiv x(2), \quad y \equiv t(2) \right\}.$$

Using Lemma (5.12) and the technique of the proof of Lemma (5.11), we can show that

$$U(R) = \left\langle \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\rangle,$$

and hence obtain a generating set for $Z(G)$. Since

$$\Psi : \begin{cases} A \to A \\ B \to A^3 B \end{cases} \quad \text{is induced by} \quad T = \begin{bmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & -1 & 0 \\ 1 & -1 & 0 & -1 \\ 1 & 1 & 1 & 0 \end{bmatrix},$$

$N(G) = \langle Z(G), G, T \rangle$.

(5.34) *Example of the Case of a Klein* 4-*Group*. Let $G = \langle A, B \mid A^2 = B^2 = (AB)^2 = I_4 \rangle$.

(a) G can be so chosen that

$$A = -I_4, \qquad B = \begin{bmatrix} I_2 & I_2 \\ 0 & -I_2 \end{bmatrix}.$$

If for $X \in M_{4\times 4}(\mathbf{Z})$, we partition X into 2×2 blocks, say

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix},$$

then by direct computation, $X \in C(G)$ if and only if $X_{21} = 0$ and $2X_{12} = X_{11} - X_{22}$. If $X \in Z(G)$, then we must have $X_{11}, X_{22} \in \mathrm{GL}(2, \mathbf{Z})$ and $X_{11} \equiv X_{22}(2)$ or, equivalently, $X_{11}X_{22}^{-1} \equiv I_2(2)$. Hence we seek the group

$$H = \left\{ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in \mathrm{GL}(2, \mathbf{Z}) \,\middle|\, a_{11} \equiv a_{22} \equiv 1(2), \quad a_{21} \equiv a_{12} \equiv 0(2) \right\}.$$

Using Lemma (5.12) and the technique in the proof of Lemma (5.11), we can show that

$$H = \left\langle \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, -I_2 \right\rangle.$$

Thus, since

$$\mathrm{GL}(2, \mathbf{Z}) = \left\langle \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\rangle,$$

$$Z(G) = \left\langle \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right], \left[\begin{array}{cc|cc} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right], \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{array}\right], \right.$$

$$\left. \left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ \hline \multicolumn{2}{c|}{0} & \multicolumn{2}{c}{I_2} \end{array}\right], \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ \hline \multicolumn{2}{c|}{0} & \multicolumn{2}{c}{I_2} \end{array}\right], \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 \\ \hline \multicolumn{2}{c|}{0} & \multicolumn{2}{c}{I_2} \end{array}\right], \left[\begin{array}{c|c} -I_2 & -I_2 \\ \hline 0 & I_2 \end{array}\right] \right\rangle.$$

Since A is in the centre of $\mathrm{GL}(4, \mathbf{Z})$, the only possible nontrivial automorphism of G induced by $N(G)$ is

$$\Psi : \begin{cases} A \to A \\ B \to AB \end{cases}$$

and $N(G)$: $Z(G) \leqq 2$. Ψ is induced by

$$T = \left[\begin{array}{c|c} I_2 & I_2 \\ \hline -2I_2 & -I_2 \end{array}\right],$$

and hence $N(G) = \langle Z(G), T \rangle$.

(5.4) *Examples of the Block Triangular Normalizer Case.* Let G be a representative of one of the 12 Bravais classes satisfying (4.11).

(a) $G = \langle A, B \mid A^2 = B^2 = (AB)^2 = I_4 \rangle$.

G is a Klein 4-group. G can be so chosen that

$$A = -I_4, \qquad B = \left[\begin{array}{c|c} I_3 & \begin{matrix} 0 \\ 0 \\ 1 \end{matrix} \\ \hline \begin{matrix} 0 & 0 & 0 \end{matrix} & -1 \end{array}\right].$$

For $X \in N(G)$, X has the block form

$$\begin{bmatrix} X_{11} & X_{12} \\ 0 & X_{22} \end{bmatrix}$$

where X_{11} is a 3×3 matrix and X_{22} is a 1×1 matrix, say $X_{11} = (a_{ij})$ and $X_{22} = (d)$.

By direct computation, X is in $C(G)$ if and only if

$$X_{12} = \begin{bmatrix} a_{13}/2 \\ a_{23}/2 \\ (a_{33} - d)/2 \end{bmatrix}.$$

Thus $X \in Z(G)$ if and only if $X \in C(G)$ and

(i) $X_{11} \in \mathrm{GL}(3, \mathbf{Z})$,

(ii) $d = \pm 1$,

(iii) $a_{13} \equiv a_{23} \equiv 0(2)$,

(iv) $a_{33} \equiv d(2)$.

The first three conditions imply condition (iv), i.e., condition (iv) is redundant.

Using Lemma (5.12) and the technique as in the proof of Lemma (5.11), we can show that

$$\{ U = (u_{ij}) \in \mathrm{GL}(3, \mathbf{Z}) \mid u_{13} \equiv u_{23} \equiv 0(2)\}$$

$$= \left\langle N_1 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad N_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad N_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}, \right.$$

$$N_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad N_5 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad N_6 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\left. N_7 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad N_8 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right\rangle.$$

Thus

$$Z(G) = \left\langle \left[\begin{array}{c|c} N_1 & \begin{matrix}1\\0\\0\end{matrix} \\ \hline 0 & 1 \end{array}\right], \left[\begin{array}{c|c} N_2 & 0 \\ \hline 0 & 1 \end{array}\right], \left[\begin{array}{c|c} N_3 & \begin{matrix}0\\1\\0\end{matrix} \\ \hline 0 & 1 \end{array}\right], \left[\begin{array}{c|c} N_4 & 0 \\ \hline 0 & 1 \end{array}\right], \right.$$

$$\left. \left[\begin{array}{c|c} N_5 & 0 \\ \hline 0 & 1 \end{array}\right], \left[\begin{array}{c|c} N_6 & 0 \\ \hline 0 & 1 \end{array}\right], \left[\begin{array}{c|c} N_7 & 0 \\ \hline 0 & 1 \end{array}\right], \left[\begin{array}{c|c} N_8 & \begin{matrix}0\\0\\-1\end{matrix} \\ \hline 0 & 1 \end{array}\right], \left[\begin{array}{c|c} I_3 & \begin{matrix}0\\0\\1\end{matrix} \\ \hline 0 & -1 \end{array}\right] \right\rangle.$$

As in Example (5.34), to determine $N(G)$ from $Z(G)$, we need only check to see if the automorphism

$$\Psi : \begin{cases} A \to A \\ B \to AB \end{cases} \text{ is induced by } N(G).$$

By direct calculation, Ψ is not induced by $N(G)$. Hence, $N(G) = Z(G)$.

(b) $G = \langle A, B, C \mid A^2 = B^2 = C^4 = (BC)^2 = [A, B] = [A, C] = I_4 \rangle$. G is isomorphic to $C_2 \times D_8$. G can be so chosen that

$$A = -I_4, \qquad B = \left[\begin{array}{c|c} I_2 & 0 \\ \hline 0 & \begin{matrix}0 & 1\\ 1 & 0\end{matrix} \end{array}\right], \qquad C = \left[\begin{array}{c|c} I_2 & \begin{matrix}0 & 1\\ 0 & 1\end{matrix} \\ \hline 0 & \begin{matrix}0 & -1\\ 1 & 0\end{matrix} \end{array}\right].$$

For $X \in N(G)$, X has the block form

$$\begin{bmatrix} X_{11} & X_{12} \\ 0 & X_{22} \end{bmatrix},$$

where each X_{ij} is a 2×2 integral matrix, say $X_{11} = (a_{ij})$.

By direct computation, $X \in C(G)$ if and only if X_{22} is a scalar matrix, say $X_{22} = dI_2$, and

$$X_{12} = \frac{1}{2}\begin{bmatrix} a_{11} + a_{12} - d & a_{11} + a_{12} - d \\ a_{21} + a_{22} - d & a_{21} + a_{22} - d \end{bmatrix}.$$

Thus $X \in Z(G)$ if and only if $X \in C(G)$ and

(i) $d = \pm 1$,

(ii) $a_{11} + a_{12} \equiv a_{21} + a_{22} \equiv 1(2)$,

(iii) $X_{11} \in \mathrm{GL}(2, \mathbf{Z})$.

For $X_{11} \in \mathrm{GL}(2, \mathbf{Z})$, condition (ii) is equivalent to

(ii′) $a_{11} \equiv a_{22}(2)$ and $a_{12} \equiv a_{21}(2)$, and a_{11} and a_{21} must be of opposite parity.

As in Example (5.33)(c),

$$\{(u_{ij}) \in \mathrm{GL}(2, \mathbf{Z}) \mid u_{11} \equiv u_{22}(2), \quad u_{12} \equiv u_{21}(2)\}$$
$$= \left\langle \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \right\rangle.$$

Hence,

$$Z(G) = \left\langle \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ -1 & 0 & -1 & -1 \\ \hline \multicolumn{2}{c|}{0} & \multicolumn{2}{c}{I_2} \end{array}\right], \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & -1 \\ \hline \multicolumn{2}{c|}{0} & \multicolumn{2}{c}{I_2} \end{array}\right], \left[\begin{array}{cc|cc} 1 & 2 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ \hline \multicolumn{2}{c|}{0} & \multicolumn{2}{c}{I_2} \end{array}\right], \left[\begin{array}{c|cc} I_2 & 1 & 1 \\ & 1 & 1 \\ \hline 0 & \multicolumn{2}{c}{-I_2} \end{array}\right] \right\rangle.$$

Let S be the subgroup of $\mathrm{Aut}(G)$ induced by $N(G)$. Since A is in the centre of $\mathrm{GL}(4, \mathbf{Z})$, it remains fixed under the action of $N(G)$. G is the direct product of its subgroups $\langle A \rangle$ and $\langle B, C \rangle$. Thus every element of G can be expressed uniquely as $A^a B^b C^c$, $a = 0, 1$; $b = 0, 1$; $c = 0, 1, 2, 3$. The elements of G with $a = 0$ and $a = 1$ are of the forms

$$\begin{bmatrix} I_2 & * \\ 0 & * \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -I_2 & * \\ 0 & * \end{bmatrix}, \text{ respectively.}$$

Since $N(G)$ is block triangular, no element of the first form can be transformed into an element of the second form and conversely under the action of $N(G)$. For if

$$\begin{bmatrix} X_{11} & X_{12} \\ 0 & X_{22} \end{bmatrix} \begin{bmatrix} I_2 & * \\ 0 & * \end{bmatrix} = \begin{bmatrix} -I_2 & * \\ 0 & * \end{bmatrix} \begin{bmatrix} X_{11} & X_{12} \\ 0 & X_{22} \end{bmatrix},$$

then $X_{11} = 0$, a contradiction.

Hence, under the action of $N(G)$, B and C must go to elements of $\langle B, C \rangle$. Therefore, S is isomorphic to a subgroup of $\mathrm{Aut}(D_{12}) \cong D_{12}$. Since S contains $\mathrm{Inn}(G)$ and $\mathrm{Aut}(D_{12}):\mathrm{Inn}(D_{12}) = 2$, to determine $N(G)$ from $Z(G)$, it suffices to check if the outer automorphism

$$\Psi : \begin{cases} A \to A \\ B \to CB \\ C \to C \end{cases} \text{is induced by } N(G).$$

By direct computation, Ψ is not induced by $N(G)$. Thus, $N(G) = \langle Z(G), G \rangle$.

The technique used in the above example to determine $N(G)$ from $Z(G)$ is also applicable to several other Bravais groups, in particular, to a Bravais group of isomorphism type $C_2 \times D_{12}$. In this case, any attempt to directly determine $N(G)$ from $Z(G)$ would be extremely difficult as $|\mathrm{Aut}(C_2 \times D_{12})| = 144$.

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