REVIEWS AND DESCRIPTIONS OF TABLES AND BOOKS

The numbers in brackets are assigned according to the indexing system printed in Volume 22, Number 101, January 1968, page 212.

1 [2.05].—Arthur Sard & Sol Weintraub, *A Book of Splines*, John Wiley & Sons, New York, 1971, x + 817 pp., 24 cm. Price $22.50.

This book consists of three descriptive and expository chapters totalling 93 pages, an appendix of 12 pages describing a Fortran program for spline approximation, a bibliography of 13 pages inclusive through March 1, 1970, and three elaborate tables of numerical data comprising 693 pages.

The point of view taken throughout is that a spline ξ which approximates an unknown function x is based on linear scalar observations $F_0x, \cdots, F_mx$ of x and the knowledge of the existence of $\int_I (D^n x)^2$ over some interval I. The splines computable from the tables are based on the particular observations $F_ix = x(c + ih)$, $i = 0, \cdots, m$, where c and h are real numbers with $h > 0$, $m = n(1)20$, and $n = 2(1)5, 8$. Chapters 1 and 2 are devoted to instruction on the use of the tables. They are self-contained and written independently of each other, though Chapter 2 is more complete. Thus, Chapter 1, through examples, instructs the reader on how to construct the basis of cardinal spline functions if $m \geqq n$ (or, alternatively, the Lagrange interpolation basis functions if $m = n - 1$) and discusses approximate differentiation and integration by the use of these exact operations on the natural spline interpolant. Chapter 2, however, includes in addition a discussion of the method of calculation of error in natural spline interpolation. The reader is shown here how to compute sharp upper bounds for $|x(t) - \xi(t)|$, $t \in I$. More complete use of the tables is required here.

Chapter 3 draws the entire book together by means of a self-contained exposition of the theory of splines. Eight distinct characterizations of ξ, four of them involving optimality, are presented here. In the interests of clarity and completeness, we shall present these now.

Let I be a compact interval of the real line and let X denote the space of functions x on I whose $(n - 1)$th derivative $D^{n-1}x$ is absolutely continuous on I and whose nth derivative $D^n x$ is square-integrable on I. Here $n \geqq 1$. Let X^* denote the set of functionals Ψ of the form

$$\Psi x = \sum_{\nu=0}^{n-1} \int_I D^\nu x(s)\, d\sigma_\nu(s) + \int_I D^n x(s)\psi(s)\, ds, \qquad x \in X,$$

where $\sigma_0, \cdots, \sigma_{n-1}$ are functions of bounded variation on I and ψ is square-integrable. X^* is precisely the space of continuous linear functionals on X when the latter is normed by

$$||x||^2 = \sum_{\nu=0}^{n-1} |D^\nu x(a)|^2 + \int_I (D^n x)^2 \qquad (a \in I).$$

Now let $F_0, \cdots, F_m$, $m \geqq n - 1$, be fixed functionals in X^* which are linearly inpendent and set $q = 2n - 1$. We define the key functions

$$f_i(t) = F_{i,s}(|t - s|^q), \qquad t \in \mathbf{R},\ i = 0, \cdots, m.$$

If $I_0 \subset I$ is the smallest closed interval which contains the support of $F_0, \cdots, F_m$, let $a \in I_0$ and define matrices P, Φ and A as follows. (Row indices are denoted by i and column indices by ν.)

$$P = F_i[(s - a)^\nu], \qquad i = 0, \cdots, m, \nu = 0, \cdots, n - 1;$$

$$\Phi = F_i(f_\nu), \qquad i = 0, \cdots, m, \nu = 0, \cdots, m;$$

$$A = \begin{bmatrix} 0 & P^* \\ P & \Phi \end{bmatrix}.$$

Here 0 is an $n \times n$ matrix of zeros and * denotes transpose. The matrices Φ and A are symmetric and A is invertible if and only if P is of full rank n. We assume A to be invertible (valid for the point functionals of the tables) and we write $B = A^{-1}$ as

$$B = \begin{bmatrix} B_{1,1} & B_{1,2} \\ B_{2,1} & B_{2,2} \end{bmatrix} = B^*,$$

where the partitioning is consistent with that of A. We introduce the notationally convenient column matrices

$$F = (F_i), \qquad i = 0, \cdots, m,$$

$$f = (f_i), \qquad i = 0, \cdots, m,$$

$$\omega = Fx = (F_i x), \qquad i = 0, \cdots, m, x \in X,$$

$$\eta = ((u - a)^i), \qquad i = 0, \cdots, m, u \in \mathbf{R},$$

of functionals, functions, scalars, and functions, respectively. Finally, we define the cardinal spline functions β_i, $i = 0, \cdots, m$, by

$$\beta = (\beta_i) = [B_{2,1}\ B_{2,2}]\begin{bmatrix} \eta \\ f \end{bmatrix}$$

and the spline projector Π on X by

$$\xi = \Pi x = \beta^* \omega.$$

The linear space M of splines is defined to be the range of the linear projector Π and is spanned by $\beta_0, \cdots, \beta_m$. This, then, is the definition of spline as presented by the authors. The remaining seven characterizing properties of ξ and/or M follow.

I. *Characterization of M Via $M^\perp$.* Define on X the Hilbert space inner product

$$(x, y) = \sum_{i=0}^{m} F_i x F_i y + \int_I D^n x D^n y.$$

Let $N = \{x \in X: F_i x = 0, i = 0, \cdots, m\}$. Then $M = N^\perp$ and $N = M^\perp$.

II. *Geometric Property of the Spline Approximation.* Let $G \in X^*$, $d \geqq 0$ and $\omega \in \mathbf{R}^{m+1}$ be prescribed; let

$$\Gamma = \{x \in X : Fx = \omega\} \quad \text{and} \quad \int_I (D^n x)^2 \leqq d^2.$$

Let $\{\xi_0\} = \Pi\Gamma$. Then the set $G\Gamma$ is the closed interval with midpoint $G\xi_0$ and length equal to twice the square root of $J[d^2 - 2(-1)^n q!\omega^* B_{2,2}\omega]$. J is given explicitly by

$$J = \frac{(-1)^n G_t G_s[|t-s|^q] - \gamma^* B\gamma}{2q!}, \qquad q = 2n - 1,$$

and

$$\gamma = \begin{bmatrix} G\eta \\ Gf \end{bmatrix}.$$

III. *Interpolating Property of the Splines.* For each $x \in X$, there is one and only one $\xi \in M$ such that $F\xi = Fx$. Furthermore, $\xi = \Pi x$.

IV. *Minimal Deviation Among Interpolants.* For each $x \in X$, the integral $\int_I (D^n y)^2$ is minimal among all $y \in X$ such that $Fy = Fx$ if and only if $y = \Pi x$.

V. *Minimal Quotient.* For any $G \in X^*$, define the set $\mathcal{H}$ of admissible approximations of G by

$$\mathcal{H} = \{H \in X^* : Hx = Gx \text{ whenever } D^n x = 0\}.$$

Set $R = G - H$ for $H \in \mathcal{H}$. For each such H there is a continuous linear functional Q on $L^2(I)$ (called the quotient of R and D^n) such that $R = QD^n$. Define $H_0 = G\Pi$. Then $H_0 \in \mathcal{H}$ and $||Q||$ is minimal among $H \in \mathcal{H}$ if and only if $H = H_0$, in which case

$$\min ||Q||^2 = J = \sup_{0 \neq \zeta \in N} \left[|G\zeta|^2 \Big/ \int (D^n \zeta)^2 \right].$$

Here N is defined in I and J agrees with the constant of II.

VI. *Minimal Deviation in M.* For each $x \in X$, the integral $\int_I (D^n x - D^n y)^2$ is minimal among $y \in M$ if and only if $D^n y = D^n \xi$ where $\xi = \Pi x$, i.e., if and only if $y - \xi$ is polynomial of degree $n - 1$.

VII. *Analytic Description of M.* The splines are precisely those functions ξ on R which are such that

(i) ξ is a linear combination of the key functions $f_0, \cdots, f_m$ plus a polynomial of degree $n - 1$, and,

(ii) $D^n \xi(u) = 0$ for all $u \in I_0$.

The authors give credit to various authors for their contributions to the characterizations I–VII.

Properties I, II, and III are due to Golomb and Weinberger (in *On Numerical Approximation*, R. E. Langer, Editor, University of Wisconsin Press, Madison, 1959, pp. 117–190), in even greater generality. Property IV, in the case of natural cubic splines, was known to Schoenberg (*Quart. Appl. Math.*, v. 4, 1946, pp. 45–99) though Holladay (*MTAC*, v. 11, 1957, pp. 233–243) appears to have supplied the first proof. Properties IV and VI, in even greater generality, are implicit in Golomb and Weinberger. Walsh, Ahlberg and Nilson (*J. Math. Mech.*, v. 11, 1962, pp. 225–234) explicitly treated a special case of IV and VI. The first explicit treatment, in the generality considered here, appears to be due to DeBoor and Lynch (*J. Math. Mech.*, v. 15, 1966, pp. 953–969).

Property V is due to Sard (*Amer. J. Math.*, v. 71, 1949, pp. 80–91, and *Linear*

Approximation, Amer. Math. Soc., Providence, R.I., 1963) although the precise calculation of J appears to be due to Secrest (*Math. Comp.*, v. 19, 1965, pp. 79–83) in the case of quadrature. Finally, Schoenberg, in a number of interesting papers between 1964 and 1968, discussed the equivalence of III, IV, V, VI, and VII in a number of important special cases.

We shall now return to the discussion of the numerical tables. The first table allows one to obtain the representations of the cardinal spline functions $\beta_i(u)$ in terms of $1, u, \cdots, u^{n-1}, |u|^q, |u-1|^q, \cdots, |u-m|^q$ (alternatively, $1, u, \cdots, u^{n-1}, u_+^q, \cdots, (u-m)_+^q$) when the uniformly spaced knots are chosen to be $0, 1, \cdots, m$. The table also produces the representation of $\beta_i(t)$ in terms of powers and absolutes or pluses when the knots are symmetrically placed about zero at unit spacing. The coefficients in these representations are given accurately to 13S and are based on 30S values obtained on a CDC 6600 computer. The coefficients were computed in three independent ways, viz., by the direct inversion of the matrix A, by the use of the duals of the functionals F_i [Golomb-Weinberger, *loc. cit.*] and by certain recursion formulas due to Greville [privately communicated by Schoenberg]. The authors report that the first of these three methods was the most efficient, despite the fact that A is not a particularly well-conditioned matrix. The authors speculate that the use of the absolute-value functions, rather than the plus functions, in obtaining A provides the symmetry which makes the inversion of A more effective than it otherwise would be. Indeed, A is symmetric about both diagonals if the knots are symmetrically placed.

Now, it is a simple matter, achieved by an elementary affine transformation, to pass to the cardinal spline functions with uniformly spaced knots in general position. In any event, Table 1, with only special exceptions, provides only the *representation* of the desired spline. If the table user wishes to avoid the calculations necessary for evaluation, differentiation, and integration, he may employ the Fortran program provided. The necessary input information for the punched data cards is presented very clearly. The numerical data are taken from Table 1. Sample output data are shown for four illustrative problems.

Table 2 gives the entries of B to 6S for the particular functionals $F_ix = x(i-p)$, $i = 0, \cdots, m = 2p$, where p is half-integral or integral. Table 3 provides the coefficients required to form the Lagrange interpolant when $m = n - 1$ and $n = 2(1)10$. It should be noted that the range of n is different here from that in Tables 1 and 2.

The reviewer was impressed with the clarity and accuracy of the exposition, both in the instructional Chapters 1 and 2 and in the theoretical Chapter 3. In the careful and impartial meting out of credits in Chapter 3, the authors have performed a considerable service. In particular, the fundamental role of the contribution of Golomb and Weinberger is made clear, a fact not sufficiently recognized in the past.

Altogether, this authoritative book should provide a valuable service to both the users of old mathematics and the makers of new mathematics.

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2 [2.05].—ARNOLD SCHÖNHAGE, *Approximationstheorie*, Walter de Gruyter & Co,. Berlin, New York, 1971, 212 pp., 24 cm. Price DM44.— or $14.—.

This concisely written text offers an excellent opportunity to become acquainted with classical problems and achievements of approximation theory. There are eight chapters covering: (i) Basic tools and generalities from functional analysis. (ii) Density theorems of Weierstrass, Stone, Bernstein and Müntz. (iii) Orthogonal polynomials. (iv) Trigonometric approximation. (v) Interpolation. (vi) Chebyshev approximation. (vii) L^1-approximation. (viii) Degree of approximation. Though moderately sized, the book contains a substantial amount of nontrivial matter in almost every chapter. One of the welcome contributions of the book is an introduction to approximation by functions of class G_ν, i.e., entire functions of exponential type and degree at most ν, bounded along and restricted to the real line. This theory, largely due to S. N. Bernstein, significantly extends and deepens trigonometric approximation; portions are worked into Chapters 2, 5 and 8.

Other highlights are: In Chapter 1, duality, strict and uniform convexity, Haar uniqueness theorem, precompactness and "uniform approximability," Bernstein's lethargy theorem, and orthogonal projections. In Chapter 3, separation and monotonicity properties of zeros of orthogonal polynomials, and a detailed discussion of Jacobi polynomials. In Chapter 4, pointwise and uniform convergence of Fourier series under the various classical hypotheses, the Kharshiladze-Lozinski theorem, and Fejér and de la Vallée-Poussin sums. In Chapter 5, approximation by interpolation, and the Markov's inequality in a refined version of Duffin and Schaeffer. Chapter 6 includes Solotarev polynomials with a weight function. Chapter 7 has a discrete analog of Nagy's criterion [1, p. 184] to be applied in the last chapter. There, one finds the generalization to G_ν-approximation of Jackson's second theorem; the latter is obtained as a corollary. Converse theorems of Bernstein and Zygmund and theorems (by Bernstein, Krein, Akhiezer and Favard) on the error in approximating classes of Lipschitz and holomorphic functions conclude the book. There is a moderate number of problems, a very brief bibliography and a usable index.

The material in this result-oriented book is tightly packed and its substance and detail should impress every reader. Proofs are usually complete and are kept as direct and elementary, hence often computational, as possible. In thus coping with limited space, the author nevertheless brings out the interplay of ideas on many occasions. The author's claim to a "modern presentation" needs some qualification since there is no reflection of the attempts by recent authors ([2], [3], etc.) to establish a unifying and more penetrating view of approximation theory. This is essentially a classical analysis book, accessible to second year graduate students in the U.S.A. Computer applications are barely mentioned in the introduction.

There are few misprints or other errors (the reviewer found an erroneous formula after (1.1) on p. 10; Theorem 1.10 is misstated; a factor $|f|$ is missing in problem 5.10; etc.). Credit is given erratically and sometimes erroneously: Bernstein's name occurs in ϵ-proportion to the great role of his results in this book. His generalization of Jackson's theorem is credited to the latter, Lozinski's theorem to Berman, and the student will never know that he learned, e.g., about Bernstein's lethargy result, Lebesgue constants, the Dini-Lipschitz condition or de la Vallée-Poussin sums.

But he will be excellently prepared to reach for more extensive treatises that include such helpful inessentials.

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1. N. I. AKHIEZER, *Theory of Approximation,* Ungar, New York, 1956.
2. P. L. BUTZER & H. BERENS, *Semigroups of Operators and Approximation,* Die Grundlehren der math. Wissenschaften, Band 145, Springer-Verlag, New York, 1967.
3. H. S. SHAPIRO, *Topics in Approximation,* Lecture Notes in Math., vol. 187, Springer-Verlag, New York and Berlin, 1971.

3 [2.05, 2.25, 2.40, 6, 7].—A. O. GEL'FOND, *Calculus of Finite Differences* (authorized English translation of the third Russian edition), Hindustan Publishing Corp., Delhi-7, India, 1971, vi + 451 pp., 23 cm. Price $10.00.

The first edition of this important treatise was published in 1952. A second, revised and enlarged, edition appeared in 1959, and the third edition (which is essentially identical with the second) in 1967, a year before the author's death. The book has been translated into several languages, including German, French, Chinese, Czechoslovakian, and Romanian. This appears to be the first translation into English. (For a review of the French edition, see Review 3, this Journal, v. 18, 1964, p. 514.)

The calculus of finite differences relates to three broad areas of analysis: interpolation and approximation, summation of functions, and difference equations. The present author places emphasis on the first of these areas, devoting to it three chapters, or about two-thirds of the book. Approximation processes in the complex plane receive particular attention.

Chapter I starts out with Lagrange's and Newton's interpolation formulas and some elementary facts on divided differences. The discussion then moves on to a general interpolation problem associated with an infinite triangular array of nodes, and to resulting interpolation series. There is a discussion of best approximation by polynomials, in preparation to a convergence result for the Lagrange interpolation process. Other polynomial approximation processes are studied, both for real and complex domains. In a final section, a general interpolation problem is conceived as a moment problem in the complex plane. Chapter II is concerned with convergence and representation properties of Newton's series. These are special interpolation series; the cases of equidistant, as well as arbitrary, interpolation points are studied in detail. Some number-theoretic applications are also included, e.g., the author's own proof of the transcendence of e and π. Chapter III, the most advanced and most technical chapter, deals with the problem of constructing an entire function from a denumerable set of data, e.g., from the function values at a sequence of points accumulating at infinity. Problems of this sort do not have unique solutions, but can be treated in a meaningful way by imposing suitable restrictions on the growth of the entire function. They have also bearing on the problem of solving linear differential equations of infinite order with constant coefficients, as is shown at the end of the chapter. The remaining two chapters return to a more elementary level, and to more standard topics, Chapter IV dealing with the problem of summation, Bernoulli numbers and Bernoulli polynomials, Euler's summation formula etc., and Chapter V dealing with the theory of linear difference equations, the usual algebraic results as well as the principal results on asymptotics, due to Poincaré and Perron.

While the effort of making this work available to the English-speaking community is commendable, the reader must be warned that the translation is seriously deficient and unreliable. The Russian language being devoid of articles, there are the usual mistakes of choosing a definite article when an indefinite one is called for, and vice versa. More seriously, there are numerous instances of semantic distortion which result in statements often totally incomprehensible. For example, on p. 23 one reads "Denote by A the identity element, which is taken with a certain number A", as compared with the original "Denote by sign A the value 1 taken with the sign of the number A"; on p. 65 one reads "This property of the power of x is known as the complete power of x in the class of functions . . ." instead of "This property of the powers of x is called completeness of the powers of x in the class of functions. . ."; on p. 231, ". . . the great Russian mathematician P. L. Chebyshev" is demoted to ". . . the talented Russian mathematician P. L. Chebyshev"; on pp. 255–256 the reader must unscramble sentences like "Let the domain D go over in the plane of a complex variable w when $w = u(z)$ is mapped onto the simply-connected domain D_1". In the face of such blatant distortions and a great many other irregularities of translation, the only advice one can give to a dismayed reader is to double-check with one of the other available translations.

W. G.

4 **[3]**.—NOEL GASTINEL, *Linear Numerical Analysis*, translated from the French, Academic Press, New York, 1971, ix + 341 pp., 23 cm. Price $15.00.

This is a translation of the author's *Analyse Numérique Linéaire*, published in 1966. The translator (unnamed) has taken a few mild liberties, but no doubt with the author's knowledge and consent. The foreword is abridged. In the original, there are chapters, sections, and some subsections, but in the translation only chapters and sections, and some of the titles are changed. One or two figures are omitted. Some theorems are formally stated and numbered in the translation that are not so stated in the original. Otherwise the translation is faithful.

The book itself is strongly algorithmic. The theory is developed from first principles (vector spaces, matrices, a postulational development of determinants) and proceeds to ALGOL programs. The theory is clearly, but succinctly, developed. There are a number of exercises, both theoretical and algorithmic.

Nearly all the standard methods for inversion, direct and iterative, are described, including some attention to SOR. For eigenvalues and eigenvectors, the coverage is a bit less complete. The chapter opens with a brief discussion of interpolative methods, not recommended, however, unless perhaps a very good initial approximation to a root is known. It is also implied in the original and explicitly stated in the translation that the root must be real, which is not strictly true.

After this, which is more or less an aside, the chapter continues with Krylov, Leverrier and Souriau's improvement, Samuelsen, "partitioning" (Bryan), Danilevskiĭ (where the author uses the "matrices of the second degree" that he had introduced in his thesis, a generalization of the reviewer's "elementary matrices"), reduction to triple-diagonal form, again by means of the matrices of second degree, Lanczos, and Givens. All these are methods of reduction. Finally come the power methods (but not backward, or the Rayleigh quotient), deflation, Jacobi, and LR, but QR receives only three lines at the end with a reference to Francis, who is named in the original but not in the translation.

It is a pity that the French consider an index of no value, and there is none in either the original or the translation. But apart from these minor quibbles, the translation is very good and the book fulfills its purpose excellently well.

A. S. H.

5 [3].—J. K. Reid, Editor, *Large Sparse Sets of Linear Equations*, Academic Press, New York, 1971, x + 284 pp., 24 cm. Price $16.00.

It is by no means uncommon that workers in disjoint fields will be faced with similar computational requirements, and that each group will develop its own techniques in ignorance of those developed by the others. A classical, and one might say glaring, example, is the method for finding eigenvalues proposed by the astronomer Leverrier in 1840, and rediscovered independently about a century later by a statistician, a psychometrician, and several mathematicians, admittedly with some improvements, even though Krylov in 1931 had described a method that was far superior.

The efficient handling of sparse matrices, a rather broader field, is another example. In 1968, a symposium was held at IBM, Yorktown Heights, in an attempt to establish communication, and a second took place there in the late summer of 1971. This volume reports the proceedings of a similar conference held at Oxford in April of 1970. The volume concludes with a somewhat discursive paper by Ralph A. Willoughby describing work at IBM and the organization of the 1968 symposium.

Principal subject matter areas represented here are structural analysis, power systems, linear programming and (more generally) optimization, and geodetics. Not surprising is the fact that Kron's "tearing", later formalized in what the reviewer calls "the method of modification", appears several times. One paper is devoted to "bi-factorisation", which is the simultaneous application on the left and on the right of what the reviewer calls "elementary matrices", those on the left differing from the identity only below the diagonal in one column, those on the right differing only to the right in one row. Joan Walsh gives a survey of direct and indirect (iterative) methods, Frank Harary discusses the use of graph theory, the editor (whose name is modestly omitted in the table of contents) discusses the method of conjugate gradients.

Some of the discussion following each paper is included, somewhat edited, as the editor confesses, and assuredly to the reader's benefit.

To develop a unified theory here may be impossible, and I, personally, was surprised to find as much unity and coherence in these papers as I did, and certainly for those interested in the subject, this volume is essential.

A. S. H.

6 [4].—SANFORD M. ROBERTS & JEROME S. SHIPMAN, *Two-Point Boundary Value Problems: Shooting Methods*, Elsevier, New York, 1972, xiv + 269 pp., 24 cm. Price $16.50.

This is the only text on the use of shooting methods to solve boundary value problems and, as a consequence, contains information and references not readily available elsewhere. It is a useful book, but it is not easy to read nor should its advice be accepted uncritically.

The level and presentation are quite uneven. The excessively long treatment of the methods of adjoints and of complementary functions would have been both shorter and clearer using vectors. In contrast, portions of the book make serious use of functional analysis. A good many sections leave the impression of being partially digested research papers. The authors are not successful in their goal of using Newton's method to clarify equivalences and relations among a collection of apparently different procedures. There is undue redundancy and quite a bit of material could have been dropped profitably as unimportant. On the other hand, some really important topics like Conte's method and multiple shooting are only sketched. It is curious that multiple shooting is not even classed as a shooting method and appears in the chapter on finite-difference methods.

To understand shooting methods, it is essential to understand the implications of the stability of the initial value problem (cf. their comments about initial conditions on p. 176). The authors spend only a couple of paragraphs on this and several pages on an irrelevant "review" of stability of some methods for numerically solving initial value problems. To use shooting methods, it is essential to have a first class code for the initial value problem. It appears that the authors used a fixed step, fourth order Runge-Kutta code in their examples. Far more attention ought to have been devoted to this matter and to other computational matters such as the storage question when solving nonlinear problems. Codes would have been desirable.

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7 [6].—JAMES ALAN COCHRAN, *Analysis of Linear Integral Equations*, McGraw-Hill Book Co., New York, 1972, xi + 370 pp., 24 cm. Price $15.95.

This book consists of a collection of important results in the theory of linear, nonsingular integral equations. The topics covered include the standard material found in most texts on integral equations as well as more recent and less accessible results. Chapters 1 to 6 cover the classical topics: Fredholm integral equations via the Schmidt theory and the Fredholm-Carleman theory, eigenvalue problems and Volterra equations. The eigenvalue problem for hermitian kernels is studied in more detail in Chapters 7 to 13. Various existence proofs are given and techniques for obtaining upper and lower bounds by the Rayleigh-Ritz procedure and the Weinstein-Aronszajn method are discussed. The last five chapters deal with the properties of non-hermitian kernels of certain types, such as nuclear and composite kernels, positive, anti-hermitian and symmetrizable kernels. Wiener-Hopf equations are briefly dealt with in the last chapter.

The book is definitely intended for the applied mathematician. While there are relatively few examples arising directly from physical problems, the selection of topics reflects the book's aim toward applications. The author has avoided highly abstract formulation and thus made the book available to a large class of readers. It is suitable also as a text for a graduate course in integral equations; the interesting exercises at the end of each chapter contribute to this aspect.

A wide variety of topics is covered in a little over 300 pages; thus the treatment is occasionally brief. However, an extensive, up-to-date bibliography helps to direct the interested reader to the appropriate research papers and more detailed coverage. The main strength of the book is that it brings together a number of important topics in integral equations in an easily accessible form. This alone should make it a useful addition to the collection of the applied mathematician.

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8 [7].—C. WILLIAM MARTZ, *Tables of the Complex Fresnel Integral*, Report NASA SP-3010, Scientific and Technical Information Division, National Aeronautics and Space Administration, Washington, D.C., 1964, v + 294 pp., 27 cm. Copies obtainable from the National Technical Information Service, Operations Division, Springfield, Virginia 22151. Price $4.00.

The title is somewhat misleading, since the tabulation is actually that of the function

$$E(z) = R(z) + iI(z) = \int_0^z \exp\left(\frac{i\pi}{2}u^2\right) du.$$

On the other hand, the complex Fresnel integrals are defined by

$$C(z) = \int_0^z \cos\left(\frac{\pi}{2}u^2\right) du = C_1(z) + iC_2(z),$$

$$S(z) = \int_0^z \sin\left(\frac{\pi}{2}u^2\right) du = S_1(z) + iS_2(z).$$

Hence, to obtain the latter quantities from the tables one must use the relations

$$C_1(z) = [R(x + iy) + R(x - iy)]/2,$$

$$C_2(z) = [I(x + iy) - I(x - iy)]/2,$$

$$S_1(z) = [I(x + iy) + I(x - iy)]/2,$$

$$S_2(z) = [-R(x + iy) + R(x - iy)]/2.$$

In these 5S tables (without differences), y extends over the range $-2.60(0.02)1.82$, while the range of x is variable, depending on the current value of y. For $y > 0$, the tabulation is carried out until both R and I are equal to 0.50000. For $y < 0$, the terminal value of x is either 20 or that value for which subsequent values of R and I are of the order of 10^8 or greater.

The arrangement of the tables is somewhat inconvenient, inasmuch as the second of the four columns on each page is not a continuation of the first column on that page but instead is that of the first column of some subsequent page.

The tables are prefaced by a description of their contents and use, their method of calculation, and means of finding values corresponding to arguments outside the tabular range.

The entries (given in floating-point decimal format) were subjected to a spot check against corresponding values computed independently by the reviewer, and no discrepancies were found.

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9 [9].—Morris Newman, *A Table of $\tau(p)$ modulo p, p prime*, $3 \leqq p \leqq 16067$, National Bureau of Standards, August 1972, 7 pp. of computer output deposited in the UMT file.

Let $\tau(n)$ denote the Ramanujan function, defined by

$$\sum_{n=1}^{\infty} \tau(n)x^n = x \prod_{n=1}^{\infty} (1 - x^n)^{24}.$$

Then $\tau(n)$ satisfies the recurrence formula

$$\tau(np) = \tau(n)\tau(p) - p^{11}\tau(n/p),$$

where p is a prime and $\tau(n/p)$ is defined to be zero if p does not divide n. Thus, if p happens to divide $\tau(p)$, then p divides $\tau(np)$ for all n.

As stated in the title, this table lists the values of $\tau(p)$ modulo p for all primes p such that $3 \leqq p \leqq 16067$. In addition to the known cases $p = 2, 3, 5$, and 7, the table shows that there is just one more prime p in the indicated range that divides $\tau(p)$; namely, $p = 2411$.

The table was computed by means of the congruence

$$\tau(n) \equiv 540 \sum_{k=1}^{n} \sigma_3(k)\sigma_3(n - k) \bmod n,$$

where $\sigma_3(n)$ denotes the sum of the cubes of the divisors of n.

This table was motivated by the unresolved question as to the existence of an n for which $\tau(n) = 0$.

Author's summary

Editorial note: From D. H. Lehmer's table of $\tau(n)$ for $n = 1(1)10000$ (*Math. Comp.*, v. 24, 1970, pp. 495–496, UMT 41) several $\tau(p)$ were selected and reduced (mod p) and no discrepancies were found when the results were compared with those in the present table. For $p = 2411$, one finds

$\tau(p)/p = 1883882662835292$, which ratio is not itself divisible by p. The ratio $\tau(p) \pmod p)/p$ appears to be distributed uniformly between 0 and 1. That implies that the number of such "Newman primes" (i.e., 2, 3, 5, 7, 2411, . . .) that do not exceed N should be asymptotic to $\sum_{p \leq N}(1/p) \sim \ln \ln N$. Since the normal order of magnitude of $\tau(p)$ is $\pm p^{11/2}$, and since there are no other Newman primes $\leqq$ 16067, it is therefore very improbable that $\tau(n)$, which is multiplicative, will have a zero.

D. S.

10 **[9]**.—Samuel Yates, *Prime Period Lengths*, RCA Defense Electronic Products, Moorestown, New Jersey. Ms. (undated) of 525 pp. deposited in the UMT file.

This voluminous unpublished table gives the length of the decimal period of the reciprocal of each of the 105000 odd primes (excluding 5) from 3 to 1370471, inclusive. This compilation evolved over the past four years from calculations performed on a succession of electronic computers such as IBM 7090, XDS Sigma 7, RCA Spectra 70/45, and (mainly) RCA Spectra 70/55 at the Moorestown computer facility.

The author has supplied supplementary detailed information relating to the density of those tabulated primes having 10 as a primitive root, from which we find, for example, that there are precisely 39447 such primes in the tabular range. On the other hand, Artin's conjecture [1] predicts a count of 39266 in the same range; however, there exists heuristic reasoning [2] to support the observation that the density of such primes generally exceeds the predicted density. (This reviewer has found the first exception to occur for the interval ending with the prime 138289.) It may be noted here that Cunningham [3] erroneously gave 3618, instead of 3617, as the count of such primes less than 10^5. Also, D. H. Lehmer & Emma Lehmer [4] reported a count of 8245 such primes below $2.5 \cdot 10^5$, attributed to Miller, but the latter in an unpublished table [5] has given this count as 8255, in agreement with one based on the present table.

The range of this new table is more than tenfold that of any of the previous tables of this type, as listed by Lehmer [6]. The table has materially assisted its author in his continuing search for new prime factors of integers of the form $10^n - 1$ [7].

J. W. W.

1. A. E. Western & J. C. P. Miller, *Tables of Indices and Primitive Roots*, Royal Society Mathematical Tables, v. 9, Cambridge Univ. Press, London, 1968, p. xli.
2. Daniel Shanks, *Solved and Unsolved Problems in Number Theory*, Spartan Books, Washington, D.C., 1962, pp. 80–83.
3. A. Cunningham, "On the number of primes of the same residuacity," *Proc. London Math. Soc.*, (2), v. 13, 1914, pp. 258–272.
4. D. H. Lehmer & Emma Lehmer, "Heuristics, anyone?," *Studies in Mathematical Analysis and Related Topics*, Stanford Univ. Press, Stanford, Calif., 1962, pp. 202–210.
5. J. C. P. Miller, *Primitive Root Counts*, University Mathematical Laboratory, Cambridge, England. Ms. deposited in UMT file; *Math. Comp.*, v. 26, 1972, p. 1024, RMT **54**.
6. D. H. Lehmer, *Guide to Tables in the Theory of Numbers*, National Research Council Bulletin No. 105, Washington, D.C., 1941, p. 15.
7. Samuel Yates, *Partial List of Primes with Decimal Periods Less than 3000*, Moorestown, New Jersey, Ms. deposited in UMT file; *Math. Comp.*, v. 26, 1972, p. 1024, RMT **55**.

11 **[10]**.—P. A. Morris, *Self-Complementary Graphs and Digraphs*, 24 pp. deposited in the UMT file.

Two graphs, G and $\bar{G}$, on the same set of nodes, are *complementary* if two nodes are joined in G if, and only if, they are not joined in $\bar{G}$. Two digraphs D and $\bar{D}$, on the same set of nodes are complementary if two nodes i and j are joined in D by a directed edge from i to j if, and only if, they are not joined in $\bar{D}$ by a directed edge from i to j. A graph or digraph which is isomorphic to its complement is said to be *self-complementary*.

Table A shows the 10 self-complementary graphs on 8 nodes, Table B the 36 self-complementary graphs on 9 nodes, and Table C the 10 self-complementary digraphs on 4 nodes. An accompanying text describes the method used.

AUTHOR'S SUMMARY

12 [12].—BORIS BEIZER, *The Architecture and Engineering of Digital Computer Complexes*, Vols. I and II, Plenum Press, New York, 1971, li + 394 pp. (Vol. I), xix + 847 pp. (Vol. II), 24 cm. Price $22.50 each volume.

This two-volume opus is a comprehensive discussion of the varied and interrelated problems that face the designer of a large computer complex. It is impressive in its scope and in the organization of its material, covering topics ranging from the mathematics of flowchart analysis and a dissection of machine instruction types to personnel management and the difficulties of handling thick cables under a computer room floor.

The first volume is mainly concerned with the components of a computer complex. Chapter 2 provides an excellent discussion of instruction repertoires, including various approaches to addressing, indexing, and instruction modification, and a classification of the types of instructions that appear in computers. Chapter 3 discusses the structural elements of a computer complex: memories, interrupt handling, controllers, and peripheral devices. There are two chapters on programming. Chapter 4 examines the programming process, considers certain selected techniques of general applicability, and shows how tradeoffs can be applied in this area. Chapter 5 is concerned with firmware, that is, the supporting programs that are required in order to make application programs work: assemblers, loaders, compilers, and utilities. Chapter 6 is on analysis, and is followed up in the second volume. It begins with an elementary discussion of statistics, and then shows how statistical techniques can be applied to estimating the behavior of programs, in terms both of time and of space. Transformation of flowcharts are shown to be a useful analytic tool, and the behavior of various statistical measures under these transformations is developed.

The second volume deals with questions of system organization. Chapter 8 considers the partitioning of tasks among hardware and software resources, as well as some of the interconnection problems. This chapter introduces a great deal of terminology, some of which, unfortunately, is rather obscure. Chapter 9 considers the functions and organization of the system executive, while Chapter 10 considers the system nucleus, whose task it is to manage storage and input-output. Chapters 11 and 12 consider the problem of system viability—that is, how to keep the system alive despite hardware and software failures and overloads. Viability has three components: *performance*, the ability of the system to handle its appointed tasks under varying loads; *reliability*, the mean time between failures; and *maintainability*, the mean time to recover from a failure. The viability executive has the task of maintaining system viability under the assumption that any part of the system, including the viability executive itself, can fail. I found the discussion of this problem and some of the solutions to it to be quite stimulating. Chapter 13 deals with system analysis, and considers questions such as the length of the basic system cycle and the response to various loading conditions. Chapter 14, the final chapter, is concerned with the process by which a system is implemented, from procurement through installation.

I recommend this book both for reference and for teaching. It is well written, though there are some unfortunate lapses of clarity. The author seems to be particularly influenced by the overseas AUTODIN system, a communications-handling system constructed by Philco for the Defense Communications Agency, and some of the material on viability, in particular, is directed towards systems of that genre. There are problems at the end of each chapter. Some of these are quite thought-provoking, though a few seemed ill-defined and insufficiently related to the preceding material. There are quite a few amusing anecdotal footnotes, which I found to be one of the most charming aspects of the book.

In sum, this book is a significant addition to the computer science literature. While it makes no great theoretical contributions, it is a cogent presentation of a wide range of pragmatic knowledge.

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