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An Asymptotic Expansion of $W_{k,m}(z)$ with Large Variable and Parameters

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Abstract. In this paper, we obtain an asymptotic expansion of the Whittaker function $W_{k,m}(z)$ when the parameters and variable are all large but subject to the growth restrictions that $k = o(z)$ and $m = o(z^{1/2})$ as $z \to \infty$. Here, it is assumed that k and m are real and $|\arg z| \leqq \pi - \delta$.

1. Introduction. In this paper, we are concerned with the asymptotic behavior of the Whittaker function $W_{k,m}(z)$. This function depends on two parameters and a variable. When the parameters k and m are fixed and the variable z is large, it is well known that a complete asymptotic expansion can be obtained; see [1, Section 7.1]. However, if the parameters k and m are allowed to increase without limit, the problem of finding asymptotic forms for $W_{k,m}(z)$ becomes much more involved and has been the subject of numerous investigations; see Buchholz [1], Chang, Chu and O'Brien [2], Kazarinoff [7], Erdélyi and Swanson [5], Slater [8] and the references given there. Although a great number of papers have been written on this subject, the treatment with two parameters and a variable is still incomplete.

In a recent paper [11], Wong and Rosenbloom have studied a certain inequality (see [4, p. 124]) connecting Whittaker functions and parabolic cylinder functions $D_\lambda(z)$, and shown that this inequality can be improved considerably. However, the above-mentioned paper contains the restriction that k and m be again fixed. The purpose of this paper is to show that this condition can be relaxed so that k and m may depend on z. Moreover, we give a complete asymptotic expansion of $W_{k,m}(z)$ when the parameters and the variable are all large, i.e.,

$$k,\ m \quad \text{and} \quad z \to \infty \tag{1.1}$$

but subject to the growth restrictions that

$$k = o(z) \quad \text{and} \quad m = o(z^{1/2}) \quad \text{as } z \to \infty . \tag{1.2}$$

Here, it is supposed that k and m are real and $|\arg z| \leqq \pi - \delta$. The term "asymptotic" is used in the sense of Erdélyi and Wyman [6], which is more general than the usual Poincaré sense. This distinction is made clear in the theorems.

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2. Two Auxiliary Results. It is well known that Hankel functions $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$ have the asymptotic expansions

$$H_\nu^{(1)}(z) = \left(\frac{2}{\pi z}\right)^{1/2} e^{i(z-\nu\pi/2-\pi/4)} \left\{\sum_{m=0}^{p-1} \frac{(-1)^m(\nu, m)}{(2iz)^m} + R_p^{(1)}\right\} \tag{2.1}$$

and

$$H_\nu^{(2)}(z) = \left(\frac{2}{\pi z}\right)^{1/2} e^{-i(z-\nu\pi/2-\pi/4)} \left\{\sum_{m=0}^{p-1} \frac{(\nu, m)}{(2iz)^m} + R_p^{(2)}\right\}, \tag{2.2}$$

where

$$(\nu, m) = \frac{\{4\nu^2 - 1\}\{4\nu^2 - 3\} \cdots \{4\nu^2 - (2m-1)^2\}}{2^{2m} m!}, \tag{2.3}$$

$$(\nu, 0) = 1, \tag{2.4}$$

and the remainders $R_p^{(1)}$ and $R_p^{(2)}$ are both $O(z^{-p})$ when ν is a fixed number. For the results to be obtained, the following estimate is needed.

LEMMA 1. *Let* arg z *be restricted to the interval* $[-\pi/2, 3\pi/2]$, *and* ν *be a real-valued function of* z *satisfying* $\nu = o(z^{1/2})$ *as* $z \to \infty$. *Then, for* $i = 1$ *and* 2,

$$R_p^{(i)} = O\{(\nu, p)/z^p\}, \quad \text{as } z \to \infty. \tag{2.5}$$

Proof. We suppose first that $\nu \geqq 0$ and Re $z \geqq 0$. Under these conditions, Weber [9, Section 7.33] showed that

$$|R_p^{(i)}| \leqq 2G^2 \, |(\nu, p)| \, \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}p + 1)}{\Gamma(\frac{1}{2}p + \frac{1}{2}) \, |2z|^p} \qquad (i = 1, 2), \tag{2.6}$$

where

$$\begin{aligned} G &= \left(1 - \frac{\nu - \frac{1}{2}}{2r}\right)^{-\nu-1/2} \qquad (\nu > \tfrac{1}{2}), \\ G &= \left(1 - \frac{\nu + \frac{3}{2}}{2r}\right)^{-\nu-5/2}\left(1 + \frac{2\nu + 2}{r}\right) \qquad (\nu \leqq \tfrac{1}{2}), \end{aligned} \tag{2.7}$$

and $|z| = r$.

Since G is clearly bounded when $0 \leqq \nu \leqq 1$ and r is sufficiently large, we may assume that $1 < \nu \leqq r^{1/2}$. A simple estimate then gives

$$(-\nu - \tfrac{1}{2}) \log(1 - 1/2r^{1/2}) \leqq (\nu + \tfrac{1}{2})/r^{1/2} \leqq \tfrac{3}{2} \tag{2.8}$$

from which it follows that

$$G \leqq (1 - 1/2r^{1/2})^{-\nu-1/2} \leqq e^{3/2}. \tag{2.9}$$

Therefore, a constant A_p exists, which is independent of ν and z, such that

$$|R_p^{(i)}| \leqq A_p \, |(\nu, p)|/|z|^p \qquad (i = 1, 2), \tag{2.10}$$

for all sufficiently large values of z. This is equivalent to (2.5).

Since (ν, p) is an even function of ν, it follows from the identities [9, Section 3.61]

$$H_{-\nu}^{(1)}(z) = e^{\nu\pi i} H_\nu^{(1)}(z), \qquad H_{-\nu}^{(2)}(z) = e^{-\nu\pi i} H_\nu^{(2)}(z) \tag{2.11}$$

and [9, Section 3.62]

$$H_\nu^{(1)}(ze^{\pi i}) = -e^{-\nu\pi i}H_\nu^{(2)}(z), \tag{2.12}$$

that the restrictions $\nu \geqq 0$ and Re $z \geqq 0$ are unnecessary. Therefore, inequality (2.10) holds for all real values of ν and complex z restricted to the sector $-\pi/2 \leqq \arg z \leqq 3\pi/2$, as long as $\nu = o(z^{1/2})$ as $z \to \infty$. This completes the proof of Lemma 1.

Remark. It should be observed that no hypothesis has been made in the estimates concerning the relative values of ν and p; in this respect, Weber's result differs from that of Schläfli [9, Section 7.4] which was used in our previous paper [11].

In [6], Erdélyi and Wyman have given an elegant proof of a result from which it is easily deduced that the parabolic cylinder function $D_{-\lambda}(z)$ has the generalized asymptotic expansion

$$z^\lambda e^{z^2/4}D_{-\lambda}(z) \sim \sum_{n=0}^{\infty} \frac{(-1)^n(\lambda)_{2n}}{n!(2z^2)^n}\ ; \qquad \left\{\left(\frac{\lambda}{z}\right)^{2n}\right\}, \tag{2.13}$$

as $z \to \infty$ in $|\arg z| \leqq \pi/2 - \Delta$, where $\lambda > 0$ and $\lambda = o(z)$. The meaning of (2.11) is

$$z^\lambda e^{z^2/4}D_{-\lambda}(z) = \sum_{n=0}^{N} \frac{(-1)^n(\lambda)_{2n}}{n!(2z^2)^n} + o\left(\left(\frac{\lambda}{z}\right)^{2N}\right) \tag{2.14}$$

as $z \to \infty$, for every fixed integer $N \geqq 0$, where the o-symbol is independent of λ and z. Unfortunately, they proved the result only for $\lambda > 0$, while, for our results, we want to use all real values of λ. Although the conditions $\lambda > 0$ and $|\arg z| \leqq \pi/2 - \Delta$ in (2.13) can be easily weakened to $|\arg \lambda| \leqq \pi/2 - \Delta$ and $|\arg z| \leqq 3\pi/2 - \Delta$, their proof does not seem readily adapted to extensions allowing λ to be negative. The following lemma shows that the condition $\lambda > 0$ is indeed unnecessary.

LEMMA 2. *The result in* (2.13) *is true if* "$\lambda > 0$" *is replaced by* "λ *real*".

Proof. We start with the contour integral representation

$$e^{z^2/4}D_{-\lambda}(z) = -\frac{\Gamma(1-\lambda)}{2\pi i}\int_\infty^{(0+)} (-t)^{\lambda-1}e^{-t^2/2-zt}\,dt, \tag{2.15}$$

where the path of integration starts at $+\infty$, goes around the origin once in the positive direction and returns to $+\infty$. The integrand is rendered one-valued by taking $-\pi \leqq \arg(-t) \leqq \pi$.

Since it has already been shown that (2.13) holds when λ is finite or $\lambda > 0$ but $\lambda = o(z)$, we shall assume that λ is large and negative. Let $r_N(t)$, $N = 0, 1, 2, \cdots$, be defined by the relation

$$e^{-t^2/2} = \sum_{n=0}^{N} \frac{(-1)^n t^{2n}}{2^n \cdot n!} + r_N(t). \tag{2.16}$$

It is evident that, if t is restricted to the path of integration, a constant B_N can be found such that

$$|r_N(t)| \leqq B_N\ |t|^{2N+2}. \tag{2.17}$$

Substituting (2.16) in (2.15) and integrating term by term, we obtain

$$e^{z^2/4}D_{-\lambda}(z) = \sum_{n=0}^{N} \frac{(-1)^n(\lambda)_{2n}}{2^n \cdot n!}z^{-(\lambda+2n)} + \Gamma(1-\lambda)\epsilon_N(\lambda, z), \tag{2.18}$$

where

$$|\epsilon_N(\lambda, z)| \leqq \frac{1}{2\pi} \int_{\infty}^{(0+)} |(-t)^{\lambda-1} r_N(t) e^{-zt}\, dt| \tag{2.19}$$
$$\leqq \frac{B_N}{2\pi} \int_{\infty}^{(0+)} |t^{\lambda+2N+1} e^{-zt}\, dt|$$

by (2.17). Since λ is negative, the transformation $zt = (-\lambda)\tau$ gives

$$\int_{\infty}^{(0+)} |t^{\lambda+2N+1} e^{-zt}\, dt| = \left|\frac{\lambda}{z}\right|^{\lambda+2N+2} \int_{\infty}^{(0+)} |\tau^{\lambda+2N+1} e^{\lambda\tau}\, d\tau| \tag{2.20}$$

when z is real and positive. It is not difficult to see that (2.20) in fact holds when $|\arg z| < \pi/2$. Hence,

$$\left|\frac{z}{\lambda}\right|^{\lambda+2N+2} |\epsilon_N(\lambda, z)| \leqq \frac{B_N}{2\pi} \int_{\infty}^{(0+)} |\tau^{2N+1} e^{\lambda(\tau+\log\tau)}\, d\tau| \tag{2.21}$$

valid when $\lambda < 0$ and $|\arg z| \leqq \pi/2 - \Delta$. To the last integral, we apply the method of steepest descents [3, Section 30]. Hence,

$$\int_{\infty}^{(0+)} |\tau^{2N+1} e^{\lambda(\tau+\log\tau)}\, d\tau| \sim e^{-\lambda}[-\pi/2\lambda]^{1/2}, \tag{2.22}$$

as $\lambda \to -\infty$. Coupling the results (2.21) and (2.22), we obtain

$$z^{\lambda}\epsilon_N(\lambda, z) = O\{(-\lambda/z)^{2N+2} e^{-\lambda}(-\lambda)^{\lambda-1/2}\}, \tag{2.23}$$

as $z \to \infty$ in $|\arg z| \leqq \pi/2 - \Delta$, where the O-symbol is independent of λ and z. Finally, by Stirling's formula

$$\Gamma(1-\lambda) z^{\lambda}\epsilon_N(\lambda, z) = O\{(\lambda/z)^{2N+2}\} \tag{2.24}$$

and so the lemma is established.

Remark. The above analysis can be used to give similar expansions for the derivatives of $D_{-\lambda}(z)$ with respect to z. In particular, we have

$$D'_{-\lambda}(z) \sim (-\tfrac{1}{2}) z^{1-\lambda} e^{-z^2/4}, \quad \text{as } z \to \infty \text{ in } |\arg z| \leqq \pi/2 - \delta, \tag{2.25}$$

where λ is real and $\lambda = o(z)$.

3. Main Theorem. It is known that the Whittaker function has the integral representation [1, Section 5.3]

$$W_{k,m}(z^2) = z e^{z^2/2 + (m+1/2-k)\pi i} \int_{-\infty}^{\infty} e^{-u^2} H_{2m}^{(1)}(2zu) u^{2k}\, du, \tag{3.1}$$

where the path of integration runs from $-\infty$ to ∞ and passes above the singularity at the origin. If we substitute (2.1) for $H_{2m}^{(1)}$, we obtain

$$W_{k,m}(z^2) = 2^{1/4-k}\sqrt{z} \left\{ \sum_{r=0}^{p-1} \frac{(2m, r)}{(2z\sqrt{2})^r} D_{2k-r-1/2}(z\sqrt{2}) + E_p(z) \right\} \tag{3.2}$$

where the remainder is given by

$$E_p(z) = \frac{1}{\sqrt{\pi}} 2^{k-1/4} e^{(1/4-k)\pi i + z^2/2} \int_{-\infty}^{\infty} e^{-u^2+2izu} u^{2k-1/2} R_p^{(1)}(2zu)\, du. \tag{3.3}$$

This result is well known [4, p. 124]. When k and m are fixed, it was shown in [11, (3.1)] that $E_p(z) = O(e^{-z^2/2} z^{2k-2p-1/2})$, uniformly in arg z, as $z \to \infty$ in $|\arg z| \leqq \pi/4 - \Delta$. When k and m are functions of z, we have the following lemma.

LEMMA 3. *Let k and m be real-valued functions of z for which $k = o(z)$ and $m = o(z^{1/2})$ as $|z| \to \infty$. If $|m| \geqq \delta > 0$ then*

$$E_p(z) = O\{2^k z^{2k-1/2} e^{-z^2/2} (m/z)^{2p}\}. \tag{3.4}$$

If $|m| \leqq \delta$ then

$$E_p(z) = O\{2^k e^{-z^2/2} z^{2k-2p-1/2}\}. \tag{3.5}$$

Both results hold uniformly in arg z, *as $z \to \infty$ in $|\arg z| \leqq \pi/2 - \Delta$, and the constants implied in O-symbols are independent of k, m, and z.*

Proof. Returning to (3.3), we let

$$I = \int_{-\infty}^{\infty} e^{-u^2+2izu} u^{2k-1/2} R_p^{(1)}(2zu)\, du. \tag{3.6}$$

In [11], it was shown that by a change of variable $u = zu'$ followed by a deformation of the contour,

$$I = z^{2k+1/2} \int_{-\infty}^{\infty} e^{-z^2(x^2+1)} (x + i)^{2k-1/2} R_p^{(1)}(2z^2(x + i))\, dx, \tag{3.7}$$

the path of integration now being a straight line joining $-\infty$ to ∞. By Lemma 1,

$$|I| \leqq A_p\, |(2m, p)|\, |e^{-z^2} z^{2k-2p+1/2}|\, J, \tag{3.8}$$

where

$$J = \int_{-\infty}^{\infty} |e^{-z^2x^2} (x + i)^{2k-p-1/2}\, dx| \tag{3.9}$$

and the constant A_p depends only on p. Since x is real, we have $|x + i| \geqq 1$, and so

$$J \leqq 2 \int_0^{\infty} e^{-(\operatorname{Re} z^2)x^2} (x^2 + 1)^k\, dx. \tag{3.10}$$

We consider separately the cases $k \leqq 0$ and $k > 0$.

When $k \leqq 0$,

$$J \leqq 2 \int_0^{\infty} e^{-(\operatorname{Re} z^2)x^2}\, dx = \left(\frac{\pi}{\operatorname{Re} z^2}\right)^{1/2}. \tag{3.11}$$

Hence, $J = O(z^{-1})$ for z restricted to $|\arg z| \leqq \pi/4 - \Delta$.

When $k > 0$,

$$J \leqq 2 \int_0^{\infty} e^{-(\operatorname{Re} z^2 - k)x^2}\, dx \tag{3.12}$$

provided that the integral exists. Since $k = o(z)$ as $|z| \to \infty$,

$$\text{(3.13)} \qquad \operatorname{Re}(z^2) - k = |z|^2 \cos(\arg z^2) - k \geqq |z|^2 \eta_k,$$

for sufficiently large z in the sector $|\arg z| \leqq \pi/4 - \Delta$, where η_k is a positive finite number and independent of $|z|$. Therefore, we again have $J = O(z^{-1})$, as $z \to \infty$ in $|\arg z| \leqq \pi/4 - \Delta$.

We have thus proved that a constant A_p' exists such that

$$\text{(3.14)} \qquad |I| \leqq A_p' \, |(2m, p)e^{-z^2}z^{2k-2p-1/2}|,$$

for large values of z in $|\arg z| \leqq \pi/4 - \Delta$. The region of validity can be extended to $|\arg z| \leqq \pi/2 - \Delta$ by a standard argument. We rotate the path of integration in (3.7) through an arbitrary angle γ, where $-\pi/4 < \gamma < \pi/4$. When z is positive, use of Cauchy's theorem easily shows that (3.7) is valid if the upper and lower limits are replaced by $\infty e^{i\gamma}$ and $-\infty e^{i\gamma}$ respectively. With this change, (3.7) holds when $|\arg (ze^{i\gamma})| \leqq \pi/4 - \Delta$. A repetition of the proof (with some slight modifications) then shows that (3.14) is also valid in this angle. By varying γ, it follows that (3.14) holds when $|\arg z| \leqq \pi/2 - \Delta$.

Since $E_p(z) = (1/\sqrt{\pi})2^{k-1/4}e^{(1/4-k)\pi i + z^2/2}I$, by (3.14),

$$\text{(3.15)} \qquad E_p(z) = O\{2^k(2m, p)e^{-z^2/2}z^{2k-2p-1/2}\}$$

for all large values of z restricted to the sector $|\arg z| \leqq \pi/2 - \Delta$. When $|m| \leqq \delta$, (3.15) is certainly equivalent to (3.5). When $|m| \geqq \delta > 0$, (3.4) follows from (3.15) in view of the fact that $(2m, p) \sim (2m)^{2p}/p\,!$.

MAIN THEOREM. *Let k and m be real-valued functions of z satisfying conditions* (1.1) *and* (1.2). *Then, for any $N \geqq 0$,*

$$\text{(3.16)} \qquad \begin{aligned} 2^{k-1/4}W_{k,m}(z) = {} & \frac{D_{2k-1/2}((2z)^{1/2})}{z^{-1/4}}\left[\sum_{s=0}^{N+1}\frac{a_s}{z^s} + o\left\{\left(\frac{m^2}{z}\right)^{2N+2}\right\}\right] \\ & + \frac{D_{2k-1/2}'((2z)^{1/2})}{z^{1/4}}\left[\sum_{s=0}^{N}\frac{b_s}{z^s} + o\left\{\left(\frac{m^2}{z}\right)^{2N+2}\right\}\right] \end{aligned}$$

as $z \to \infty$ in $|\arg z| \leqq \pi - \delta$, uniformly with respect to $\arg z$. *The coefficients a_s and b_s depend on k and m, and are explicitly given in* (3.24).

Proof. Clearly, $\{(m^2/z)^{2n}\}$ is an asymptotic sequence under the hypothesis $m = o(z^{1/2})$ as $|z| \to \infty$. Let N be an arbitrary but fixed positive integer, and set

$$\text{(3.17)} \qquad S = \sum_{r=0}^{2N+2}\frac{(2m, r)}{(2(2z)^{1/2})^r} D_{2k-r-1/2}((2z)^{1/2}).$$

The following lemma is given in [10].

LEMMA. *For each $r \geqq 0$ we have*

$$\text{(3.18)} \qquad (-1)^r(-\lambda)_r D_{\lambda-r}(z) = D_\lambda(z)P_r(z) + D_\lambda'(z)Q_{r-1}(z)$$

where $P_r(z)$ and $Q_{r-1}(z)$ are polynomials of the form

$$\text{(3.19)} \qquad P_r(z) = \sum_{s=0}^{[r/2]} p_{r,s}z^{r-2s},$$

$$\text{(3.20)} \qquad Q_{r-1}(z) = \sum_{s=0}^{[(r-1)/2]} q_{r-1,s}z^{r-(2s+1)}.$$

The coefficients $p_{r,s}$ and $q_{r-1,s}$ can be successively determined from the recurrence relations

$$P_{r+1}(z) = zP_r(z) + (-\lambda + r - 1)P_{r-1}(z), \tag{3.21}$$

$$Q_r(z) = zQ_{r-1}(z) + (-\lambda + r - 1)Q_{r-2}(z), \tag{3.22}$$

with $P_0(z) = 1$, $P_1(z) = z/2$, $Q_{-1}(z) = 0$ and $Q_0(z) = 1$.

Now, let $|k| \geqq N + 1$ so that $2k - \frac{1}{2} \neq 0, 1, \cdots, 2N + 1$, and hence $(\frac{1}{2} - 2k)_r \neq 0$ for $r = 0, 1, \cdots, 2N + 2$. It follows from (3.17) that the sum S can be rearranged in the form

$$S = D_{2k-1/2}((2z)^{1/2}) \sum_{s=0}^{N+1} \frac{a_s}{z^s} + D'_{2k-1/2}((2z)^{1/2}) \sum_{s=0}^{N} \frac{b_s}{z^{s+1/2}} \tag{3.23}$$

where

$$\begin{aligned} a_s &= \frac{1}{2^s} \sum_{r \geqq 2s}^{2N+2} \frac{(-1)^r (2m, r)}{2^r(\frac{1}{2} - 2k)_r} p_{r,s} \quad \text{and} \\ b_s &= \frac{1}{2^{s+1/2}} \sum_{r \geqq 2s+1}^{2N+2} \frac{(-1)^r (2m, r)}{2^r(\frac{1}{2} - 2k)_r} q_{r-1,s}. \end{aligned} \tag{3.24}$$

Therefore

$$\begin{aligned} W_{k,m}(z) = 2^{1/4-k} z^{1/4} \Big\{ & D_{2k-1/2}((2z)^{1/2}) \sum_{s=0}^{N+1} \frac{a_s}{z^s} \\ & + D'_{2k-1/2}((2z)^{1/2}) \sum_{s=0}^{N} \frac{b_s}{z^{s+1/2}} + E_{2N+3}(\sqrt{z}) \Big\} \end{aligned} \tag{3.25}$$

for any fixed integer $N \geqq 0$.

Now, it only remains to consider the remainder E_{2N+3}. By Lemmas 2 and 3, we have

$$E_{2N+3}(\sqrt{z}) = O\{(m^2/z)^{2N+3} D_{2k-1/2}((2z)^{1/2})\}, \tag{3.26}$$

and, similarly,

$$E_{2N+3}(\sqrt{z}) = O\{(m^2/z)^{2N+3} z^{-1/2} D'_{2k-1/2}((2z)^{1/2})\} \tag{3.27}$$

by (3.26). Both results hold uniformly with respect to arg z, as $z \to \infty$ in $|\arg z| \leqq \pi - \delta$.

We have thus proved that, for any integer $N \geqq 0$,

$$\begin{aligned} 2^{k-1/4} W_{k,m}(z) = & \frac{D_{2k-1/2}((2z)^{1/2})}{z^{-1/4}} \left[\sum_{s=0}^{N+1} \frac{a_s}{z^s} + O\left\{ \left(\frac{m^2}{z} \right)^{2N+3} \right\} \right] \\ & + \frac{D'_{2k-1/2}((2z)^{1/2})}{z^{1/4}} \left[\sum_{s=0}^{N} \frac{b_s}{z^s} + O\left\{ \left(\frac{m^2}{z} \right)^{2N+3} \right\} \right], \end{aligned} \tag{3.28}$$

as $z \to \infty$ in $|\arg z| \leqq \pi - \delta$, uniformly with respect to arg z, which certainly implies the required result.

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