

# Chebyshev Approximation by Exponentials on Finite Subsets

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**Abstract.** This paper is concerned with Chebyshev approximation by exponentials on finite subsets. We take into account that varisolvency does not hold for exponentials in general. A bound for the derivatives of exponentials is established and convergence of the solutions for the discrete problems is proved in the topology of compact convergence on the open interval.

**1. Introduction.** In a recent note, Rosman [9] studied the convergence of best exponential Chebyshev approximation on finite subsets. Unfortunately, his investigations heavily depend on results of Rice from 1962 [6], [8], and he assumed that the family of exponentials

$$(1) \quad V_n = \left\{ E(x) = \sum_{i=1}^l \sum_{j=0}^{m_i} p_{ij} x^j e^{t_i x}; p_{ij}, t_i \in \mathbb{R}, \sum_{i=1}^l (1 + m_i) \leq n \right\}$$

has the varisolvency property. But, as was shown by the author in 1967 [1], [3], varisolvency holds only for the special exponentials of the form

$$(2) \quad \sum_{i=1}^n \alpha_i e^{t_i x}.$$

Moreover, there are two different definitions of varisolvency in the literature. The exponentials of the form (2) are varisolvent in the sense of Rice's papers [6], [7], [8], but not in the sense of Hobby and Rice [5]. For the study of Rosman's proof, this difference cannot be neglected.

In this note, we will present a different proof, using ideas in Werner's [11] and Schmidt's [10] proof for an existence theorem. At first, we establish an estimation of the derivatives of exponentials similar to Bernstein's inequality for polynomials. Computational methods are not considered here; for this, we refer to [2], [8], [12].

**2. Estimation of Derivatives.** The main result of this section is an estimation of the derivatives of exponentials mentioned in [4]. But the major part will be concerned with the lemmas preparing the convergence theorem in the next section.

**LEMMA 1.** Let  $x_0 < x_1 < \cdots < x_n$ . If  $f \in C^n[x_0, x_n]$ , and if

$$(3) \quad |f(x_i)| \leq M, \quad i = 0, 1, \cdots, n,$$

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holds, then there exists a point  $z \in [x_0, x_n]$ , such that

$$(4) \quad |f^{(n)}(z)| \leq M \cdot n! \cdot \sum_{i=0}^n \prod_{j=0; j \neq i}^n 1/|x_i - x_j|.$$

*Proof.* Consider the polynomial  $p(x)$  of degree  $n$  which interpolates  $f(x)$  at  $x_0, x_1, \dots, x_n$ . Since  $f - p$  has  $n + 1$  zeros, there is at least one zero of  $(f - p)^{(n)}$ . Observe that the right-hand side of (4) represents an upper bound of the  $n$ th derivative of the Lagrangian interpolating polynomial.  $\square$

By a special choice of points, we get an estimation for functions defined on an interval. For each compact set  $Y$ , we define the Chebyshev norm on  $C(Y)$  as

$$\|f\|_Y = \sup_{x \in Y} |f(x)|.$$

**COROLLARY 2.** Let  $X = [\alpha, \beta]$  and  $d = \beta - \alpha$ . For each  $f \in C^n(X)$ , there exists a point  $z \in X$ , satisfying

$$(5) \quad |f^{(n)}(z)| \leq 2^{n-1} n! \cdot d^{-n} \cdot \|f\|_X.$$

*Proof.* Let  $x_i = \frac{1}{2}(\alpha + \beta) - (d/2)\cos(\pi i/n)$  for  $i = 0, 1, \dots, n$ . By applying Lemma 1 to the transformed Chebyshev polynomial  $f(\frac{1}{2}(\alpha + \beta) + dx/2) = T_n(x)$ , we obtain the equal sign in (4). From  $T_n^{(n)}(x) = 2^{n-1}n!$ , we determine the factor of  $M$  in (4). This yields the theorem.  $\square$

Now, we have established a priori estimates which are necessary for the application of the main lemma that generalizes a theorem of Schmidt [10]. Notice that derivatives of exponentials are exponentials, too. Thus, they have at most  $n - 1$  zeros or vanish identically.

To each (finite) sequence of distances  $d_1, d_2, \dots, d_n$  and to the corresponding a priori constants  $M_1, M_2, \dots, M_n$ , there are associated  $n + 1$  numbers by a recursive process

$$(6) \quad \begin{aligned} K_{n+1} &= 0, \\ K_\nu &= M_\nu + d_\nu K_{\nu+1}, \quad \nu = n, n-1, \dots, 1. \end{aligned}$$

**LEMMA 3.** Let  $d_1, d_2, \dots, d_n$  be positive numbers, satisfying

$$(7) \quad d_1 + d_2 + \dots + d_n < d.$$

Let  $X \supset [x_0 - d, x_0 + d]$  and  $f \in C^{n+1}(X)$ . Suppose that  $f^{(n+1)}$  has at most  $n - 1$  zeros or vanishes identically in  $X$ , and, moreover, that in each subinterval of length  $d_\nu$  ( $\nu = 1, 2, \dots, n$ ), there is a point  $z$  such that

$$(8) \quad |f^{(\nu)}(z)| \leq M_\nu.$$

Then

$$(9) \quad |f'(x_0)| \leq K_1$$

holds with  $K_1$  defined by the recursion relation (6).

*Proof.* Suppose to the contrary that (9) is violated and, say,  $f'(x_0) > K_1$  holds. By an inductive proof we will show that, for  $\nu = 1, 2, \dots, n$ , there are points  $\xi_\nu, \eta_\nu$  such that

$$(10) \quad x_0 - \sum_{\mu=1}^{\nu} d_{\mu} \leq \xi_{\nu} \leq \eta_{\nu} \leq x_0 + \sum_{\mu=1}^{\nu} d_{\mu},$$

$$(11) \quad f^{(\nu+1)}(\xi_{\nu}) > K_{\nu+1}, \quad (-1)^{\nu} f^{(\nu+1)}(\eta_{\nu}) > K_{\nu+1},$$

and  $f^{(\nu+1)}$  has  $\nu$  distinct zeros in  $[\xi_{\nu}, \eta_{\nu}]$ .

Let  $\nu = 1$ . By assumption,  $|f'(z_1)| \leq M_1$  holds for a point  $z_1 \in [x_0 - d_1, x_0]$ . From Rolle's theorem, we obtain a point  $\xi_1 \in [z_1, x_0]$ , satisfying

$$f''(\xi_1) = \frac{f'(x_0) - f'(z_1)}{x_0 - z_1} > \frac{K_1 - M_1}{d_1} = K_2.$$

A corresponding construction yields  $\eta_1 \in [x_0, x_0 + d_1]$  with the postulated properties. By virtue of (11), there is a zero of  $f''(x)$  in  $(\xi_1, \eta_1)$ .

Assume that the statement holds for  $\nu - 1 \leq n - 1$ . Denote by  $x_1, x_2, \dots, x_{\nu-1}$  the zeros of  $f^{(\nu)}$ . By assumption, we have  $|f^{(\nu)}(z)| \leq M_{\nu}$  for a point  $z \in [\xi_{\nu-1} - d_{\nu}, \xi_{\nu-1}]$ . Let  $f^{(\nu)}$  attain its maximum in  $[z, x_1]$  at  $z_1$ . From  $f^{(\nu)}(z_1) \geq f^{(\nu)}(\xi_{\nu-1}) > K_{\nu} \geq M_{\nu} \geq f^{(\nu)}(z)$  and  $f^{(\nu)}(x_1) = 0$ , we conclude that  $z_1 \in (z, x_1)$  and  $f^{(\nu+1)}(z_1) = 0$ . Set  $z_2 = \min(z_1, \xi_{\nu-1})$ . By virtue of Rolle's theorem, there exists  $\xi_{\nu}$ , satisfying

$$f^{(\nu+1)}(\xi_{\nu}) = \frac{f^{(\nu)}(z_2) - f^{(\nu)}(z)}{z_2 - z} > \frac{K_{\nu} - M_{\nu}}{d_{\nu}} = K_{\nu+1}.$$

Construct  $\eta_{\nu} \in [\eta_{\nu-1}, \eta_{\nu-1} + d_{\nu}]$  by an analogous procedure. Hence,  $f^{(\nu+1)}$  has at least  $\nu - 2$  zeros between  $x_1$  and  $x_{\nu-1}$ . Moreover, two zeros are determined in  $(\xi_{\nu}, x_1)$  and  $(x_{\nu-1}, \eta_{\nu})$ , respectively, and the induction is complete. As a consequence, for  $\nu = n$ , there is a contradiction to the assumption on the zeros of  $f^{(n+1)}$ .  $\square$

Now we are ready to prove the desired estimation.

**THEOREM 4.** Let  $X = [\alpha, \beta]$  and  $2d \leq \beta - \alpha$ . There exists a constant  $c = c_n$ , such that, for each exponential  $E$  of degree  $\leq n$ ,

$$(12) \quad |E'(x)| \leq (c_n/d) \cdot \|E\|_X \quad \text{for } x \in [\alpha + d, \beta - d].$$

*Proof.* It is sufficient to prove the theorem for  $E(x) \neq 0$ . Given  $x_0 \in [\alpha + d, \beta - d]$ , set

$$f(x) = (1/\|E\|_X) \cdot E(x_0 + dx), \quad -1 \leq x \leq +1.$$

Obviously,  $f(x)$  is an exponential and  $\|f\|_{[-1, +1]} \leq 1$  holds. Let  $d_1, d_2, \dots, d_n$  be positive numbers, the sum of which is 1. Set

$$(13) \quad \begin{aligned} K_{n+1} &= 0, \\ K_{\nu} &= 2^{2\nu-1} \cdot \nu! \cdot d_{\nu}^{-\nu} + d_{\nu} K_{\nu+1}, \quad \nu = n, n-1, \dots, 1, \\ c_n &= K_1. \end{aligned}$$

By virtue of Corollary 2 and Lemma 3 we obtain  $|f'(0)| \leq c_n$ . From this, the inequality (12) is evident.  $\square$

**3. Approximation on Finite Subsets.** Let  $X$  be a compact interval on the real line and let  $X_r$  be a set of  $r$  distinct points in  $X$ . Then the density of  $X_r$  in  $X$  is measured by

$$\Delta_r = \max_{x \in X} \min_{y \in X_r} |x - y|.$$

We consider a sequence of subsets  $\{X_r\}$ , satisfying  $\Delta_r \rightarrow 0$  as  $r$  tends to infinity. Since  $X$  is compact, this is equivalent to the assumption in [9] that, given  $x \in X$ , there is an  $x_r \in X_r$  such that  $x_r \rightarrow x$ .

As usual,  $E_r$  is called a best approximation to  $f$  on  $X_r$ , if the functional  $\|f - E\|_X$ , attains its minimum on  $V_n$  at  $E_r$ . It is known that best approximations need not exist on finite point sets [8] and unicity of the solution cannot be ensured [1]. From the computational point of view, it is reasonable to assume existence anyway. However, for a rigorous proof of the convergence theorem, we avoid this difficulty by the definition of nearly best approximations. Let  $E^*$  be a best approximation to  $f(x)$  on  $X$ .  $E_r$  is called a nearly best approximation to  $f(x)$  on  $X_r$ , if

$$\|f - E_r\|_{X_r} \leq \|f - E^*\|_X.$$

Obviously, each best approximation on  $X_r$  is a nearly best approximation.

**THEOREM 5.** *Let  $X = [\alpha, \beta]$ , and let  $X_r$  be a sequence of finite subsets, such that  $\Delta_r \rightarrow 0$ . Then, each sequence of nearly best approximations  $\{E_r\}$  contains a subsequence that converges to a best approximation  $E^*$  on  $X$  uniformly on each compact subinterval of  $(\alpha, \beta)$ . If  $E^*$  has the maximal degree  $n$ , then convergence is uniform on the total interval  $X$ .*

*Proof.* Let  $Y = [\alpha_1, \beta_1]$  be a compact subinterval of  $(\alpha, \beta)$ . Set  $Y_1 = [\frac{1}{2}(\alpha + \alpha_1), \frac{1}{2}(\beta + \beta_1)]$ . From Corollary 2, we know that, in any interval  $I$  of length  $d$ , we can find  $n + 1$  points  $x_0, x_1, \dots, x_n$  such that the sum in inequality (4) has the value  $2^{2n-1}d^{-n}$ . Since  $\Delta_r$  tends to zero, for sufficiently large  $r$ , we may choose  $n + 1$  points in  $I \cap X_r$  such that the sum can be bounded by  $2^{2n} \cdot d^{-n}$ . By virtue of Lemma 3 and by  $\|E_r\|_{X_r} \leq \|f - E_r\|_{X_r} + \|f\|_{X_r} \leq 2\|f\|_X$ , there is a constant  $c$  such that, for sufficiently large  $r$ ,

$$|E'_r(x)| \leq c \cdot \|E_r\|_{X_r} \leq 2c \|f\|_X \quad \text{for } x \in Y_1.$$

Since for each  $x \in X$  there is a point in  $X_r$  with a distance not greater than  $\Delta_r$ , we obtain

$$\|E_r\|_{Y_1} \leq (2 + 2c \cdot \Delta_r) \cdot \|f\|_X.$$

Hence,  $\{E_r\}$  is bounded on  $Y_1$ . By Corollary 1 in [10], there exists a subsequence converging uniformly on  $Y \subset Y_1$  to an exponential  $E^*$ . By standard arguments, we conclude that this subsequence converges uniformly to  $E^*$  on each compact subset of  $(\alpha, \beta)$ . Obviously,  $E^*$  is a best approximation to  $f$  on  $X$ . Moreover, from Theorem 4 in [10], we obtain uniform convergence on  $X$ , if  $E^*$  has maximal degree.  $\square$

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