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A Search Procedure and Lower Bound for Odd Perfect Numbers

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Abstract. An infinite tree-generating "q-algorithm" is defined, which if executed would enumerate all odd perfect numbers (opn's). A truncated execution shows that any opn has either some component $p^a > 10^{18}$, with a even, or no divisor < 7; hence any opn must be $> 10^{36}$.

1. Introduction. It is unknown whether any odd perfect numbers (opn's) exist. For a history, see McCarthy [2]. Kanold [3] gave a lower bound of 10^{20} for possible opn's. Muskat [5] showed that every opn must have a divisor $p^a > 10^{12}$.

The present paper is a condensed and clarified version of [6], which was announced in [7], and submitted to this journal in 1968. The completion of the requested revision has been delayed until now.

The chief results are:

(I) A "q-algorithm" is given, which defines a countably infinite tree, on which, if enumerated, every opn (if any) would be recognized at some node.

(II) A finite truncation of the tree was computed, which shows that (1) every opn must satisfy the known restrictions defined at some one of the truncation-nodes implied by this tree; (2) every opn must have either (a) some component $p^a > u = 10^{18}$, with a even, or (b) no prime divisor < 7; (3) hence any opn must be $> u^2 = 10^{36}$.

This "10^{36}-tree", occupying 9 pages, has been deposited in the UMT file [8] (and occurs in an earlier arrangement in [6]). To convey the spirit of that tree and of the algorithm, an analogous "10^{16}-tree" (based on $u = 10^8$), occupying 2 pages, is included in this paper. Statements about the 10^{36}-tree will typically hold equally well for the 10^{16}-tree.

The bound of 10^{36} has been superseded in a recent paper by Hagis [9]. Nevertheless, there are enough different approaches in the two papers to warrant the present publication. For omitted proofs and details, see [6].

2. Notation. Let N, P, Ω be the sets of all positive integers (n, m, h, d, etc.), primes (p or q), and opn's, respectively. Let $a(p, n) \geqq 0$ be the multiplicity of p in n, and let $h(p, n) = a(p, n) + 1 \geqq 1$. With Sylvester, we call $p^{a(p,n)} > 1$ a *component* of n. Let $A(n) = \{p : p \mid n\}$. For $n > 1$, define $p_1(n) = \min \{p \in A(n)\}$; the other $p \in A(n)$ may be numbered $p_2(n), \cdots, p_{r(n)}(n)$ in any convenient order. We may then represent n in any of the forms

$$n = \prod_{i=1}^{r(n)} p_i(n)^{a_i(n)} = \prod_{i=1}^{r(n)} p_i(n)^{h_i(n)-1} = \prod_p p^{a(p,n)} = \prod_p p^{h(p,n)-1}$$

where the range of p in the last two products is indifferently P or $A(n)$.

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Given p^a, write $h = a + 1$. Then $\sigma(p^a) = \sigma(p^{h-1}) = (p^h - 1)/(p - 1) = \prod_{d \in D} F_d(p)$ where $F_d(\cdot)$ is the dth cyclotomic polynomial (see Nagell [1]), and $D = \{d : d \mid h \wedge d \neq 1\}$. If $h_1 \mid h_2$, then $\sigma(p^{h_1 - 1}) \mid \sigma(p^{h_2 - 1})$ (proof in [6, p. 23]). Thus in using $\sigma(\cdot)$ and in studying opn's, it is more effective to deal with h and its prime divisors rather than with a. It will prove convenient to write its prime factorization as $h = q_1 q_2 \cdots q_s$, where $q_1 \leqq q_2 \leqq \cdots \leqq q_s$. By the jth prime divisor of h we will mean q_j. (Thus if $h = 45 = 3 \cdot 3 \cdot 5$, then the 1st, 2nd, 3rd prime divisors of h are 3, 3, 5.) For $h = h(p, n)$, we define $s(p, n)$ and $q_j(p, n)$ like s and q_j.

We say n_1 *h-divides* n_2, $n_1 \mid_h n_2$, if for all p, $h(p, n_1) \mid h(p, n_2)$; n_1 *q-divides* n_2, $n_1 \mid_q n_2$, if for all p, $s(p, n_1) \leqq s(p, n_2)$ and $q_j(p, n_1) = q_j(p, n_2)$ for $1 \leqq j \leqq s(p, n_1)$. Clearly, $n_1 \mid_q n_2$ implies $n_1 \mid_h n_2$.

LEMMA. *If* $n_1 \mid_q n_2$, *or if* $n_1 \mid_h n_2$, *then* $n_1 \mid n_2$ *and* $\sigma(n_1) \mid \sigma(n_2)$.

This result accounts for the usefulness of the concepts $\mid_h$ and $\mid_q$. The proof is easy.

In the context of opn's, we define a prime power $p^a > 1$ to be *ordinary* if $a \equiv 0 \bmod 2$ (i.e. h is odd); *exceptional* if $p \equiv 1 \bmod 4$, and $a \equiv 1 \bmod 4$ (i.e., h is singly even); *inadmissible* otherwise. For $k = 0, 1, 2, \cdots$, define $E_k = \{n :$ exactly k of the components of n are exceptional, and the rest are ordinary$\}$. Euler proved that $\Omega \subset E_1$.

3. Trees. Let C be a set of *choices* c; λ a fixed symbol; $C^* = \{\nu : \nu = \lambda c_1 c_2 \cdots c_l$ where $c_i \in C\}$ ($l = l_\nu \geqq 0$ is called the *level* of ν); and $C(\cdot)$ a function $C(\cdot) : C^* \to 2^C$ (we denote the image $C(\nu)$ by C_ν). Then a unique tree $T = T_{C(\cdot)}$, rooted at λ, is defined as the smallest set T of *nodes* ν such that $\lambda \in T$, and such that if $\nu = \lambda c_1 c_2 \cdots c_l \in T$, then $\nu c = \lambda c_1 c_2 \cdots c_l c_{l+1} \in T$ for all $c = c_{l+1} \in C_\nu$.

4. Opn-Trees. We will define such a T, called an opn-tree, with the following properties.

At each $\nu \in T$, of level l, there will be defined a sequence $R_\nu = (\rho_1, \cdots, \rho_l)$ of l *restrictions* (truth-valued functions) $\rho_k(n)$ on the variable n. Let $N_\nu = \{n : \rho_k(n)$ is true for all $\rho_k \in R_\nu\}$; let $\Omega_\nu = N_\nu \cap \Omega$. Clearly $R_\lambda = \varnothing$, $N_\lambda = N$, $\Omega_\lambda = \Omega$. At some ν, it might become known that $\Omega_\nu = \{n\}$ where n is known to be an opn. We will call this ν an *opn-node*. At some other ν it may become known that $\Omega_\nu = \varnothing$. In either case, ν is a *terminal node*, and we do not branch from it. Otherwise, we will *select*, and *branch on*, some function $f_\nu(n)$, known to be defined for all $n \in \Omega_\nu$, with "adequate" *range* C_ν, i.e., such that $f_\nu(\Omega_\nu) \subseteq C_\nu \subseteq C$. Thus, if we define the sons of ν to be all νc for $c \in C_\nu$, then at each νc we define $R_{\nu c} = (\rho_1, \cdots, \rho_l, \rho_{l+1})$ where $\rho_{l+1}(n)$ is '$f_\nu(n) = c$'. Clearly, $\{\Omega_{\nu c} : c \in C_\nu\}$ is a disjoint partitioning of Ω_ν. Hence, every $m \in \Omega$ (if any) lies on a unique path from λ down T.

At $\nu = \lambda$, we branch on $p_{\min}(n)$, with $C_\nu = P - \{2\}$. At every other $\nu \in T$ of level $l > 0$, we will branch (if at all) on $q_{j_l}(\hat{p}_l, n)$, i.e., on the j_lth prime divisor of $h(\hat{p}_l, n)$, for suitable $\hat{p}_l$, j_l (both depending on ν) and C_ν. As a result, at every $\nu = \lambda c_1 c_2 \cdots c_l \in T$ of level $l > 0$ we will have $R_\nu = (\rho_1, \rho_2, \cdots, \rho_l)$ where

$$\rho_1 \quad \text{is} \quad `p_{\min}(n) = c_1\text{'},$$

$$\rho_k \quad \text{is} \quad `q_{j_{k-1}}(\hat{p}_{k-1}, n) = c_k\text{'} \quad \text{for} \quad k = 2, 3, \cdots, l.$$

Of course, $\hat{p}_{k-1}$, j_{k-1}, and a $C_{\nu_{k-1}}$ were defined at $\nu_{k-1} = \lambda c_1 c_2 \cdots c_{k-1}$, and $c_k \in C_{\nu_{k-1}}$.

The sequence $\hat{p}_1, \hat{p}_2, \cdots, \hat{p}_{l-1}$ may contain repetitions. Let its distinct elements, in order of first appearance, be $p_1, p_2, \cdots, p_r$, and let $A_\nu = \{p_i\}$. The later conditions on branching will ensure that $p_1 = c_1$ and that, for each p_i, the subsequence of the ρ_k $(2 \leqq k \leqq l)$, for which $\hat{p}_{k-1} = p_i$, will have consecutive values $1, 2, \cdots, s_i$ $(i \geqq 1)$ of j_{k-1}, and nondecreasing prime values of c_k. Consequently, we can replace R_ν by the equivalent

$$p_{\min}(n) = p_1,$$

$$q_j(p_i, n) = q_{ij} \quad \text{for} \quad i = 1, 2, \cdots, r; j = 1, 2, \cdots, s_i,$$

where $p_1 = c_1$, r and the s_i are known ($\sum s_i = l - 1$), and the q_{ij} are a known permutation of the c_k $(k > 1)$; or equivalently

$$p_{\min}(n) = p_1,$$

$$m \mid_q n,$$

where $m = m_\nu = \prod_{i=1}^{r} p_i^{h_i - 1}$ and $h_i = \prod_{j=1}^{s_i} q_{ij}$.

All of $l, \hat{p}_{k-1}, j_{k-1}, c_k, r, p_i, s_i, q_{ij}, h_i, m$ are functions of ν. Since we generally consider a typical ν and its sons νc or father ν^-, we omit this dependence on ν from the notation, except that for $l = l_\nu$ and $m = m_\nu$ it is optional. Such sets as N_ν, Ω_ν, A_ν will always bear the subscript.

5. The Computation of $\sigma(m)$. For $l = 1$, $\sigma(m) = m = 1$. For $l > 1$, we are assisted in calculating $\sigma(m)$ by the assumed previous factorization of $\sigma(m^-)$, where $m^- = m_{\nu^-}$. For that i such that $p_i = \hat{p}_{l-1}$, write p for p_i, s for s_i, q for q_{is}, h for h_i/q. Then p appears in m^- as p^{h-1} (possibly as $p^0 = 1$) and in m as p^{hq-1}. Thus, $\sigma(m)/\sigma(m^-) = \sigma(p^{hq-1})/\sigma(p^{h-1}) = (p^{hq} - 1)/(p^h - 1) = \prod_{d \in D} F_d(p)$, where $D = \{d : d \mid hq \wedge d \nmid h\}$. The factorization of $\sigma(m)$ may therefore be found from that of $\sigma(m^-)$ and of each $F_d(p)$. In the important special case $s = 1$, $h = 1$, $p \nmid m^-$ we have simply $\sigma(m)/\sigma(m^-) = \sigma(p^{q-1}) = F_q(p)$. In the 10^{36}-tree, only this case arose.

If we define $b(p, m) = a(p, \sigma(m))$, then $\sigma(m) = \prod_p p^{b(p,m)}$. Define $B_\nu = \{p \neq 2 : b(p, m) > 0\}$. We will ensure, inductively, that $m \in E_0 \cup E_1$. (For $l \leqq 1$, $m = 1 \in E_0$.) If $m \in E_0$, then no $q_{ij} = 2$, and $b(2, m) = 0$. If $m \in E_1$, then just one $q_{ij} = 2$, for some i, and $j = 1$; p_i is the exceptional prime, and $b(2, m) = 1$.

6. More Details on Branching. At each node $\nu \neq \lambda$ we define a set P_ν of p which are *admissible to be* $\hat{p} = \hat{p}_l$. If $l = 1$, let $P_\nu = \{p_1\}$, where $p_1 = c_1$. If $l > 1$, let $P_\nu = \{p \neq 2 : b(p, m) > a(p, m)\}$, i.e., the set of $p \neq 2$ which are "over-generated" by $\sigma(\cdot)$. The proof of our next result may be found on p. 29 of [6].

THEOREM. *If $l_\nu \geqq 1$, and $P_\nu = \varnothing$, then $m = m_\nu$ is an opn, and $\Omega_\nu = \{m\}$. Thus ν is an opn-node.*

Otherwise, consider any $p \in P_\nu \neq \varnothing$. For every $n \in \Omega_\nu$, $a(p, n) = b(p, n) \geqq b(p, m)$. If $l > 1$, $b(p, m) > a(p, m)$. Hence, $a(p, n) > a(p, m)$. The latter also holds for $l = 1$. Thus, $h(p, n) > h(p, m)$; and since $m \mid_q n$, $s(p, n) > s(p, m)$. Thus, $q_{s(p,m)+1}(p, n)$ is defined for all $n \in \Omega_\nu$, so that it is *admissible to branch* on it. There are two cases.

If $p \notin A_\nu$ (in particular, if $l = 1$), then $s(p, m) = 0$. It is thus admissible to branch on $q_1(p, n)$, with $C_\nu = P$ if p is *admissible as the exceptional prime*, i.e., if $p \equiv 1 \bmod 4$ and $m \in E_0$. Otherwise, $C_\nu = P - \{2\}$.

If $p \in A_\nu$, say $p = p_i$, then $s(p_i, m) = s_i > 0$. It is then admissible to branch on $q_{s_i+1}(p_i, n)$, with $C_\nu = \{q \in P : q \neq 2 \wedge q \geqq q_{is_i}\}$.

The use of any of the above p as $\hat{p}_l$ preserves, at all νc, the properties assumed at ν. In particular, if $c = 2$, then $\nu \in E_0$ and $\nu 2 \in E_1$; otherwise, ν and νc both belong to E_0 or both belong to E_1. Thus the property $m \in E_0 \cup E_1$ at ν is preserved at all νc.

Any well-defined global selection rule(s) (i.e., for all $\nu \in T$) for $\hat{p} \in P_\nu \neq \varnothing$ will define a particular tree. A set of such rules is given on p. 40 of [6]. In the 10^{36}-tree, these rules always reduced (for $l > 1$) to $\hat{p} = \max\{p \in (B_\nu - B_{\nu^-})\}$; and always $\hat{p} \notin A_\nu$. Hence, the branching was always on $q_1(\hat{p}, n)$, so that always $s_i = 1, p_i = \hat{p}_i$, $h_i = q_{i1}$ $(i = 1, 2, \cdots, r)$, $r = l - 1$; but the more general case may be needed for the infinite tree.

Thus an opn-tree is defined. Its previously assumed properties hold, by induction on l. It is enumerable, in fact by various *admissible sequences*, in which the father ν of any νc is processed before νc.

If n is any opn, it lies on a unique path from λ, and will be recognized at the node ν on this path which has level $l = 1 + \sum_{i=1}^{r(n)} s(p_i, n)$.

7. Contradiction-Nodes. This tree can be pruned, during construction, by taking into consideration two conditions which can give contradictions to $n \in \Omega_\nu$, thus showing that $\Omega_\nu = \varnothing$, so that we can make ν a terminal node. These are discussed in the next two paragraphs.

For every $p \in B_\nu$, $p \,|\sigma(m)|\, \sigma(n) = 2n$, for all $n \in \Omega_\nu$. Hence if $\exists\ p \in B_\nu$ such that $p < p_1$, this contradicts $p_1 = p_{\min}(n)$, and we have a *least-element-contradiction-node* at ν.

For every $n \in \Omega_\nu$, (1) $m \,|_q\, n$, (2) $\sigma(m) \mid 2n$, (3) $n \in E_1$. Let $N'_\nu = \{n : (1) \wedge (2) \wedge (n \in E) \wedge (n \in E_1 \text{ if } m \in E_1)\}$, where $E = \bigcup_{k=0}^{\infty} E_k$. Let $m' = \min\{n \in N'_\nu\} = \gcd\{n \in N'_\nu\}$; m' is readily determined by its $\{a(p, m')\}$ in terms of $\{a(p, m)\}$ and $\{b(p, m)\}$. (Cf. [6, pp. 37–39]; and note, for example, that $2 = a(p, m) < b(p, m)\ (\leqq 8)$ implies $a(p, m') = 3^2 - 1 = 8$, a strong contribution to m', and to $\sigma(m')/m'$, especially for $p = 3$.) Now $m' \mid n$ for all $n \in \Omega_\nu$. If we compute $\sigma(m')/2m'$, and if this is > 1, then m' is abundant, and so are its multiples n. Hence, $\Omega_\nu = \varnothing$, and ν is an *abundance-contradiction-node*. (The 10^{36}-tree was first computed without the use of abundance-contradictions and had about twice as many nodes, and about twice the depth, as with their use.)

Although not used as such in [6], the above m' is a lower bound for all $n \in \Omega_\nu$; and a "better" bound is $m'' = \min\{n : (1) \wedge (2) \wedge (n \in E_1)\}$, which is $> m'$ if $m' \notin E_1$.

8. Truncated Execution. For computation, the opn-tree must be truncated to a finite tree. For simplicity, and because the chief computational effort was in factoring large $F_q(p)$, it was decided to truncate primarily by omitting all νq (*truncation nodes*) for which $p^{hq-1} > u$ or $> 2^{1/2}u$, according as hq is odd, or even and > 2, for a chosen u. Any opn at such a node is $> u^2$. To obtain the same bound at level 1, we used results of Norton [4] and derived (pp. 35–36 of [6]) these lower bounds $w(p_{\min}(n))$: $w(3) = 10^3$; $w(5) = 10^{13}$; $w(7) = 10^{41}$; $w(11) = 10^{90}$; $w(13) = 10^{154}$; etc. We let λp be truncation nodes as soon as $w(p) > u^2$. Using $u = 10^{18}$ and then $u = 10^8$, with $\max(p_{\min}) = 5$ in both cases, yielded the 10^{36}- and 10^{16}-trees.

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          ODD-PERFECT-NUMBER SEARCH TREE TO 10**16

          (REPRESENTATIVE OF SEARCH TREE TO 10**36)

NODE      L E F

          0 - ALL ODD PERFECT NUMBERS
3         1 - LEAST ELEMENT 3, HENCE N > 10**3
33        2 - F( 3,3) = 13
332       3 * F( 2,13) = 2.7                               '7'
3323      4   F( 3,7) = 3.19
33233     5   F( 3,19) = 3.127
332333    6   F( 3,127) = 3.5419
3323333   7   F( 3,5419) = 3.31.313.1009
3323333*          3**8.(7.19.31)**2.13**1                        +.024283
33235     5   F( 5,19) = 151.911
332353    6   F( 3,911) = 830833
33237     5   F( 7,19) = 701.70841
3325      4   F( 5,7) = 2801
33253     5   F( 3,2801) = 37.43.4933
332533    6   F( 3,4933) = 3.127.193.331                 '331'
3325333   7   F( 3,331) = 3.7.5233
33253333  8   F( 3,5233) = 3.7.31.42073
33253333*         3**8.7**4.(31.37)**2.13**1             "331"  +.000601
3327      4   F( 7,7) = 29.4733
33273     5   F( 3,4733) = 22406023                       "7"
333       3 - F( 3,13) = 3.61
3332      4 * F( 2,61) = 2.31                             '31'
33323     5   F( 3,31) = 3.331                           '331'
333233    6   F( 3,331) = 3.7.5233
3332333   7   F( 3,5233) = 3.7.31.42073                  "331"
33325     5   F( 5,31) = 5.11.17351
33325*            (3.5.11.13)**2                          "31"  +.063835
3333      4 - F( 3,61) = 3.13.97
33332     5 * F( 2,97) = 2.7**2                           '7'
333323    6   F( 3,7) = 3.19
333323*           3**8.(7.13.19.61)**2                           +.013531
333325    6   F( 5,7) = 2801
3333253   7   F( 3,2801) = 37.43.4933
33332533  8   F( 3,4933) = 3.127.193.331
33332533*         3**8.7**4.(13.37.43.61)**2                     +.013367
333327    6   F( 7,7) = 29.4733
3333273   7   F( 3,4733) = 22406023                       "7"
33333     5 - F( 3,97) = 3.3169
333332    6 * F( 2,3169) = 2.5.317
333332*           3**8.(5.13)**2                                 +.006965
333333    6 - F( 3,3169) = 3.3348577
3333332   7 * F( 2,3348577) = 2.1674289
33335     5 - F( 5,97) = 11.31.262321
333352    6 * F( 2,262321) = 2.31.4231
3333523   7   F( 3,4231) = 3.601.9931
33335233  8   F( 3,9931) = 3.211.155821
3335      4 - F( 5,61) = 5.131.21491
```

The number of nodes in the 10^{36}-tree could have been reduced somewhat, and the amount of computation reduced somewhat more, by using the m'' mentioned earlier. This was not done, partly for simplicity, partly to obtain the statement $p^a > u = 10^{18}$ in the Introduction, and partly not to limit the search for possible opn's unnecessarily soon.

9. Description of the Listings. The nodes are in lexicographic order. The field "NODE" shows $c_1c_2 \cdots c_l$ (omitting λ), with the primes 11, 13, 17, $\cdots$ abbreviated as A, B, C, $\cdots$. "L" shows l. "E" contains ($-$, $*$, blank) according as

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335          3 - F( 5,13) = 30941
3352         4 * F( 2,30941) = 2.3**4.191
33523        5   F( 3,191) = 7.13**2.31                            '31'
335233       6   F( 3,31) = 3.331                                 '331'
3352333      7   F( 3,331) = 3.7.5233
33523333     8   F( 3,5233) = 3.7.31.42073                        "331"
335235       6   F( 5,31) = 5.11.17351
335235*            (3.5.11.13)**2                                  "31"  +.063835
337          3 - F( 7,13) = 5229043
35           2 - F( 5,3) = 11**2
353          3 - F( 3,11) = 7.19
353*               3**4.(7.11.19)**2                                     +.007905
355          3 - F( 5,11) = 5.3221
3552         4 * F( 2,3221) = 2.3**2.179
3552*              (3.11)**4.5**2                                        +.018609
3553         4 - F( 3,3221) = 10378063
357          3 - F( 7,11) = 43.45319
37           2 - F( 7,3) = 1093
372          3 * F( 2,1093) = 2.547
3723         4   F( 3,547) = 3.163.613
37233        5   F( 3,613) = 3.7.17923
373          3 - F( 3,1093) = 3.398581
3732         4 * F( 2,398581) = 2.17.19.617
37323        5   F( 3,617) = 97.3931
373233       6   F( 3,3931) = 3.7.31.23743
373233*            3**6.(7.17.19.31)**2                                  +.010190
3A           2 - F(11,3) = 23.3851
3A3          3 - F( 3,3851) = 13.1141081
3A32         4 * F( 2,1141081) = 2.337.1693
3A323        5   F( 3,1693) = 3.13.151.487
3A3233       6   F( 3,487) = 3.7.11317
3A3233*            3**10.(7.13.23.151.337.487.1693.3851)**2 +.000022
3B           2 - F(13,3) = 797161
3B2          3 * F( 2,797161) = 2.398581
3C           2 - F(17,3) = 1871.34511
5            1 - LEAST ELEMENT 5, HENCE N > 10**13
52           2 * F( 2,5) = 2.3                                       3<5
53           2 - F( 3,5) = 31
533          3 - F( 3,31) = 3.                                       3<5
535          3 - F( 5,31) = 5.11.17351
55           2 - F( 5,5) = 11.71
553          3 - F( 3,71) = 5113
5532         4 * F( 2,5113) = 2.2557
55323        5   F( 3,2557) = 3.                                     3<5
5533         4 - F( 3,5113) = 3.                                     3<5
555          3 - F( 5,71) = 5.11.211.2221
5552         4 * F( 2,2221) = 2.11.101
55523        5   F( 3,101) = 10303
5553         4 - F( 3,2221) = 3.                                     3<5
57           2 - F( 7,5) = 19531
5A           2 - F(11,5) = 12207031
7+           1 - LEAST ELEMENT >= 7, HENCE N > 10**41
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the exceptional prime has (not yet, just, already) been chosen. "F" shows $F(q, p) = F_q(p)$ and its factorization, performed on an IBM 7094. If ν has an abundance-contradiction, the succeeding line ν^* shows an abundant $\bar{m} \mid m'$, and the value of $\ln(\sigma(\bar{m})/2\bar{m}) > 0$. For a fuller explanation, see pp. 37–39 and 45 of [6]. I am indebted to a referee for four minor corrections to the hand-computed $\bar{m}$. Least-element-contradictions are indicated by "$3 < 5$". Any branch-nodes not shown, of other than terminal nodes, are truncation nodes. The presentation of the 10^{36}-tree [8] completes the demonstration of the assertions in the Introduction.

For $\hat{p} = 7, 31, 331$ there are 2, 2, 3 nodes $\nu^{(\hat{p}, i)}$ having the same branching function, range, and $p_{\min}(n)$ for each i. For example, at both $\nu^{(7,1)} = \lambda 332$ and $\nu^{(7,2)} = \lambda 33332$, the branching is on $q_1(7, n)$. For any such $\hat{p}$, the subtrees rooted at each $\nu^{(\hat{p},i)}$ will have identical branchings and factorizations (at least initially), aside from abundance-contradictions which merely cause different prunings of these subtrees.

The major benefit of identifying these subtrees is to eliminate duplicate execution of some factorizations. A lesser potential benefit could be to shorten the listings. In [6], some shortening was obtained by "overlaying" all the subtrees for each such $\hat{p}$ onto one of them, on which the separate cases were carried along. In the present listings, for greater clarity these subtrees have not been overlaid. However, each has been demarcated by '$\hat{p}$' at its root, and "$\hat{p}$" at its lexicographically last node.

Some space, but not much computation, could be saved by subsuming the cases for such a $\hat{p}$ into one appropriate new case (represented on a separate tree) which includes appropriate common restrictions. For example, the cases for $\hat{p} = 7$ could be subsumed into a case with the restrictions $p_{\min}(n) = 3$, $m = 3^2 \cdot (13^1 \text{ or } 97^1) \mid_a n$ (hence $m \in E_1$), $2^1 \cdot 3^1 \cdot 7^1 \cdot 13^1 \mid \sigma(n)$; branch on $q_1(7, n)$. Analogs of m'_ν and m''_ν could be defined; for example, let $m' = m'' = 3^2 \cdot 7^2 \cdot 13^1$ at the root.

10. A Comparison. The above q-algorithm is a modification of a simpler "a-algorithm" [6] which uses branchings on $a(\hat{p}, n)$. The "complication" is more than compensated for by the fewer branches and factorizations. For example, with $u = 10^{18}$, the branches at $\lambda 3$, with $\hat{p} = 3$, are reduced from 18 cases of $\hat{p}^a < u$ to 11 cases of $\hat{p}^{q_1 - 1} < u$; at $\lambda 33$, with $\hat{p} = 13$, from 12 cases to 7; at $\lambda 5$, with $\hat{p} = 5$, from 19 cases to 9. Each omitted case, i.e., one for which $h = a + 1$ is composite, is subsumed in the case of the least $q \mid h$, with retention of the useful common factorization of $F_q(\hat{p})$, and omission of the not-yet-needed further branching and computation.

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