Four Large Amicable Pairs

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Abstract. This note gives a report of systematic computer tests of Euler's rule and several Thabit-ibn-Kurrah-rules, in search of large amicable pairs. The tests have yielded four amicable pairs, which are much larger than the largest amicable pair thus far known.

1. The pair of 25-digit numbers

(45222 6553454520 8537974785, 45398 0132623392 8286140415)

has been the largest known amicable pair since 1946 ([8], [10]). This note gives four new amicable pairs with 32-, 40-, 81-, and 152-digit numbers, as a result of systematic computer tests by Euler's rule (Section 2) and several Thabit-ibn-Kurrah-rules (Sections 3 and 4).

In this research, primality of very large numbers N had to be established, where $N + 1$ can be easily factorized; this was done by use of the following:

THEOREM (LUCAS-LEHMER [11, p. 442]). *Let P and Q be relatively prime integers and let $U_0 = 0$, $U_1 = 1$, $U_{i+1} = PU_i - QU_{i-1}$ for $i \geq 1$. If N is a natural number, relatively prime to $2P - 8Q$, and if $U_{N+1} \bmod N = 0$, while $U_{(N+1)/p} \bmod N \neq 0$ for each prime p dividing $N + 1$, then N is prime.*

It is convenient to choose $P = 1$, while Q has to be chosen such that $D^{(N-1)/2} \bmod N = -1$, where $D = P - 4Q$.

In the sequel, the indication " $(Q = A)$ " after a number means that primality of that number was established by use of this Lucas-Lehmer theorem, with $Q = A$. The computations were carried out on the Electrologica-X8 computer of the Mathematical Centre; the value of $U_i \bmod N$ was computed in $O(\log i)$ steps by use of the binary method (see [6, p. 360 (Exercise 15) and p. 421 (Exercise 26)]).

2. Euler's rule [4] for amicable numbers is given by: $2^n pq$ and $2^n r$ are amicable numbers, if the three integers $p = 2^{n-m}f - 1$, $q = 2^n f - 1$ and $r = 2^{2n-m}f^2 - 1$ are primes, with $f = 2^m + 1$ and $n > m \geq 1$. For $m = 1$, this rule is due to Thabit ibn Kurrah and yields amicable numbers for $n = 2, 4, 7$, but for no other value $n \leq 1000$ (see [12, p. 874]).* Only one more solution of Euler's rule was known thus far, viz., $m = 7$, $n = 8$ (Legendre, Chebyshev).

A systematic computer search for triples (p, q, r) such that both p, q and r are primes was carried out for all values of n, m with $n > m > 1$ and $r < 10^{132}$; this search yielded just one new solution, viz., $m = 11$, $n = 40$. Thus we have the new 40-digit amicable numbers:

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* This corrects an error in [2, p. 571, footnote]. W. Borho has asked me to point out here, that his quotation of a correct, private communication of E. J. Lee was incorrect.

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$$m_1 = 2724918040\ 3937065577\ 8575224081\ 9405848576 = 2^{40}pq,$$

$$m_2 = 2724918040\ 3961848563\ 0625803878\ 7235905536 = 2^{40}r,$$

with

$$p = 2^{29}3 \cdot 683 - 1 = 110\ 0048498687 \quad (Q = -1),$$

$$q = 2^{40}3 \cdot 683 - 1 = 225289\ 9325313023 \quad (Q = -13),$$

$$r = 2^{69}3^2 683^2 - 1 = 24782985\ 2050580016\ 6853312511 \quad (Q = -4)$$

and

$$m_1/m_2 \approx 1 - 2^{-40}.$$

3. Definition. A Thabit-ibn-Kurrah-rule or Thabit-rule

$$T(b_1, b_2, p, c_1X - 1, c_2X - 1),$$

with given natural numbers b_1, b_2 , a prime p , and linear polynomials

$$c_1X - 1, c_2X - 1 \in \mathbb{Z}[X]$$

is a statement of the form:

$p^n b_1 (c_1 p^n - 1)$ and $p^n b_2 (c_2 p^n - 1)$ are amicable numbers, if $q_i = c_i p^n - 1$ is prime and prime to b_i for $i = 1, 2$ ($n = 1, 2, \dots$).

For a more general definition see [2].

Walter Borho [2] presents a list of fifteen Thabit-rules, which are constructed from those amicable numbers of the form au, as (with $(a, us) = 1$, s prime), for which $p = u + s + 1$ is prime. Table 1 presents another seven Thabit-rules, constructed in the same way; this completes the list of Thabit-rules which can be constructed from the (at least) 67 published ([8], [9], [10], [2]) amicable pairs of the form au, as with $(a, us) = 1$, s prime.

TABLE 1

Seven new Thabit-rules $T(au, a, p, (u + 1)X - 1, (u + 1)\sigma(u)X - 1)$ obtained from amicable pairs au, as (with $(a, us) = 1$, s prime) such that $p = u + s + 1$ is prime.

No.	a	u	$\sigma(u)$	p	obtained from pair no.
(i)	$3^{27} 13 \cdot 19 \cdot 29$	$41 \cdot 173 = 7093$	7308	14401	(33) of [3]
(ii)	$3^4 5 \cdot 11^{27} 1$	$709 \cdot 2129 = 1509461$	1512300	3021761	(31) of [3]
(iii)	$3^{27} 11 \cdot 19 \cdot 43 \cdot 89$	$293 \cdot 22961 = 6727573$	6750828	13478401	(8) of [5] top of p. 168
(iv)	$2^3 31$	$17 \cdot 107 \cdot 4339 = 7892641$	8436960	16329601	(34) of [3]
(v)	2^8	$257 \cdot 33023 = 8486911$	8520192	17007103	(17) of [3]
(vi)	$2^3 19 \cdot 137$	$83 \cdot 218651 = 18148033$	18366768	36514801	(2) of [7]
(vii)	$2^7 263$	$4271 \cdot 280883 = 1199651293$	1199936448	2399587741	(18) of [3]

In the fifteen Thabit-rules of Borho, and the seven, given here, the numbers $q_1 = (u + 1)p^n - 1$ and $q_2 = (u + 1)\sigma(u)p^n - 1$ were tested for primality, for all values of $n \geq 1$ such that $q_2 < 10^{120}$. Both q_1 and q_2 appeared to be prime in only three cases; these cases, together with those of Borho and Lee (see [2]) are listed in

Table 2. Table 2 also mentions the discoverers of the amicable pairs from which the Thabit-rules were obtained.

TABLE 2
Five cases in which Thabit-rules yield amicable pairs.

<i>Thabit-rule</i>	<i>obtained from an amicable pair discovered by</i>	<i>value of n for which both q_1 and q_2 are primes</i>	<i>discovered by</i>
1 of [2]	Pythagoras (?)	2	Borho
6 of [2]	Euler	1	Lee
6 of [2]	Euler	19	te Riele
(i) of Table 1	Escott	8	te Riele
(ii) of Table 1	Escott	1	te Riele

Next follow the details of the three new amicable pairs. Thabit-rule 6 of [2], $n = 19$, yields the 152-digit amicable numbers:

$$\begin{aligned}
 m_1 &= 86 \ 2593766501 \ 4359638769 \ 0953818787 \ 1666597148 \ 4088835777 \\
 &\quad 4281383581 \ 6831022646 \ 6591332953 \ 3162256868 \ 3649647747 \\
 &\quad 2706738497 \ 3129580885 \ 3683841099 \ 1321499127 \ 6380031055 \\
 &= 3^{45} \cdot 11 \cdot 5281^{19} \cdot 29 \cdot 89 \cdot q_1, \\
 m_2 &= 90 \ 2364653062 \ 3313066515 \ 5201592687 \ 0786444130 \ 4548569003 \\
 &\quad 8961540360 \ 5363719932 \ 5828701918 \ 5759580345 \ 2747004992 \\
 &\quad 7532312907 \ 0333233826 \ 7840675607 \ 3892061566 \ 6452384945 \\
 &= 3^{45} \cdot 11 \cdot 5281^{19} \cdot q_2,
 \end{aligned}$$

with

$$\begin{aligned}
 q_1 &= 2 \cdot 1291 \cdot 5281^{19} - 1 = \quad 13917 \ 5701888775 \ 9763088555 \ 3289918626 \\
 &\quad 7927088632 \ 5517442305 \ 8328801872 \ 3382689621 \\
 &\quad (Q = -7), \\
 q_2 &= 2^3 3^3 5^2 1291 \cdot 5281^{19} - 1 = \quad 37577439 \ 5099695136 \ 0339099388 \ 2780292340 \\
 &\quad 3139307889 \ 7094225748 \ 7765055313 \ 3261979399 \\
 &\quad (Q = -4)
 \end{aligned}$$

and $m_1/m_2 = .955926$.

Thabit-rule (i), $n = 8$, yields the 81-digit amicable numbers:

$$\begin{aligned}
 m_1 &= 5 \ 4392258330 \ 0492702317 \ 4526035140 \ 9264518101 \ 4270450011 \\
 &\quad 0522977234 \ 9031452833 \ 4873070667 \\
 &= 3^{27} 2^{13} \cdot 19 \cdot 29 \cdot 14401^8 \cdot 41 \cdot 173 \cdot q_1, \\
 m_2 &= 5 \ 6040973336 \ 5289816514 \ 3019352151 \ 4014539485 \ 6000068942 \\
 &\quad 0342360436 \ 3362262293 \ 3462548533 \\
 &= 3^{27} 2^{13} \cdot 19 \cdot 29 \cdot 14401^8 \cdot q_2,
 \end{aligned}$$

with

$$\begin{aligned}
 q_1 &= 2 \cdot 3547 \cdot 14401^8 - 1 = 13122977 \ 5913527520 \ 5302674846 \ 4924755893 \\
 &\quad (Q = -15),
 \end{aligned}$$

$$q_2 = 2^3 3^{27} \cdot 29 \cdot 3547 \cdot 14401^8 - 1 \\ = 95902720237 \ 6059120035 \ 1947778167 \ 0116073351 \ (Q = -24)$$

and $m_1/m_2 = .970580$.

Thabit-rule (ii), $n = 1$, yields the 32-digit amicable numbers:

$$m_1 = 72 \ 3874114476 \ 8207595520 \ 9400624355 = 3^4 5 \cdot 11^{271} \cdot 3021761 \cdot 709 \cdot 2129 \cdot q_1, \\ m_2 = 72 \ 5235579669 \ 5217952865 \ 9056738845 = 3^4 5 \cdot 11^{271} \cdot 3021761 \cdot q_2,$$

with

$$q_1 = 2 \cdot 3^3 27953 \cdot 3021761 - 1 = 456 \ 1233402581 \ (Q = -3), \\ q_2 = 2^3 3^4 5^2 71^2 27953 \cdot 3021761 - 1 = 689795327 \ 4724758599 \ (Q = -3)$$

and $m_1/m_2 = .998123$.

Remark. The two previous examples, of size 81D and 152D, offer contrary evidence to the conjecture [1] that if there exists an infinity of amicable pairs (m_1, m_2) with $m_1 < m_2$, then $\lim_{m_1 \rightarrow \infty} m_1/m_2 = 1$.

4. Table 2 of [2] lists five more Thabit-rules, which differ slightly from the Thabit-rules mentioned above in Section 3. The numbers q_1 and q_2 occurring in these five Thabit-rules were also tested for primality, for all $n \geq 1$ with $q_2 < 10^{120}$. The results were negative in the sense that no pairs (q_1, q_2) were found with both q_1 and q_2 prime.

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