A Quadratic Partition of Primes $\equiv 1 \pmod{7}$

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Abstract. The solutions of a quadratic partition of primes $p \equiv 1 \pmod{7}$, in terms of which the author and P. A. Leonard have given the cyclotomic numbers of order seven and also necessary and sufficient conditions for 2, 3, 5 and 7 to be seventh powers $(\text{mod } p)$, are obtained for all such primes < 1000 .

Let p be a prime $\equiv 1 \pmod{7}$. P. A. Leonard and the author [4] have given necessary and sufficient conditions for 2, 3, 5 and 7 to be seventh powers $(\text{mod } p)$ (see also [1], [6]), in terms of the solutions of the following quadratic partition of p :

$$(1) \quad 72p = 2x_1^2 + 42(x_2^2 + x_3^2 + x_4^2) + 343(x_5^2 + 3x_6^2),$$

$$(2) \quad \begin{aligned} 12x_2^2 - 12x_4^2 + 147x_5^2 - 441x_6^2 + 56x_1x_6 + 24x_2x_3 - 24x_2x_4 \\ + 48x_3x_4 + 98x_5x_6 = 0, \end{aligned}$$

$$(3) \quad \begin{aligned} 12x_3^2 - 12x_4^2 + 49x_5^2 - 147x_6^2 + 28x_1x_5 + 28x_1x_6 + 48x_2x_3 \\ + 24x_2x_4 + 24x_3x_4 + 490x_5x_6 = 0. \end{aligned}$$

It was shown in [2], [5] that the system (1)–(3) has exactly eight solutions $(x_1, x_2, x_3, x_4, x_5, x_6)$ with $x_1 \equiv 1 \pmod{7}$. (The negatives of these eight solutions, each satisfying $x_1 \equiv -1 \pmod{7}$, are the only other solutions.) Of the eight solutions with $x_1 \equiv 1 \pmod{7}$, two solutions, namely $(x_1, x_2, x_3, x_4, x_5, x_6) = (-6t, \pm 2u, \pm 2u, \mp 2u, 0, 0)$, where $p = t^2 + 7u^2$, $t \equiv 1 \pmod{7}$, are regarded as trivial. If $(x_1, x_2, x_3, x_4, x_5, x_6)$ is one of the six nontrivial solutions with $x_1 \equiv 1 \pmod{7}$, all six such solutions are given by (*) where $0 \leq k \leq 5$. In this paper, a nontrivial solution of (1)–(3) with $x_1 \equiv 1 \pmod{7}$ is given for each of the 28 primes $p < 1000$ with $p \equiv 1 \pmod{7}$ (see Table 2 below). These solutions were computed from a prime factor λ of p in the unique factorization domain $Z[\alpha]$, $\alpha = \exp(2\pi i/7)$, where the values of λ were obtained from an old

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$$(*) \quad (x_1, x_2, x_3, x_4, x_5, x_6) \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & -\frac{3}{2} & -\frac{1}{2} \end{pmatrix}^k,$$

table of Kummer [3], as follows: for each λ an associate π of λ was found such that

$$(4) \quad \pi_1 \pi_4 \pi_5 \equiv -1 \pmod{(1-\alpha)^2},$$

where $\pi_i = \sigma_i(\pi)$ and σ_i is the automorphism of $\mathcal{Q}(\alpha)$ defined by $\sigma_i(\alpha) = \alpha^i$ ($1 \leq i \leq 6$). Then, if

$$(5) \quad \pi_1 \pi_4 \pi_5 = c_1 \alpha + c_2 \alpha^2 + c_3 \alpha^3 + c_4 \alpha^4 + c_5 \alpha^5 + c_6 \alpha^6,$$

a solution $(x_1, x_2, x_3, x_4, x_5, x_6)$ of (1)–(3) is given by

$$\begin{aligned} x_1 &= -c_1 - c_2 - c_3 - c_4 - c_5 - c_6 \quad (x_1 \equiv 1 \pmod{7}), \\ x_2 &= c_1 - c_6, \\ x_3 &= c_2 - c_5, \\ (6) \quad x_4 &= c_3 - c_4, \\ 7x_5 &= c_1 + c_2 - 2c_3 - 2c_4 + c_5 + c_6, \\ 7x_6 &= c_1 - c_2 - c_5 + c_6. \end{aligned}$$

(Alternatively, as $\pi_1 \pi_4 \pi_5$ is a Jacobi sum of order 7, the c_i could have been obtained from tables of Jacobi sums.) The solutions $(x_1, x_2, x_3, x_4, x_5, x_6)$ obtained are listed in Table 2 below and each one was shown directly to satisfy (1)–(3).

In view of the relative inaccessibility of Kummer's paper [3], we list for convenience his values of λ in Table 1.

Two mistakes were noted in Kummer's table. For $p = 337$, he gives the incorrect value $\lambda = 2 + \alpha - \alpha^2 - \alpha^4$ (which is a factor of 344) and, for $p = 617$, he gives the incorrect value $\lambda = 2 + \alpha + \alpha^2 - \alpha^5$ (which is a factor of 113). The respective correct values $\lambda = 3 - 4\alpha + 2\alpha^2 - 5\alpha^4 + 4\alpha^5 - 8\alpha^6$ and $\lambda = 5 + 5\alpha - 4\alpha^3 - 3\alpha^4 + 2\alpha^6$ (given below) are taken from a table of Reuschle [7]. (Kummer's table was used rather than Reuschle's, as Kummer's values of λ are in general simpler than those of Reuschle. Two errors were noted in Reuschle's table: the factor of 29 given is incorrect (it is a factor of 1093), and the twelfth prime p listed should be 421 not 431.)

TABLE 1. *Prime factors λ in $Z[\alpha]$ of primes $p \equiv 1 \pmod{7}$, $p \leq 1000$*

p	λ	p	λ
29	$1 + \alpha - \alpha^2$	491	$3 + \alpha + \alpha^3 - \alpha^5$
43	$2 + \alpha$	547	$3 + \alpha$
71	$2 + \alpha + \alpha^3$	617	$5 + 5\alpha - 4\alpha^3 - 3\alpha^4 + 2\alpha^6$
113	$2 - \alpha + \alpha^5$	631	$2 + 2\alpha - \alpha^2 + \alpha^3 + \alpha^6$
127	$2 - \alpha$	659	$2 + 2\alpha - \alpha^2 + \alpha^5$
197	$3 + \alpha + \alpha^5 + \alpha^6$	673	$4 + 3\alpha + 2\alpha^2 + \alpha^4 + 2\alpha^6$
211	$3 + \alpha + 2\alpha^2$	701	$3 + \alpha + \alpha^4 - \alpha^5 + \alpha^6$
239	$3 + 2\alpha + 2\alpha^2 + \alpha^3$	743	$3 + 2\alpha - \alpha^3 - \alpha^4$
281	$2 - \alpha - 2\alpha^3$	757	$3 + 2\alpha + \alpha^3$
337	$3 - 4\alpha + 2\alpha^3 - 5\alpha^4 + 4\alpha^5 - 8\alpha^6$	827	$2 + 2\alpha - \alpha^4 - \alpha^6$
379	$3 + 2\alpha + \alpha^2$	883	$2 - \alpha^2 - 2\alpha^3 - \alpha^5$
421	$3 + \alpha + \alpha^2$	911	$3 + 2\alpha - \alpha^3 + \alpha^4$
449	$2 + \alpha - \alpha^3 - \alpha^6$	953	$3 + \alpha - \alpha^2 - \alpha^3$
463	$3 + 2\alpha$	967	$2 + 2\alpha - \alpha^3 + 2\alpha^5$

TABLE 2. *Solutions of (1)–(3)*

p	x_1	x_2	x_3	x_4	x_5	x_6
29	1	-2	-3	-2	-1	1
43	1	-6	-1	-2	-1	1
71	15	0	3	-2	-3	-1
113	-27	6	-4	3	0	-2
127	29	0	12	-1	-2	0
197	-13	-6	1	-8	-5	1
211	-55	0	13	-4	1	-1
239	57	-11	0	6	3	-1
281	57	6	7	12	-3	-1
337	-13	15	-10	4	-5	-1
379	-13	10	13	-12	-5	1
421	-55	-4	3	18	-5	1
449	-41	0	10	19	-4	2
463	1	0	9	22	-1	-3
491	-69	6	9	20	3	1
547	43	2	15	0	-1	5
617	-55	-6	-1	-16	1	-5

(continued)

TABLE 2 (*continued*)

p	x_1	x_2	x_3	x_4	x_5	x_6
631	8	-6	-18	14	-8	0
659	-27	-4	-9	-30	3	1
673	22	20	8	-12	-4	-4
701	-125	20	3	-4	-1	1
743	-27	20	12	-3	-6	4
757	-27	14	-13	4	9	3
827	15	26	3	-6	-3	-5
883	15	-4	-13	-32	3	-3
911	29	-6	-10	-31	-2	4
953	50	12	8	-28	4	4
967	127	15	-6	20	-1	3

From Table 2, we see that x_1 is *even* only for $p = 631, 673, 953$, so that (see [4]) 2 is a seventh power (mod p) for primes $p \equiv 1 \pmod{7}$ less than 1000 only for these primes. Indeed, we can show directly that $2 \equiv 196^7 \pmod{631}$, $2 \equiv 128^7 \pmod{673}$, $2 \equiv 120^7 \pmod{953}$.

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