REVIEWS AND DESCRIPTIONS OF TABLES AND BOOKS

The numbers in brackets are assigned according to the revised indexing system printed in Volume 28, Number 128, October 1974, pages 1191–1194.

1 [7].—DANIEL SHANKS & JOHN W. WRENCH, JR., *Sums of Reciprocals to* 1,000,000, 1961, ms. of 20 computer sheets deposited in the UMT file.

Herein are tabulated values of the partial sums $\Sigma_1^N n^{-1}$ of the harmonic series for $N = 10^4(10^4)10^6$, truncated to 1060D. These were computed on an IBM 7090 system at the same time that we evaluated π [1] and e [2], and they were intended to be used by the second author in computing Euler's constant, γ, by means of the Euler-Maclaurin formula. However, Knuth [3] computed γ to higher precision before this was completed.

For the sake of comparison we list these sums truncated to 1000D for $N = 10^4$, 10^5, and 10^6, respectively, with $^*(M)^*$ denoting the omission of M digits:

$$9.7876060360\ 4438226417\ ^*(960)^*\ 9216446619\ 7618373424,$$
$$12.0901461298\ 6342794736\ ^*(960)^*\ 7602452004\ 8801442625,$$
$$14.3927267228\ 6572363138\ ^*(960)^*\ 3436083266\ 8760078693.$$

Our value corresponding to $N = 10^4$ agrees in its entirety with the value found to 1275D by Knuth, which has been deposited in the UMT file along with his unpublished table of Bernoulli numbers mentioned on p. 277 of [3].

D. S., J. W. W.

1. DANIEL SHANKS & JOHN W. WRENCH, JR., "Calculation of π to 100,000 decimals," *Math. Comp.*, v. 16, 1962, pp. 76–99.

2. UMT 46, *Math. Comp.*, v. 23, 1969, pp. 679–680.

3. DONALD E. KNUTH, "Euler's constant to 1271 places," *Math. Comp.*, v. 16, 1962, pp. 275–281.

2 [9].—DAVID BALLEW, JANELL CASE & ROBERT N. HIGGINS, *Table of* $\phi(n) = \phi(n + 1)$, South Dakota School of Mines and Technology, 1974, ii + 3 pages, deposited in the UMT file.

There are listed here the 88 solutions of $\phi(n) = \phi(n + 1)$ from $n = 3$ to $n = 2792144$. (Previous tables have listed $n = 1$ also; counting this, there are 89 solutions for $n < 2.8 \cdot 10^6$.) This extends the tables of the 36 solutions to $n = 10^5$ by Lal and Gillard [1] and the 56 solutions to $n = 5 \cdot 10^5$ by Miller [2]. Note that Miller is wrong in stating that the next solution is $n = 525986$. She has omitted $n = 524432$.

A propos my editorial note to [2], there is only one further case in this extension (if I did it correctly). For $n = 2539004$, multiplication (mod n) is isomorphic to multiplication (mod $n + 1$). That is a much more stringent requirement; I do not know if anyone has made a heuristic estimate of whether there are infinitely many such n.

D. S.

1. M. LAL & P. GILLARD, "On the equation $\phi(n) = \phi(n + k)$," *Math. Comp.*, v. 26, 1972, pp. 579–583.
2. KATHRYN MILLER, UMT 25, *Math. Comp.*, v. 27, 1973, pp. 447–448.

3 [9].—B. D. BEACH, H. C. WILLIAMS & C. R. ZARNKE, *Some Computer Results on Units of Quadratic and Cubic Fields,* Scientific Report 31, University of Manitoba, Winnipeg, July 1971.

The table in the appendix lists the class number H and fundamental unit ϵ_0 $(0 < \epsilon_0 < 1)$ of the pure cubic fields $Q(\rho)$ where $\rho = D^{1/3}$. For each cube-free D between 2 and 998 there is listed $H, U, V, W, T,$ and J where

$$\epsilon_0 = (U + V\rho + W\rho^2)/T \tag{1}$$

and J is the length of the period of Voronoi's algorithm. The largest U here is a 330-decimal number for $D = 951$ where $H = 1$. Here, $J = 1352$, and for large U one finds that $J/\log_{10} U \approx 4.1$. Presumably, the mean value of this ratio is analogous to Lévy's constant but its identity is not known to me. The largest H equals 162 here for $D = 813$. Some fields are given twice: e.g., $Q((12)^{1/3}) = Q((18)^{1/3})$ and so its ϵ_0 is given in two forms. Happily, the H then agree—in all cases that I checked.

A direct comparison with Wada's units to $D = 249$, see [1], is not possible since Wada gives the reciprocal $\epsilon = 1/\epsilon_0 = (A + B\rho + C\rho^2)/E$ instead. It is of some interest to argue which unit is preferable. Usually, U, V, W have only one-half the decimals of A, B, C; for example, for $D = 239$, U has 94 decimals while A has 188. But for applications, ϵ is usually preferable. Thus, in evaluating the regulator $R = |\log \epsilon_0|$, the formula (1) can suffer catastrophic loss of significance since ϵ_0 may be exceedingly small. Of course, one can obtain ϵ from ϵ_0 by

$$\epsilon = (U^2 - DVW) + (W^2D - UV)\rho + (V^2 - UW)\rho^2$$

if $T = 1$. So, for such large $U, V, W, R = \log(3U^2 - 3DVW)$ will be very accurate.

The text describes Voronoi's algorithm and refers to earlier, less extensive tables by Markov, Cassels, Selmer, etc.

D. S.

1. H. WADA, RMT 15, *Math. Comp.*, v. 26, 1972, pp. 302–303.

4 [9].—RICHARD P. BRENT, *Tables Concerning Irregularities in the Distribution of Primes and Twin Primes,* Computer Centre, Australian National University, Canberra, 1974, 11 computer sheets deposited in the UMT file.

These are the tables referred to repeatedly in Brent's paper [1]. The numbers $\pi(n)$, $\pi_2(n)$ and $B^*(n)$ and

$$r_i(n),\ s_i(n),\ R_i(n, n'),\ \rho_i(n, n')$$

for $i = 1, 2, 3$ are defined in [1]. They are listed in Table 1 for 533 values of n:

$$10^4\ (10^4)\ 10^6\ (10^5)\ 10^7\ (10^6)\ 10^8\ (10^7)\ 10^9\ (10^8)\ 10^{10}\ (10^9)\ 83\cdot 10^9.$$

Table 2 (1 page long) lists n, $\pi_2(n)$, $B(n)$, and $B^*(n)$ with some auxiliary functions for

$$10^5\ (10^5)\ 10^6\ (10^6)\ 10^7\ (10^7)\ 10^8\ (10^8)\ 10^9\ (10^9)\ 10^{10}\ (10^{10})\ 8\cdot 10^{10}:$$

The author indicates that he has much more detailed tables and is continuing to 10^{11}.

Section 3 of [1] ends with the same conclusion given earlier in our [2]: that the unpredictable fluctuations of $\pi_2(n)$ around the Hardy-Littlewood approximation makes it difficult to compute Brun's constant accurately. But his Fig. 3 allows for a posteriori judgment; although we do not know where $s_3(n)$ is going, we know where it's been! We see that Fröberg's low value at $\log_{10}n = 6.02$, our high value at $\log_{10}n = 7.51$ and Bohman's low value at $\log_{10}n = 9.30$ all correlate (inversely) with the peaks and valleys of Fig. 3. In fact, Fig. 3 between $\log_{10}n = 6.63$ and 7.19 gives a crude, distorted, upside-down version of our Fig. 1 [2] and $\log_{10}n$ between 7.19 and 7.51 continues with our Fig. 2. Thus, for Brun's constant, it does appear that $n = 8\cdot 10^{10}$ is a good time to quit since $s_3(n)$ is then very small.

Concerning the negative peaks in Brent's Fig. 1 at $\log_{10}n = 8.04$ and 8.25, it would be nice to know when they are exceeded. As Brent is aware, if a likely n were known that is not too large, one could restart his tables of $r_i(n)$ and $s_i(n)$ for $i = 1, 2$ by computing a fiducial mark $\pi(n)$ by Lehmer's method.

D. S.

1. RICHARD P. BRENT, "Irregularities in the distribution of primes and twin primes," *Math. Comp.*, v. 29, 1975, pp. 43–56 (this issue).

2. DANIEL SHANKS & JOHN W. WRENCH, JR., "Brun's constant," *Math. Comp.*, v. 28, 1974, pp. 293–299; "Corrigendum", *ibid*, p. 1183.

5 [9].—CARL-ERIK FRÖBERG, *Kummer's Förmodan,* Lund University, 1973, 133 pages of computer output deposited in the UMT file.

The Kummer Sum

$$(1)\qquad S_p = \sum_{n=0}^{p-1} \exp(2\pi i n^3/p) = 1 + 2\sum_{n=1}^{(p-1)/2} \cos(2\pi n^3/p)$$

for a prime $p \equiv 1 \pmod 3$ equals one of the three real roots of

$$x^3 = 3px + pA \tag{2}$$

where $4p = A^2 + 27B^2, A \equiv 1 \pmod 3$. On the basis of only the 45 primes $p < 500$, Kummer conjectured that S_p occurs as the minimum, median, or maximum root of (2) in the proportions: 1, 2, 3. Subsequent work of von Neumann [1] and Emma Lehmer [2] suggested that as $p \longrightarrow \infty$ there may be equidistribution instead, and Vinogradov once thought [3] that he had proven this.

Fröberg [4] computed S_p for the 8988 $p < 2 \cdot 10^5$ and found 2370, 2990, and 3628 solutions, respectively, with the maximal roots now down to 40.4%, the minimal roots up to 26.4% and the median roots remaining very close to 33%. There is deposited here a listing of these 8988 primes: p, A, B, S_p (to 6D), and an asterisk in the appropriate column labelled MIN, MED, MAX. S_p has rounding errors (example below) but this accuracy is not needed here since it suffices to know where S_p lies in the three intervals: $I_1 < -\sqrt{p} < I_2 < +\sqrt{p} < I_3$. Note also that it is unnecessary to compute A and B separately, since $A = (S_p^3 - 3pS_p)/p$.

After extrapolating the three empirical percentage functions $\%(P)$, for $p < P$, according to the proposed formulas

$$\%(P) = a + b \exp(-cP), \tag{3}$$

Fröberg conjectures that the asymptotic proportions are 4, 5, 6—that is, that the limiting percentages are $26\frac{2}{3}, 33\frac{1}{3}$, and 40%, respectively. This reviewer is skeptical for two reasons: (A) No rationale, even heuristic, is given to support (3) and the exponential there tends to leave the purported asymptotic values a near his final empirical values at $P = 2 \cdot 10^5$. Whereas, any logarithmic function in place of (3) would make equidistribution more plausible. (B) If 4, 5, 6 are the true asymptotic proportions, it should be possible to find some reasonably simple heuristic argument that suggests these proportions. I know of none.

There are 51 cases here with $A > 0, B = 1$. Here the two smaller roots are nearly equal, being approximately $-\sqrt{p} \mp 1\frac{1}{2}$, while the largest root is nearly $+2\sqrt{p}$. If there is a difference in the ultimate proportion of MIN and MAX one might expect to see it here since the dissymmetry is maximized. One does not; there are 16, 18, and 17 cases, respectively. In the 53 cases with $A < 0, B = 1$, there is the opposite dissymmetry with the two larger roots close together near $\sqrt{p} \mp 1\frac{1}{2}$. One now finds 16, 19, 18 cases. (For more on the cyclic cubic fields with $B = 1$, see [5].) In the 74 cases here with $A = +1, -2, +4$ or -5, where the median root is $\approx -A/3$ while the extreme roots are $\approx \mp\sqrt{3p}$, one has the greatest symmetry. Here one finds 24, 21, and 29 cases. These are all small numbers but they seem to suggest equidistribution; certainly nothing here suggests that the MAX are 50% more numerous than the MIN. But if there is equidistribution, why are the MAX more common when p is small? A good, quantitative explanation is wanted.

$|S_p|$ is bounded below by $1/3$. The smallest S_p here is one of the aforementioned $A = 1$, namely, $p = 170647, A = 1, B = 159, S_p = -0.3333334056$. (The table lists $S_p = -0.335414$ for this p, showing that four decimals are corrupted in adding up the 85 thousand cosines.) The existence of such small S_p illustrates the marked distinction between these cubic sums and the quadratic Gauss Sums with n^2 instead of n^3 in (1). Then, $|S_p| = \sqrt{p}$, as is well known. For other recent work, see Cassels [6] and the references cited there.

D. S.

1. J. v. NEUMANN & H. H. GOLDSTINE, "A numerical study of a conjecture of Kummer," *MTAC*, v. 7, 1953, pp. 133–134.

2. EMMA LEHMER, "On the location of Gauss sums," *MTAC*, v. 10, 1956, pp. 194–202.

3. A. I. VINOGRADOV, "On the cubic Gaussian sum," *Izv. Akad. Nauk SSSR Ser. Mat.*, v. 31, 1967, pp. 123–148. (Russian)

4. C.-E. FRÖBERG, "New results on the Kummer conjecture," *BIT*, v. 14, 1974, pp. 117–119.

5. DANIEL SHANKS, "The simplest cubic fields," *Math. Comp.*, v. 28, 1974, pp. 1137–1152.

6. J. W. S. CASSELS, "On Kummer sums," *Proc. London Math. Soc.*, v. 21, 1970, pp. 19–27.

6 [9].—MARIE NICOLE GRAS, "Méthodes et algorithmes pour le calcul numérique du nombre de classes et des unités des extensions cubiques cycliques de Q," Institut de mathématiques pures, Grenoble, 1972–1973. Tables 1–4.

For any product $m = p_1 \cdot p_2 \cdot \cdots \cdot p_n$ of distinct primes $p \equiv 1 \pmod 3$ there are 2^{n-1} distinct cyclic cubic fields of discriminant m^2 and for $m = 9 \cdot p_1 \cdot \cdots \cdot p_n$ there are 2^n such fields. Altogether there are 630 fields with $m < 4000$. Table 1 lists each such m with (A) its prime decomposition; (B) its appropriate representation $4m = a^2 + 27b^2$; (C) its class number h; and, in most cases, (D) $\mathrm{tr}(\epsilon)$ and $\mathrm{tr}(\epsilon^{-1})$. These latter integers give the equation

$$x^3 = \mathrm{tr}(\epsilon)x^2 - \mathrm{tr}(\epsilon^{-1})x + 1$$

satisfied by the fundamental units and having a discriminant m^2k^2 for some index $k \geqslant 1$. When $\mathrm{tr}(\epsilon)$ and $\mathrm{tr}(\epsilon^{-1})$ are too large, they are omitted here since they were not obtained with the precision used. (These large units are only missing from Table 1 for some cases of $h = 1$ or 3 when $\zeta_k/\zeta(1)$ is relatively large because one or more small primes split in the field. The first units missing are those for $m = 919$ which has $h = 1$ and both 2 and 3 as splitting primes.)

This table, and those that follow, were computed by a new, interesting method described in Marie Gras's paper [1]. The tables are more easily extended to larger m by this method if h is large. There are known criteria for $9|h$ and $4|h$, [2], [3]. Table 2 continues with 154 more $m < 10^4$ having $9|h$ while Table 3 contains 119 $m < 10^4$ having $4|h$. These two tables overlap some. Sometimes, units are missing, as before.

Table 4 contains all m between $4 \cdot 10^3$ and $2 \cdot 10^4$ having a representation $4m = a^2 + 27$ or $1 + 27b^2$ or $9 + 27b^2$. In these 89 fields, $\mathrm{tr}(\epsilon)$ and $\mathrm{tr}(\epsilon^{-1})$ are never missing since they are known a priori. They equal $\pm 1/2(a \mp 3)$, $\pm 3/2(9b \mp 1)$ and $\pm 3/2(3b \mp 1)$, respectively. These units are relatively small and the class numbers, correspondingly, are relatively large. The largest is $h = 129$ for $m = 97 \cdot 181 = (1 + 27 \cdot 51^2)/4$.

These tables of cyclic cubic fields go far beyond earlier tables of Hasse, Cohn and Gorn, and Godwin. For the "simplest cubics", having $4m = a^2 + 27$, the reviewer has gone further [4] using an entirely different method.

D. S.

1. MARIE NICOLE GRAS, "Méthodes et algorithmes pour le calcul numérique du nombre de classes et des unités des extensions cubiques cycliques de Q," *Crelle's J.* (To appear.)

2. G. GRAS, "Sur les l-classes d'ideaux dans les extensions cycliques relatives de degre premier l," Thèse, Grenoble, 1972.

3. MARIE-NICOLE MONTOUCHET, "Sur le nombre de classes du sous-corps cubique de $Q^{(p)}$ ($p \equiv 1(3)$)," Thèse, Grenoble, 1971.

4. DANIEL SHANKS, "The simplest cubic fields," *Math. Comp.*, v. 28, 1974, pp. 1137–1152.

7 [9].—WELLS JOHNSON, *The Irregular Primes to* 30000 *and Related Tables*, ms. of 28 computer pages (+ 1 introductory page), deposited in the UMT file, June 1974.

This unpublished table constitutes an appendix to a paper published elsewhere in this issue. The 13-column table presents the complete list of 1619 irregular pairs $(p, 2k)$ with $p < 30000$ together with some computations which depend upon this list. The table shows that Fermat's Last Theorem is true for all prime exponents $p < 30000$. In addition, the tables of [1], [2], [3] are completed to 30000, so that the cyclotomic invariants of Iwasawa are completely determined for primes within this range. The computations were performed on the PDP-10 computer at Bowdoin College.

AUTHOR'S SUMMARY

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1. K. IWASAWA & C. SIMS, "Computation of invariants in the theory of cyclotomic fields," *J. Math. Soc. Japan*, v. 18, 1966, pp. 86–96. MR 34 #2560.

2. W. JOHNSON, "On the vanishing of the Iwasawa invariant μ_p for $p < 8000$," *Math. Comp.*, v. 27, 1973, pp. 387–396.

3. W. JOHNSON, "Irregular prime divisors of the Bernoulli numbers," *Math. Comp.*, v. 28, 1974, pp. 653–657.

8 [9].— J. P. KULIK, *Magnus Canon Divisorum* ⋯ , 8 ms. volumes (v. 2 now missing), deposited in the Library of the Academy of Sciences, Vienna in 1867.

A photostatic copy of that portion of v. 1 consisting of pages 260 through 416 has been deposited by D. H. Lehmer in the UMT file. This portion of Kulik's monumental table gives the least prime factor for all integers not divisible by 2, 3, or 5 between 9,000,000 and 12,642,600. The deposited copy includes handwritten corrections by Professor Lehmer inserted in the margins.

A detailed description of the complete table has been published by Joffe [1], superseding that of D. N. Lehmer [2].

In 1948 an announcement [3] was made that the Carnegie Institution of Washington had made in the preceding year a negative microfilm of this same portion of v. 1 and that it was prepared to supply positive microfilm copies at a nominal charge ($1.00 per copy at that time).

J. W. W.

1. UMT 48, *MTAC*, v. 2, 1946, pp. 139–140.
2. D. N. LEHMER, *Factor Table for the First Ten Millions*, Washington, D. C., 1909; also *List of Prime Numbers from* 1 *to* 10,006,721, Washington, D. C., 1914. (Both reprinted by Hafner Publishing Co., New York, 1956.)
3. *MTAC*, v. 3, 1948, p. 222, N 93.

9 [9].— SIGEKATU KURODA, *Table of Class Numbers,* $h(p)$ *Greater than* 1, *for Fields* $Q(\sqrt{p})$, $p \equiv 1 \pmod 4 \leqslant 2776817$, University of Maryland, 1965, copy deposited in the UMT file.

The table consists of 88 Xeroxed computer sheets containing class numbers $h(p)$ for primes of the form $p = 4n + 1$. The purpose of the computation was not simply to calculate $h(p)$, but to test a conjectured method of doing so. It is well known that every ideal class of $Q(\sqrt{p})$ contains an integral ideal with norm $< B = \frac{1}{2}\sqrt{p}$. Also the class number h is odd, the nonprincipal classes (if $h > 1$) occurring in conjugate pairs $C_i, C_i' = C_i^{-1}$, $1 \leqslant i \leqslant h' = \frac{1}{2}(h - 1)$. The conjecture states that the h' classes C_i can be represented by integral ideals *all having distinct norms less than* B.

Thus, only those p with $h(p) > 1$ were printed. There are $22 \cdot 2^{10} = 22528$ such primes up to $p = 2776817$. A printed count shows that altogether 100811 cases were computed; thus 78283 cases with $h = 1$ are omitted. The table lists the primes p and the class numbers $h(p)$ on alternate pages. Every other page contains 512 primes in 16 columns and 32 rows, each position identified by a number in base 32. The class number $h(p)$ is found on the next sheet and in the same position as p, cf. [1]. The class numbers are written in base 32 and followed by a symbol (P, Q, C or D) indicating that the conjecture was verified for that case. The primes p were printed both in decimal and in base 32. The copy deposited contains the decimal version except for the first two pages. These were missing from the reviewer's copy of the table and so are given in base 32. For statistics about the class number distribution see the reviewer's paper in this issue [2].

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1. R. B. LAKEIN & S. KURODA, UMT 38, *Math. Comp.*, v. 24, 1970, pp. 491–493.
2. R. B. LAKEIN, "Computation of the ideal class group of certain complex quartic fields. II," *Math. Comp.*, v. 29, 1975, pp. 137–144 (this issue).

10 [9].—RICHARD B. LAKEIN, *Class Numbers of* 5000 *Quartic Fields* $Q(\sqrt{\pi})$, SUNY at Buffalo, 1973, ms. of 21 computer sheets deposited in the UMT file.

Let P be a rational prime $\equiv 1 \bmod 8$, and $\pi = a + bi$ a Gaussian prime with norm $a^2 + b^2 = P$, normalized so that $a, b > 0, b \equiv 0 \bmod 4$. Then $K = Q(\sqrt{\pi})$ is a totally complex quartic field with no quadratic subfield other than $Q(i)$. The arithmetic of K has many strong analogies to that of a real quadratic field with prime discriminant. In particular, the class number $h(\pi)$ of K is odd.

This table lists the first 5000 primes $P \equiv 1 \bmod 8$ (from $P = 17$ through $P = 226241$), the (normalized) Gaussian prime factor π of P, and the class number $h(\pi)$ of the quartic field $K = Q(\sqrt{\pi})$. The final page of the table lists the cumulative distribution of class numbers for each successive 1000 fields. The distribution of class numbers is very close to that for the first 5000 real quadratic prime discriminants [2]. Details of the method of calculation, as well as the class number distribution, are contained in [1].

AUTHOR'S SUMMARY

1. R. B. LAKEIN, "Computation of the ideal class group of certain complex quartic fields," *Math. Comp.*, v. 28, 1974, pp. 839–846.
2. D. SHANKS, UMT 10, *Math. Comp.*, v. 23, 1969, pp. 213–214.

11 [9].—MORRIS NEWMAN, *A Table of the Coefficients of the Modular Invariant* $j(\tau)$, National Bureau of Standards, Washington, D. C., ms. of 14 pages deposited in the UMT file.

The absolute modular invariant $j(\tau)$, defined by

$$j(\tau) = x^{-1} \prod_{n=1}^{\infty} (1 - x^n)^{-24} \left\{ 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) x^n \right\}^3$$
$$= \sum_{n=-1}^{\infty} c(n) x^n = x^{-1} + 744 + 196884x + \cdots,$$

where $x = \exp 2\pi i\tau$ and $\sigma_r(n) = \Sigma_{d/n} d^r$, is the Hauptmodul of the classical modular group Γ. Its coefficients possess many remarkable arithmetic properties, which are set forth in the appended references. For example, the congruence

$$(n+1)c(n) \equiv 0 \pmod{24},$$

due to D. H. Lehmer [2], implies that $c(n)$ is even except possibly when $n \equiv 7 \pmod 8$. In this case it may be shown that $c(n)$ assumes both even and odd values infinitely often, although necessary and sufficient conditions for $c(n)$ to be odd are still unknown.

The coefficients were first computed for $-1 \leqslant n \leqslant 24$ by H. S. Zuckerman [7] and then for $-1 \leqslant n \leqslant 100$ by A. van Wijngaarden [6]. Here we tabulate the coefficients for $-1 \leqslant n \leqslant 500$. There would seem to be little point in extending the table further, since $c(500)$ is already a number of 120 digits.

The coefficients were computed, using residue arithmetic, by means of the following formula [5]:

$$c(n) = p_{-24}(n+1) + \frac{65520}{691} \sum_{k=0}^{n} \sigma_{11}(k+1)p_{-24}(n-k), \qquad n \geqslant 1,$$

where $\Sigma_{n=0}^{\infty} p_{-24}(n)x^n = \Pi_{n=1}^{\infty} (1-x^n)^{-24}$.

The total computation time on a UNIVAC 1108 system was approximately four minutes.

AUTHOR'S SUMMARY

1. O. KOLBERG, "Congruences for the coefficients of the modular invariant $j(\tau)$ modulo powers of 2," *Univ. Bergen Arbok Naturvit. Rekke*, v. 16, 1961.

2. D. H. LEHMER, "Properties of the coefficients of the modular invariant $J(\tau)$," *Amer. J. Math.*, v. 64, 1942, pp. 488–502.

3. J. LEHNER, "Divisibility properties of the Fourier coefficients of the modular invariant $J(\tau)$," *Amer. J. Math.*, v. 71, 1949, pp. 136–148.

4. J. LEHNER, "Further congruence properties of the Fourier coefficients of the modular invariant $J(\tau)$," *Amer. J. Math.*, v. 71, 1949, pp. 337–386.

5. M. NEWMAN, "Congruences for the coefficients of modular forms and for the coefficients of $j(\tau)$," *Proc. Amer. Math. Soc.*, v. 9, 1958, pp. 609–612.

6. A. VAN WIJNGAARDEN, "On the coefficients of the modular invariant $J(\tau)$," *Nederl. Akad. Wetensch. Proc. Ser. A*, v. 16, 1953, pp. 389–400.

7. H. S. ZUCKERMAN, "The computation of the smaller coefficients of $J(\tau)$," *Bull. Amer. Math. Soc.*, v. 45, 1939, pp. 917–919.

12 [9].—DANIEL SHANKS, *Table of the Greatest Prime Factor of* $N^2 + 1$ *for* $N = 1(1)185000$, 1959, two ms. volumes, each of 185 computer sheets, bound in cardboard covers and deposited in the UMT file.

This table was calculated in 1959 on an IBM 704 system by the p-adic sieve method described completely in [1]. The method is extraordinarily efficient: each division performed is known a priori to have a zero remainder. From the complete factorization of $n^2 + 1$ for $n = 1(1)185000$ I then tabulated only the greatest prime factors, 500 per page, arranged in an obvious format. (One can see at once which $n^2 + 1$ are prime by the relative size of the corresponding listed factors.)

These factorizations relate to questions concerning reducible numbers, primes of the form $n^2 + 1$, formulas for π, and other questions surveyed in [1].

In [2] and [3] similar p-adic sieves were run for $n^2 \pm 2, n^2 \pm 3, n^2 + 4, n^2 \pm 5, n^2 \pm 6$, and $n^2 \pm 7$ for $n = 1(1)180000$, but only statistical information was preserved, not the complete table of greatest prime factors.

D. S.

1. DANIEL SHANKS, "A sieve method for factoring numbers of the form $n^2 + 1$," *MTAC*, v. 13, 1959, pp. 78–86.
2. DANIEL SHANKS, "On the conjecture of Hardy & Littlewood concerning the number of primes of the form $n^2 + a$," *Math. Comp.*, v. 14, 1960, pp. 321–332.
3. DANIEL SHANKS, "Supplementary data and remarks concerning a Hardy-Littlewood conjecture," *Math. Comp.*, v. 17, 1963, pp. 188–193.

13 [9].—J. D. SWIFT, *Table of Carmichael Numbers to* 10^9, University of California at Los Angeles, ms. of 20 pages, $8\frac{1}{2}'' \times 11''$, deposited in the UMT file.

A Carmichael number, CN, is a composite number n such that $a^{n-1} \equiv 1 \pmod{n}$ whenever $(a, n) = 1$. Carmichael numbers are starred in Poulet's table [1] of pseudoprimes less than 10^8. The present table corrects that table and extends the range to 10^9. The CN's are given with their prime factors.

Calculations were performed on a CDC 1604 made available by IDA, in Princeton. The computer programs used depended explicitly on congruence properties of CN's with respect to their component primes rather than on the pseudoprimality with respect to any particular base. A different routine was run for each possible number of primes occurring in the factorization, from 3 (the absolute minimum) to 6 (the effective maximum defined by the upper limit of the table).

For example, consider $n = p_1p_2p_3 = (r_1 + 1)(r_2 + 1)(r_3 + 1)$. The basic criteria are that $r_i \mid p_jp_k - 1$ where i, j, k is a permutation of 1, 2, 3. For a fixed choice of p_1 (assuming $p_1 < p_2 < p_3$), bounds on the limits of the calculation are obtained. In this simplest case an explicit bound is available:

$$p_1p_2p_3 \leqslant (p_1^6 + 2p_1^5 - p_1^4 - p_1^3 + 2p_1^2 - p_1)/2;$$

and this is actually a CN for $p_1 = 3, 5, 31, \cdots$ (?).

The total number of CN's less than or equal to each power of 10 is as follows:

x	CN(x)	ratio
10^4	7	
		2.3
10^5	16	
		2.7
10^6	43	
		2.4
10^7	105	
		2.4
10^8	255	
		2.5
10^9	646	

The known results thus appear to suggest an asymptotic relation for CN(x) as of the order of $Cx^{0.4}$, which is much smaller than has been conjectured by Erdös [2]. In this connection, the change from 2.43 to 2.53 in the ratios of the last orders of magnitude computed may be significant.

AUTHOR'S SUMMARY

1. P. POULET, "Table des nombres composés vérifiant le théorème de Fermat pour le module 2 jusqu'à 100.000.000," *Sphinx*, v. 8, 1938, pp. 42–52. For corrections see *Math. Comp.*, v. 25, 1971, pp. 944–945, MTE 485; v. 26, 1972, p. 814, MTE 497.

2. P. ERDÖS, "On pseudoprimes and Carmichael numbers," *Publ. Math. Debrecen*, v. 4, 1956, pp. 201–206.

14 [9]—H. C. WILLIAMS & C. R. ZARNKE, *A Table of Fundamental Units for Cubic Fields*, Scientific Report 63, University of Manitoba, Winnipeg, January 1973.

Table 1 gives the fundamental unit $\epsilon_0 = (U + V\rho + W\rho^2)/T$ for all irreducible cubics $\rho^3 = Q\rho + N$ having $|Q|$, $N \leqslant 50$ and a discriminant $D < 0$. Table 3 gives ϵ_0 for $\rho^3 = A\rho^2 + B\rho + C$ with A, $|B|$, $|C| \leqslant 10$ and $D < 0$. For $D > 0$ there are two fundamental units and Tables 2 and 4 give both of them for the same range of Q, N and $A, B, C,$ respectively.

These are the most extensive tables of cubic units known to me although for special types, such as cyclic or pure cubic fields, units have been computed that are not included here.

No attempt is made here to identify different Q, N or A, B, C that give the same field. That would be a valuable addition, especially if it gave the transformation taking one ρ into another.

D. S.

15 [9].—KENNETH S. WILLIAMS & BARRY LOWE, *Table of Solutions* (x, u, v, w) *of the Diophantine System* $16p = x^2 + 50u^2 + 50v^2 + 125w^2, xw = v^2 - 4uv - u^2,$ $x \equiv 1 \pmod 5$ *for Primes* $p < 10000, p \equiv 1 \pmod 5$, Carleton University, Ottawa, 1974, manuscript of 13 pages deposited in the UMT file.

The authors tabulate the values (x, u, v, w) of one of the four solutions of the Diophantine system in the title for all primes $p \equiv 1 \pmod{10}$ less than 10000, the remaining three solutions being $(x, -u, -v, w)$, $(x, v, -u, -w)$, and $(x, -v, u, -w)$. These solutions are obtained from the coefficients of the Jacobi function of order five which have been tabulated by Tanner [1] for $p < 10000$. Two errors in Tanner's tables are noted and one in earlier tables.

A derivation of the well-known linear relationship between these coefficients (which are in fact Jacobsthal sums) and the solutions x, u, v, w is also given.

It should be pointed out that Joseph Muskat has obtained the solutions (x, u, v, w) a number of years ago for all $p \equiv 1 \pmod{10}$ for $p < 50000$ from the corresponding values of the cyclotomic numbers of order five, which he also tabulated together with the ratios of x^2, $50u^2$, $50v^2$, $125w^2$ to $16p$. This comprises 26 computer sheets and could probably still be obtained from the author.

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1. H. W. LLOYD TANNER, *Proc. London Math. Soc.*, v. 18, 1886–1887, pp. 214–234; v. 24, 1892–1893, pp. 223–272.