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On Lower and Upper Bounds of the Difference Between the Arithmetic and the Geometric Mean

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Abstract. Lower and upper bounds of the difference between the arithmetic and the geometric mean of n quantities are given here in terms of n, the smallest value a and the largest value A of given n quantities. Also, an upper bound for the difference, independent of n, is given in terms of a and A. All the bounds obtained are sharp.

1. Introduction. Let $a_1, \ldots, a_n$ be n quantities such that $0 < a \equiv a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n \equiv A$. Let A_n be their arithmetic mean, G_n their geometric mean. Trivial lower and upper bounds of the difference $A_n - G_n$ are 0 and $A - a$ respectively. A nice upper bound has been obtained in [2]. Here we shall prove the following inequalities:

$$n^{-1}(\sqrt{A} - \sqrt{a})^2 \leqslant A_n - G_n \leqslant ca + (1 - c)A - a^c A^{1-c}$$

where

$$c = \frac{\log[(A/(A - a))\log A/a]}{\log A/a}.$$

The inequalities give lower and upper bounds of the difference $A_n - G_n$ in terms of the smallest value a and the largest value A of the given n quantities. Instead of a discrete method, a continuous and analytic approach is used to obtain the inequalities.

2. Lower Bounds. We consider the lower bound of n quantities $0 < a \equiv a_1 \leqslant \cdots \leqslant a_{k-1} \leqslant a_{k+1} \leqslant \cdots \leqslant a_n \equiv A$ with $a_k \equiv x$, $1 < k < n$, to be a variable in the interval $[a, A]$. Let the arithmetic and the geometric means of the fixed $n - 1$ quantities be A_{n-1} and G_{n-1}, respectively. Then

$$A_n - G_n = n^{-1}\{(n - 1)A_{n-1} + x\} - \{G_{n-1}^{n-1}x\}^{1/n} \equiv D_n(x).$$

Since $D_n'(x) = 0$ at $x = G_{n-1}$, the lower bound of $D_n(x)$ for x in the interval $[a, A]$ is

$$D_n(G_{n-1}) = ((n - 1)/n)(A_{n-1} - G_{n-1}).$$

This result can also be found in [1, p. 12], but the method used here seems to be simpler and more straightforward. By repeating this process, we have

$$A_n - G_n \geqslant \frac{n-1}{n}(A_{n-1} - G_{n-1}) \geqslant \frac{n-1}{n}\frac{n-2}{n-1}(A_{n-2} - G_{n-2}) \geqslant \cdots$$

$$\geqslant \frac{2}{n}(A_2 - G_2) = \frac{2}{n}\left(\frac{a + A}{2} - \sqrt{aA}\right) = \frac{1}{n}(\sqrt{A} - \sqrt{a})^2.$$

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Equality holds only if $a_2 = \cdots = a_{n-1} = \sqrt{a_1 a_n} = \sqrt{aA}$.

3. Upper Bounds. Now we investigate the upper bound of $A_n - G_n$. The maximum of $D_n(x)$ on $[a, A]$ is attained at the endpoint a or A. Thus, the maximum of $A_n - G_n$ of n quantities is attained when $a \equiv a_1 = \cdots = a_k \leqslant a_{k+1} = \cdots = a_n \equiv A$ for some k, $1 < k < n$, with the form

$$\frac{ka + (n-k)A}{n} - \{a^k A^{n-k}\}^{1/n} = a\left[\frac{k + (n-k)A/a}{n} - \{A/a\}^{(n-k)/n}\right].$$

For the sake of simplicity, let $a = 1$ and consider the function

$$D(x) = \frac{x + (n-x)A}{n} - A^{(n-x)/n}.$$

Through straight calculation, we have $D'(x) = 0$ for $x = cn$, where

$$c = \frac{\log[(A/(A-1))\log A]}{\log A}.$$

Thus, the upper bound for $A_n - G_n$ is $D(cn) = c + (1-c)A - A^{1-c} \equiv U$ which is independent of n. By repeated application of L'Hospital's rule, we have

$$\lim_{A\to 1} c = \frac{1}{2} \quad \text{and} \quad \lim_{A\to\infty} c = 0.$$

The upper bound is attained only at its limiting case $A \longrightarrow 1$ or $n \longrightarrow \infty$. But this is the best possible bound independent of the positive integer $n \geqslant 2$. For a fixed n, the sharp upper bound of $A_n - G_n$ is attained by $D(k_n)$ with $k_n = [cn]$ or $[cn] + 1$, where $[cn]$ denotes the largest integer not greater than cn. Therefore, we have lower and upper bounds of $A_n - G_n$, both dependent and independent of n, in terms of $a = 1$ and A as follows:

$$0 \leqslant n^{-1}(\sqrt{A} - 1)^2 \leqslant A_n - G_n \leqslant D(k_n) \leqslant c + (1-c)A - A^{1-c}.$$

Some numerical data are shown below.

TABLE ($a = 1$)

A	c	k_{10}	$D(k_{10})$	U
1	0.5		0	0
1.001	0.499925	5	1.25×10^{-7}	1.25×10^{-7}
1.1	0.496029	5	1.191152×10^{-3}	1.191227×10^{-3}
2	0.471234	5	0.085786	0.086071
e	0.458675	5	0.210420	0.211867
5	0.434331	4	0.773472	0.777337
10	0.407973	4	2.418928	2.419591
100	0.333805	3	45.18114	45.45570
10^5	0.212238	2	7.00002×10^4	7.00906×10^4
10^{10}	0.136222	1	8.00000×10^9	8.20349×10^9

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2. C. LOEWNER & H. B. MANN, "On the difference between the geometric and the arithmetic mean of n quantities," *Advances in Math.*, v. 5, 1971, pp. 472–473. MR 43 #4982.