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# Triples of Sixth Powers With Equal Sums

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**Abstract.** The diophantine equation $x^6 + y^6 + z^6 = u^6 + v^6 + w^6$ is shown to have a two-parameter solution which is homogeneous of degree four. The solution also satisfies $x^2 + y^2 + z^2 = u^2 + v^2 + w^2$; and in addition, $3x + y + z = 3u + v + w$.

The diophantine equation

$$x^6 + y^6 + z^6 = u^6 + v^6 + w^6 \tag{1}$$

is a particular instance of the much-studied problem of finding equal sums of like powers of integers, surveyed by Lander, Parkin and Selfridge in 1967 [4]. The smallest nontrivial solution was published by Subba Rao in 1934 [5], namely, $3^6 + 19^6 + 22^6 = 10^6 + 15^6 + 23^6$. Early editions of Hardy and Wright [3] referred to this result as "an isolated curiosity". However, Lander, Parkin and Selfridge [4] discovered that (1) has ten primitive solutions in the range up to $2.5 \times 10^{14}$, and that all but one of these also satisfy

$$x^2 + y^2 + z^2 = u^2 + v^2 + w^2. \tag{2}$$

In [1] it was shown that there are infinitely many primitive solutions to (1), each also satisfying (2) and

$$v = y - z, \qquad w = y + z. \tag{3}$$

Subsequently, in [2] the complete solution to (1), (2) and (3) was obtained in terms of an infinite cyclic group of rational points on a cubic curve. (Regrettably, the solution $5P$ appeared in [2] with transcription errors in the values of $x$ and $w$; it should read $x = 165809277507$, $y = 151561337462$, $z = 23038103009$, $u = 63175337782$, $v = 128523234453$ and $w = 174599440471$.)

The principal aim of this paper is to exhibit the following explicit solution to (1) in terms of parameters $m$, $n$:

$$\begin{aligned} x &= 2m^4 + 4m^3n - 5m^2n^2 - 12mn^3 - 9n^4, \\ y &= 3m^4 + 9m^3n + 18m^2n^2 + 21mn^3 + 9n^4, \\ z &= -m^4 - 10m^3n - 17m^2n^2 - 12mn^3, \\ u &= m^4 - 3m^3n - 14m^2n^2 - 15mn^3 - 9n^4, \\ v &= 3m^4 + 8m^3n + 9m^2n^2, \\ w &= 2m^4 + 12m^3n + 19m^2n^2 + 18mn^3 + 9n^4. \end{aligned} \tag{4}$$

This solution also satisfies (2); and in addition,

$$3x + y + z = 3u + v + w. \tag{5}$$

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TABLE 1

| $m$ | $n$ | $m'$ | $n'$ | $x$ | $y$ | $z$ | $u$ | $v$ | $w$ | $d$ | $d'$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 1 | −1 | −1 | 1 | 0 | −1 | 0 | 1 | 9 | 4 |
| 1 | 0 | 3 | −1 | 2 | 3 | −1 | 1 | 3 | 2 | 1 | 36 |
| 1 | 1 | 3 | −2 | −1 | 3 | −2 | −2 | 1 | 3 | 20 | 45 |
| 2 | −1 | 3 | 1 | −1 | 3 | 4 | 1 | 4 | −3 | 5 | 180 |
| 1 | −2 | 3 | −5 | −74 | 33 | 47 | −73 | 23 | 54 | 1 | 36 |
| 1 | 2 | 9 | −7 | −50 | 81 | −37 | −65 | 11 | 78 | 5 | 180 |
| 2 | 1 | 9 | −5 | 11 | 243 | −188 | −103 | 148 | 249 | 1 | 36 |
| 1 | −3 | 3 | −4 | −23 | 15 | 10 | −22 | 3 | 19 | 20 | 45 |
| 1 | 3 | 6 | −5 | −271 | 372 | −127 | −317 | 27 | 356 | 4 | 9 |
| 2 | −3 | 3 | −7 | −65 | 15 | 52 | −67 | 36 | 37 | 5 | 180 |
| 4 | −1 | 9 | −1 | 43 | 81 | 32 | 55 | 80 | −3 | 5 | 180 |
| 1 | −4 | 9 | −11 | −326 | 243 | 107 | −311 | 23 | 282 | 5 | 180 |
| 3 | 2 | 5 | −3 | −26 | 225 | −169 | −121 | 111 | 230 | 9 | 4 |
| 5 | −1 | 6 | −1 | 169 | 276 | 65 | 179 | 275 | 36 | 4 | 9 |
| 5 | 1 | 9 | −4 | 389 | 891 | −590 | −46 | 775 | 831 | 4 | 9 |
| 4 | −3 | 3 | 5 | −409 | 93 | 512 | −293 | 528 | −271 | 1 | 36 |
| 5 | −2 | 9 | 1 | 86 | 729 | 655 | 431 | 775 | −426 | 1 | 36 |
| 6 | 1 | 7 | −3 | 71 | 147 | −92 | 1 | 132 | 133 | 45 | 20 |
| 7 | −1 | 9 | −2 | 163 | 243 | 14 | 142 | 245 | 75 | 20 | 45 |

Of the ten smallest primitive solutions to (1), listed in [4], all but the sixth satisfy (2). Only the second satisfies (3), while (4) gives rise to all except the seventh (and, of course, the sixth).

It should be noted that a particular choice of $m$ and $n$ does not necessarily yield a *primitive* solution in (4), even if $m$ and $n$ are coprime. Indeed, suppose $(m, n) = 1$ and $d = (x, y, z, u, v, w)$. It is not difficult to prove that (i) $2|d$ just if $m \equiv n \pmod 2$, and then $2^2 \| d$; (ii) $3|d$ just if $m \equiv 0 \pmod 3$, and then $3^2 \| d$; and (iii) $5|d$ just if $m \equiv n$ or $2m \equiv n \pmod 5$, and then $5^1 \| d$. Moreover, suppose $p|d$ for some prime $p > 5$. Clearly, $p \nmid mn$, so $v \equiv 0 \pmod p$ yields $3m^2 + 8mn + 9n^2 \equiv 0 \pmod p$. With $y - v \equiv 0 \pmod p$ this leads to $10(m + 3n) \equiv 0 \pmod p$, and finally with $z \equiv 0 \pmod p$ this yelds $72n^4 \equiv 0 \pmod p$, which is impossible. Hence, $d$ has no prime factor greater than 5.

Consider the transformation to (4) which results from replacing $m, n$ by $m', n'$ satisfying

$$m' : n' = -3(m + n) : (m + 3n). \tag{6}$$

If $x, y, z, u, v, w$ is the solution corresponding to $m, n$ and $x', y', z', u', v', w'$ is the solution corresponding to $m', n'$, then

$$x' : y' : z' : u' : v' : w' = u : v : w : x : y : z. \tag{7}$$

It follows that any particular primitive solution to (1), (2) and (5) obtained from (4) actually arises from two distinct ratios $m:n$.

Next, we remark that any solution to (1), (2) and (5) has an interesting geometrical interpretation. The points $(x, y, z)$ and $(u, v, w)$ in $E^3$ are lattice points which simultaneously lie on a sphere $X^2 + Y^2 + Z^2 = a$, a concentric closed surface $X^6 + Y^6 + Z^6 = b$, and a double cone with vertex at the origin and axis in the direction $3:1:1$. It is intriguing to speculate that the solutions might turn out to have some physical interpretation.

Finally, in Table 1 are listed all primitive solutions obtained from (4) with the property that $\max\{|x|, |y|, |z|\} < 10^3$. As remarked earlier, these include all but two of the numerical examples given in Table IX of [4].

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