A Rational Approximation to Weierstrass' $\wp$ -Function

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Abstract. A rational approximation to Weierstrass' $\wp$ -function in the equianharmonic case for unit period parallelogram is given. With a third-degree numerator polynomial and a fourth-degree denominator polynomial the maximal error for $|z| < 1/\sqrt{3}$ becomes $3 \cdot 10^{-14}$.

The approximation of $\wp(z)$ is then used to calculate a rational approximation to $\wp'(z)$ together with an error bound.

Weierstrass' elliptic functions are frequently used in mathematical physics ([2], [6], [7], [8], [12], [13]). A very interesting new field of application is given in the theory of type-II superconductors [4] and of superfluid helium II [11]. In contrast to other special functions of mathematical physics, however, there are only very few approximations of these functions available. The classical way is to compute them by Laurent series or ϑ -series. Recently, Emersleben [5] gave approximations of generalized zeta functions and Southard [9], [10] published economized polynomial representations.

In this paper, a rational approximation to Weierstrass' $\wp$ -function for the equianharmonic case is presented. This approximation was computed by a method for evaluating lattice sums in solid state physics. In [4], Weierstrass' zeta function was approximated in exactly the same manner.

In the equianharmonic case, the period lattice of Weierstrass' elliptic functions is given in the following way: With the complex numbers

$$(1) \quad r_j = e^{j\pi i/3}, \quad j = 0, 1, \dots,$$

the lattice points are defined by

$$(2) \quad w = w_{mn} = mr_1 + nr_5, \quad m, n = 0, 1, 2, \dots$$

In the more mathematical literature on Weierstrass' elliptic functions, the r_j are pre-multiplied by a factor $3.059908074 \dots$ (see [10, 18.13.15]). In view of the physical applications we prefer to have $|r_j| = 1$.

For any complex z , Weierstrass' $\wp$ -function is defined as follows:

$$(3) \quad \wp(z) = \frac{1}{z^2} + \sum_{w \neq 0} \left[\frac{1}{(z-w)^2} - \frac{1}{w^2} \right].$$

This double sum is unconditionally convergent. We prescribe a fixed succession of summation by renumbering the w_{mn} .

Let

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$$(4) \quad w = w_{n,k,j} = nr_j + kr_{j+2},$$

$$n = 1, 2, 3, \dots, \quad k = 1, 2, \dots, n, \quad j = 0, 1, 2, 3, 4, 5.$$

We denote

$$(5) \quad \sum_{n=\alpha}^{\beta} {}^{(6)} = \sum_{n=\alpha}^{\beta} \sum_{k=1}^n \sum_{j=0}^5,$$

i.e. we are summing over concentric hexagonal shells. This results in the following simpler conditionally convergent series:

$$(6) \quad \wp(z) = \frac{1}{z^2} + \sum_{n=1}^{\infty} {}^{(6)} \frac{1}{(z-w)^2}.$$

We split this series into three parts

$$(7a) \quad \wp(z) = \frac{1}{z^2} + 6 \cdot z^4 \cdot \frac{5 + z^6}{(1 - z^6)^2}$$

$$(7b) \quad + \sum_{n=2}^N {}^{(6)} \frac{1}{(z-w)^2}$$

$$(7c) \quad + \sum_{n=N+1}^{\infty} {}^{(6)} \frac{1}{(z-w)^2}.$$

(7.b) is a rational function:

$$(8) \quad \sum_{n=2}^N {}^{(6)} \frac{1}{(z-w)^2} = 6 \cdot z^4 \cdot \tilde{P}(z)/\tilde{Q}(z);$$

$\tilde{P}$ and $\tilde{Q}$ are polynomials whose coefficients can be calculated conveniently and safely (e.g. PL1-FORMAC rational arithmetic).

First, we examine (7.c).

$$(9) \quad \begin{aligned} \sum_{n=N+1}^{\infty} {}^{(6)} \frac{1}{(z-w)^2} &= - \sum_{n=N+1}^{\infty} {}^{(6)} \frac{d}{dz} \frac{1}{z-w} \\ &= - \sum_{n=N+1}^{\infty} \sum_{k=1}^n \frac{d}{dz} \frac{6z^5}{z^6 - (n + kr_2)^6} \\ &= - \sum_{n=N+1}^{\infty} \frac{d}{dz} \left[\frac{6z^5}{n^6} \cdot \frac{1}{\left(1 + \frac{k}{n}r_2\right)^6} \cdot \sum_{\nu=0}^{\alpha-1} \frac{(z/n)^{6\nu}}{\left(1 + \frac{k}{n}r_2\right)^{6\nu}} \right. \\ &\quad \left. + \frac{6z^5}{n^6} \cdot \frac{(z/n)^{6\alpha}}{\left(1 + \frac{k}{n}r_2\right)^{6\alpha}} \cdot \frac{1}{\left(1 + \frac{k}{n}r_2\right)^6 - (z/n)^6} \right] \\ &= \sum_{\nu=0}^{\alpha-1} (36\nu + 30)z^{6\nu+4} \sum_{n=N+1}^{\infty} \frac{1}{n^{6\nu+5}} \frac{1}{n} \sum_{k=1}^n \frac{1}{\left(1 + \frac{k}{n}r_2\right)^{6\nu+6}} + R_N^{(\alpha)}, \end{aligned}$$

where

$$(10) \quad R_N^{(\alpha)} = \sum_{n=N+1}^{\infty} \sum_{k=1}^n \left[\frac{1}{n^{6\alpha+6}} \frac{1}{\left(1 + \frac{k}{n} r_2\right)^{6\alpha}} \frac{(36\alpha + 30) z^{6\alpha+4}}{\left(1 + \frac{k}{n} r_2\right)^6 - \left(\frac{z}{n}\right)^6} \right. \\ \left. + \frac{1}{n^{6\alpha+12}} \frac{1}{\left(1 + \frac{k}{n} r_2\right)^{6\alpha}} \frac{36z^{6\alpha+10}}{\left[\left(1 + \frac{k}{n} r_2\right)^6 - \left(\frac{z}{n}\right)^6\right]^2} \right].$$

For $k = 1, 2, \dots, n$ is

$$(11) \quad \frac{\sqrt{3}}{2} \leq \left| 1 + \frac{k}{n} r_2 \right| \leq 1.$$

We confine ourselves to argument values z with

$$(12) \quad |z| \leq (N+1)/2$$

and define

$$(13) \quad \sigma_N^{(k)} := \sum_{n=N+1}^{\infty} \frac{1}{n^k} = \zeta_0(k) - \sum_{n=1}^N \frac{1}{n^k},$$

$\zeta_0(k)$ being the Riemann zeta function. This gives the following estimate for $R_N^{(\alpha)}$:

$$(14) \quad |R_N^{(\alpha)}| \leq (36\alpha + 30) \left(\frac{64}{27}\right)^{\alpha} \frac{64}{26} \sigma_N^{(6\alpha+5)} |z|^{6\alpha+4} \\ + 36 \left(\frac{64}{27}\right)^{\alpha} \frac{64^2}{26^2} \sigma_N^{(6\alpha+11)} |z|^{6\alpha+10}.$$

In (9), the following sum appears:

$$(15) \quad S_n^{(\nu)} = \frac{1}{n} \sum_{k=1}^n \frac{1}{\left(1 + \frac{k}{n} r_2\right)^{6\nu}} = \frac{1}{n} \sum_{k=0}^n{}' \frac{1}{\left(1 + \frac{k}{n} r_2\right)^{6\nu}}.$$

The prime indicates the first and the last term of the sum being multiplied by a factor $\frac{1}{2}$. Denoting by

$$(16) \quad f_{\nu}(t) = (1 + t \cdot r_2)^{-6\nu},$$

$S_n^{(\nu)}$ is exactly the trapezoid approximation for the integral

$$(17) \quad I_{\nu} = \int_0^1 f_{\nu}(t) dt = -\frac{1}{6\nu-1}, \quad \nu = 1, 2, \dots$$

$S_n^{(\nu)}$ can be expressed by I_{ν} by means of the Euler-Maclaurin formula [1, 23.1.30]. In order to do so we denote by B_{2m} the $2m$ th Bernoulli number and

$$(18) \quad \Delta_k^{(\nu)} = f_{\nu}^{(2k-1)}(1) - f_{\nu}^{(2k-1)}(0), \quad k = 1, 2, \dots$$

Then

$$(19) \quad S_n^{(\nu)} = I_{\nu} + \sum_{k=1}^{m_{\nu}-1} \frac{B_{2k}}{(2k)!} \frac{1}{n^{2k}} \Delta_k^{(\nu)} \\ + \frac{B_{2m_{\nu}}}{(2m_{\nu})!} \frac{1}{n^{2m_{\nu}}} \frac{1}{n} \sum_{k=0}^{n-1} f_{\nu}^{(2m_{\nu})} \left((1 + \theta) \frac{k}{n} \right), \quad 0 < \theta < 1.$$

Abbreviating

$$(20) \quad c_m = \frac{B_{2m}}{(2m)!}, \quad \gamma_m^{(\nu)} = \max_{0 \leq t \leq 1} |f_v^{(2m)}(t)|,$$

we get

$$(21) \quad \left| \frac{B_{2m_\nu}}{(2m_\nu)!} \frac{1}{n} \sum_{k=0}^{n-1} f_v^{(2m_\nu)} \left((1 + \theta) \frac{k}{n} \right) \right| \leq c_{m_\nu} \cdot \gamma_{m_\nu}^{(\nu)}.$$

(9), (13) and (19) yield

$$(22) \quad \sum_{n=N+1}^{\infty} \frac{1}{(z-w)^2} = \sum_{\nu=0}^{\alpha-1} (36\nu + 30) z^{6\nu+4} \sum_{k=1}^{m_\nu-1} \frac{B_{2k}}{(2k)!} \Delta_k^{(\nu+1)} \sigma_N^{(6\nu+2k+5)} \\ - 6 \sum_{\nu=0}^{\alpha-1} \sigma_N^{(6\nu+5)} z^{6\nu+4} + \rho_{N,m_\nu}^{(\alpha)}.$$

The error $\rho_{N,m_\nu}^{(\alpha)}$ can be estimated by means of (14), (19) and (21) as follows:

$$(23) \quad |\rho_{N,m}^{(\alpha)}| \leq \sum_{\nu=0}^{\alpha-1} (36\nu + 30) c_{m_\nu} \gamma_{m_\nu}^{(\nu)} \sigma_N^{(6\nu+2m_\nu+5)} |z|^{6\nu+4} \\ + (36\alpha + 30) \left(\frac{64}{27} \right)^\alpha \frac{64}{26} \sigma_N^{(6\alpha+5)} |z|^{6\alpha+4} \\ + 36 \left(\frac{64}{27} \right)^\alpha \frac{64^2}{26^2} \sigma_N^{(6\alpha+11)} |z|^{6\alpha+10}.$$

For $\alpha = 1$ we have in particular

$$(24) \quad \sum_{n=N+1}^{\infty} \frac{1}{(z-w)^2} = 30 z^4 \left[-0.2 \sigma_N^{(5)} - 0.5 \sigma_N^{(7)} - \frac{14}{15} \sigma_N^{(9)} \right. \\ \left. - \sigma_N^{(11)} + 3.3 \sigma_N^{(13)} + \frac{91}{3} \sigma_N^{(15)} + \frac{2764}{30} \sigma_N^{(17)} + \dots \right] \\ + \rho_{N,m}^{(1)}$$

and

$$(25) \quad |\rho_{N,m}^{(1)}| \leq 30 \cdot c_m \gamma_m^{(1)} \cdot \sigma_N^{(2m+5)} |z|^4 \\ + 66 \cdot \frac{64^2}{26 \cdot 27} \cdot \sigma_N^{(11)} \cdot |z|^{10} + 36 \cdot \frac{64^3}{27 \cdot 26^2} \sigma_N^{(17)} |z|^{16} \\ \leq 30 \cdot c_m \gamma_m^{(1)} \sigma_N^{(2m+5)} |z|^4 + 385.095 \cdot \sigma_N^{(11)} \cdot |z|^{10} \\ + 517.050 \cdot \sigma_N^{(17)} |z|^{16}.$$

Tables for c_m , $\gamma_m^{(1)}$ and $\sigma_N^{(k)}$ can be found in [4].

In order to calculate $\wp(z)$ efficiently by the approximation given here, one takes

advantage of the fact that this function is doubly periodic, i.e.

$$(26) \quad \wp(z + r_j) = \wp(z).$$

Thus, it is sufficient to know $\wp(z)$ in the unit circle. Using all symmetries of the $\wp$ -function in the equianharmonic case, it suffices to know $\wp(z)$ for $|z| \leq 1/\sqrt{3}$ (see [10], 18.2.22 to 18.2.23 and 18.13.1 to 18.13.8]). It depends, however, on the computer used, whether this latter reduction will be economical.

Having bounded the argument in this manner, one is able to reduce the polynomials $\tilde{P}$ and $\tilde{Q}$, as defined in (8). If we put for abbreviation

$$(27) \quad K_{N,\nu}^{(\alpha)} = (6\nu + 5) \cdot \sum_{k=1}^{m_\nu-1} \frac{B_{2k}}{(2k)!} \cdot \Delta_k^{(\nu+1)} \cdot \sigma_N^{(6\nu+2k+5)} - \sigma_N^{(6\nu+5)},$$

we have

$$(28) \quad \wp(z) = \frac{1}{z^2} + 6z^4 \cdot \frac{5 + z^6}{(1 - z^6)^2} + 6z^4 \cdot \frac{\tilde{P}(z)}{\tilde{Q}(z)} + 6z^4 \cdot \sum_{\nu=0}^{\alpha-1} K_{N,\nu}^{(\alpha)} \cdot z^{6\nu}.$$

Let

$$(29) \quad \begin{aligned} \hat{P}(z) &= \tilde{P}(z) + \tilde{Q}(z) \cdot \sum_{\nu=0}^{\alpha-1} K_{N,\nu}^{(\alpha)} \cdot z^{6\nu} =: \sum \alpha_j z^{6j}, \\ \tilde{Q}(z) &=: \sum \beta_j z^{6j}. \end{aligned}$$

With increasing j the coefficients α_j and β_j become quickly small in modulus. Therefore, we choose integers k_N and k_D and define the truncated polynomials

$$(30) \quad P(z) = \sum_{j=0}^{k_N} \alpha_j z^{6j}, \quad Q(z) = \sum_{j=0}^{k_D} \beta_j z^{6j}.$$

The truncation error

$$(31) \quad \delta(z) = \hat{P}(z)/\tilde{Q}(z) - P(z)/Q(z)$$

can easily be estimated. For the approximation

$$(32) \quad \wp(z) \approx \frac{1}{z^2} + 6z^4 \cdot \frac{5 + z^6}{(1 - z^6)^2} + 6z^4 \cdot \frac{P(z)}{Q(z)}$$

we thus have an error of

$$(33) \quad \epsilon(z) = \rho_{N,m}^{(\alpha)} + 6z^4 \cdot \delta(z).$$

To find an approximation to $\wp(z)$ one first has to decide whether one wishes to confine to $|z| \leq 1$ or $|z| \leq 1/\sqrt{3}$. After that, N is fixed so that the desired accuracy is achieved with a minimum of computer time. In most cases $\alpha = 1$ is preferable. m is chosen in such a way that the error in (19) has nearly the same order of magnitude as $R_N^{(\alpha)}$. Finally, k_N and k_D are chosen such that $6z^4\delta(z)$ has the same order of magnitude as $\rho_{N,m}^{(\alpha)}$.

TABLE I

*The coefficients α_j and β_j of the representation (32)
for $N = 20$, $\alpha = 1$, $m = 4$, and $j = 0, \dots, 9$.*

j	α_j	β_j
0	- 0.11414 02554 78832 25212	1.00000 00000 00000 00000
1	0.01246 21470 34158 86306	0.04565 55348 48201 06097
2	0.00007 33190 74452 31537	- 0.00056 44480 18476 46390
3	0.00000 12310 17921 46078	- 0.00002 61983 29204 21111
4	0.00000 00026 84924 69697	0.00000 02553 31457 29808
5	- 0.00000 00000 44046 10396	0.00000 00006 22521 91170
6	0.00000 00000 00135 87306	0.00000 00000 04512 35409
7	0.00000 00000 00000 21151	- 0.00000 00000 00000 97714
8	0.00000 00000 00000 00002	0.00000 00000 00000 00184
9	---	- 0.00000 00000 00000 00004

TABLE II

*The coefficients $\alpha_j^{(1)}$ and $\beta_j^{(1)}$ of the representation (37)
for $N = 20$, $\alpha = 1$, $m = 4$, $j = 0, \dots, 10$.*

j	$\alpha_j^{(1)}$	$\beta_j^{(1)}$
0	- 0.45656 10219 15329 00848	1.00000 00000 00000 00000
1	0.13504 37391 64781 42656	0.09131 10696 96402 12194
2	0.00293 35592 15775 43401	0.00095 55318 25322 37367
3	0.00003 27611 02419 05813	- 0.00010 39370 10763 52269
4	0.00000 40036 54572 23584	- 0.00000 15629 32983 73684
5	- 0.00000 00446 07821 71793	0.00000 00541 34822 32985
6	- 0.00000 00004 92565 29248	0.00000 00004 63977 63268
7	- 0.00000 00000 02669 74929	- 0.00000 00000 13671 20407
8	0.00000 00000 00007 09648	0.00000 00000 00027 39656
9	- 0.00000 00000 00000 27512	0.00000 00000 00000 08265
10	0.00000 00000 00000 00024	0.00000 00000 00000 00274

All numbers occurring in the calculations can be computed very accurately. $\sigma_N^{(k)}$ is calculated according to (13) by using Table 23.3 in [1]. This gives a maximum accuracy of about 10^{-20} . For greater accuracy Table 33 of [3] is preferable.

As an example, an approximation of the $\wp$ -function for $N = 20$, $\alpha = 1$ and $m = 4$ was computed. The author is very thankful to Mrs. I. Oellers for excellently written programs. All computations were performed on the IBM 360/67 of the Central Institute for Applied Mathematics of the Nuclear Research Center, Jülich. The coefficients α_j and β_j according to (29) and (30) are listed in Table I. For $|z| \leq 1$ is $|\rho_{20,4}^{(1)}| \leq 3.050 \cdot 10^{-12}$. Choosing $k_N = k_D = 5$, one has

$$(34) \quad |\epsilon(z)| \leq 7.678 \cdot 10^{-12}.$$

For $|z| \leq 1/\sqrt{3}$, $k_N = 3$, $k_D = 4$ is

$$(35) \quad |\epsilon(z)| \leq 3.131 \cdot 10^{-14}.$$

This approximation compares very well with Southard's approximation [9] and [10, 18.13.67].

By differentiating all relevant formulas, one can derive very directly approximations and error estimates for derivatives of Weierstrass' $\wp$ -function. We sketch this procedure for the first derivative. Putting (9) into (7) and differentiating, we have as an estimate for the error

$$\left| \frac{d}{dz} R_N^{(\alpha)} \right| + \sum_{\nu=0}^{\alpha-1} (6\nu + 4) (36\nu + 30) \cdot c_{m\nu} \gamma_{m\nu}^{(\nu)} \cdot \sigma_N^{(6\nu+2m\nu+5)} \cdot |z|^{6\nu+3};$$

in particular, for $\alpha = 1$ the error is less than

$$(36) \quad 120 c_m \gamma_m^{(1)} \sigma_N^{(2m+5)} |z|^3 + 3.851 \cdot 10^3 \sigma_N^{(11)} |z|^9 + 8.28 \cdot 10^3 \sigma_N^{(17)} |z|^{15}.$$

With $\hat{P}$ and $\tilde{Q}$ from (29) the corresponding approximation is

$$(37) \quad \begin{aligned} \wp'(z) &\approx -\frac{2}{z^3} + 12z^3 \cdot \frac{10 + 25z^6 + z^{12}}{(1 - z^6)^3} \\ &+ 6z^3 \cdot \frac{4 \cdot \hat{P} \cdot \tilde{Q} + z \cdot \hat{P}' \cdot \tilde{Q} - z \cdot \hat{P} \cdot \tilde{Q}'}{\tilde{Q}^2} \\ &= -\frac{2}{z^3} + 12z^3 \cdot \frac{10 + 25z^6 + z^{12}}{(1 - z^6)^3} + 6z^3 \cdot \frac{P_1(z)}{Q_1(z)}. \end{aligned}$$

The coefficients of $P_1(z) = \sum \alpha_j^{(1)} \cdot z^{6j}$ and $Q_1(z) = \sum \beta_j^{(1)} \cdot z^{6j}$ can be computed easily using the coefficients α_j and β_j . For $N = 20$, $\alpha = 1$ and $m = 4$ they are given in Table II. The corresponding error estimates are:

For $|z| \leq 1$, $k_N = k_D = 6$ is

$$|\epsilon(z)| \leq 1.067 \cdot 10^{-10},$$

for $|z| \leq 1/\sqrt{3}$, $k_N = k_D = 4$ we have

$$|\epsilon(z)| \leq 3.244 \cdot 10^{-13}.$$

FIGURE 1. *The error in approximating $\wp(z)$ on the real axis for*

$$N = 20, \alpha = 1, m = 4.$$

- (1) Bound for $\epsilon(x)$ according to (25) and (33) for $k_N = k_D = 5$,
- (2) Error for $k_N = k_D = 5$,
- (3) Error for $k_N = 3, k_D = 4$; see (35),
- (4) Error for $k_N = k_D = 10$.

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