Positivity of the Weights of Extended Gauss-Legendre Quadrature Rules

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Abstract. We show that the weights of extended Gauss-Legendre quadrature rules are all positive.

1. Introduction. We consider extended Gauss-Legendre quadrature formulas, i.e., integration rules of the type

$$(1) \quad \int_{-1}^1 f(x) dx = \sum_{i=1}^n A_i^{(n)} f(\xi_i^{(n)}) + \sum_{j=1}^{n+1} B_j^{(n)} f(x_j^{(n)}) + R_n(f),$$

where $\xi_i^{(n)}$, $i = 1, \dots, n$, are the zeros of the n th degree Legendre polynomial $P_n(x)$, while the nodes $x_j^{(n)}$, $j = 1, 2, \dots, n+1$, and the weights $A_i^{(n)}$, $B_j^{(n)}$ are chosen so that (1) has degree of exactness $p = 3n + 1$ ($3n + 2$ if n is odd), i.e., $R_n(f) = 0$ whenever f is a polynomial of degree up to p . If we denote by $E_{n+1}(x)$ the polynomial of degree $n + 1$, whose zeros are the abscissas $x_j^{(n)}$, $j = 1, 2, \dots, n + 1$, then $E_{n+1}(x)$ has to satisfy the following orthogonality relation

$$\int_{-1}^1 P_n(x) E_{n+1}(x) x^k dx = 0, \quad k = 0, 1, \dots, n.$$

Szegő [4] has studied $E_{n+1}(x)$ in a different context and gives some very interesting results. For instance, he proves that the nodes $x_j^{(n)}$ are in $(-1, 1)$ and interlace with the zeros of $P_n(x)$.

Formulas for the computation of the weights $A_i^{(n)}$ and $B_j^{(n)}$ are given in [2], [3]. In [2] it is shown that the $B_j^{(n)}$'s are positive; however, nothing has been said about the sign of $A_i^{(n)}$. In this note we show that the weights $A_i^{(n)}$ are also positive.

2. Positivity of $A_i^{(n)}$. We consider the Legendre function of second kind

$$(2) \quad Q_n(x) = \frac{1}{2} \int_{-1}^1 \frac{P_n(t)}{x-t} dt, \quad n \geq 1,$$

defined for any x in the complex plane cut along the segment $[-1, 1]$; we introduce the function

$$(3) \quad \bar{Q}_n(x) = \frac{1}{2} \lim_{\epsilon \rightarrow +0} [Q_n(x + i\epsilon) + Q_n(x - i\epsilon)],$$

which is analytic on $(-1, 1)$. It is known [5, p. 78] that

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$$(4) \quad \lim_{\epsilon \rightarrow +0} [Q_n(x + i\epsilon) - Q_n(x - i\epsilon)] = -i\pi P_n(x), \quad -1 < x < 1.$$

From (2), (3) and (4), and recalling Lebesgue's convergence theorem, it then follows that at the zeros $\xi_i^{(n)}$, $i = 1, \dots, n$, of $P_n(x)$ we have

$$(5) \quad \bar{Q}_n(\xi_i^{(n)}) = \frac{1}{2} \int_{-1}^1 \frac{P_n(t)}{\xi_i^{(n)} - t} dt.$$

Let now

$$E_{n+1}(\cos \phi) = \lambda_0 \cos(n+1)\phi + \lambda_1 \cos(n-1)\phi + \dots + \begin{cases} \lambda_{n/2} \cos \phi, & n \text{ even,} \\ \frac{1}{2}\lambda_{(n+1)/2}, & n \text{ odd,} \end{cases}$$

and

$$e_{n+1}(\phi) = \lambda_0 \sin(n+1)\phi + \lambda_1 \sin(n-1)\phi + \dots + \begin{cases} \lambda_{n/2} \sin \phi, & n \text{ even,} \\ 0, & n \text{ odd,} \end{cases}$$

where $x = \cos \phi$, $0 < \phi < \pi$, and, as known [4], $\lambda_0 = (2n+1)!/(2^{2n}(n!)^2)$. Then, Szegő in his paper [4, p. 507] gives the following inequality

$$(6) \quad \bar{Q}_n(\cos \phi)E_{n+1}(\cos \phi) + \frac{\pi}{2} P_n(\cos \phi)e_{n+1}(\phi) > 1, \quad 0 < \phi < \pi,$$

which implies that at the nodes $\xi_i^{(n)}$

$$(7) \quad |E_{n+1}(\xi_i^{(n)})| > |\bar{Q}_n(\xi_i^{(n)})|^{-1}, \quad i = 1, \dots, n.$$

We are now ready to prove the following

THEOREM. *The weights $A_i^{(n)}$ and $B_j^{(n)}$ of the extended Gauss-Legendre rules are always positive.*

Proof. The positivity of $B_j^{(n)}$ has already been proved in [2]. In that paper, the following expression for the weights $A_i^{(n)}$ has also been given

$$(8) \quad A_i^{(n)} = H_i^{(n)} - \frac{h_n}{k_n |P'_n(\xi_i^{(n)})| |q_{n+1}(\xi_i^{(n)})|}, \quad i = 1, \dots, n,$$

where $H_i^{(n)} = 2|\bar{Q}_n(\xi_i^{(n)})|/|P'_n(\xi_i^{(n)})|$ are the weights of the n -point Gauss-Legendre rule, $h_n = 2/(2n+1)$, $k_n = (2n)!/(2^n(n!)^2)$ and $q_{n+1}(x) = 1/(2^n\lambda_0)E_{n+1}(x)$. Recalling (7), from (8) we have

$$A_i^{(n)} > H_i^{(n)} \left(1 - \frac{h_n}{k_n} 2^{n-1}\lambda_0\right) = 0,$$

which proves the theorem.

What follows is an immediate consequence (see for example [5, Theorem 15.2.2]) of the theorem we have just proved.

COROLLARY. *The quadrature process defined by (1) is convergent for every function $f(x)$ which is Riemann-integrable in $[-1, 1]$, i.e., $\lim_{n \rightarrow \infty} R_n(f) = 0$.*

Remark. In his paper, Szegő derives, although not explicitly stated, the analogue of (6) for rules of type (1) with a weight function of the form $(1-x^2)^{\mu-1/2}$, when

$0 < \mu < 1$. In a way very similar to the Legendre case, it may then be shown that the weights of that type of rules are positive, too. For $\mu = 0, 1$ see [2].

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