The Condition of Polynomials in Power Form*

By Walter Gautschi

Abstract. A study is made of the numerical condition of the coordinate map M_n which associates to each polynomial of degree $\leq n-1$ on the compact interval $[a, b]$ the n -vector of its coefficients with respect to the power basis. It is shown that the condition number $\|M_n\|_\infty \|M_n^{-1}\|_\infty$ increases at an exponential rate if the interval $[a, b]$ is symmetric or on one side of the origin, the rate of growth being at least equal to $1 + \sqrt{2}$. In the more difficult case of an asymmetric interval around the origin we obtain upper bounds for the condition number which also grow exponentially.

1. Introduction. Let $M_n: \mathbf{R}^n \rightarrow \mathbf{P}_{n-1}$ be the linear map associating to each vector $u^T = [u_1, u_2, \dots, u_n] \in \mathbf{R}^n$ the polynomial

$$p(x) = \sum_{k=1}^n u_k x^{k-1} \in \mathbf{P}_{n-1}, \quad n \geq 2.$$

For any $p \in \mathbf{P}_{n-1}$ we shall write $u_p = M_n^{-1}p$, where M_n^{-1} is the inverse map of M_n . We define the *condition* of the map M_n , relative to the compact interval $[a, b]$, by

$$(1.1) \quad \text{cond}_\infty M_n = \|M_n\|_\infty \|M_n^{-1}\|_\infty,$$

where the norms are $\|u\|_\infty = \max_{1 \leq k \leq n} |u_k|$ (in $\mathbf{R}^n$) and $\|p\|_\infty = \max_{a \leq x \leq b} |p(x)|$ (in $\mathbf{P}_{n-1}[a, b]$). We are interested in the growth rate of $\text{cond}_\infty M_n$ as $n \rightarrow \infty$, and how this growth depends on the particular interval $[a, b]$ chosen.

The answer is relatively straightforward for symmetric intervals $[-\omega, \omega]$ and for intervals $[a, b]$ with $0 \leq a < b$, in which cases the condition number in (1.1) can be expressed explicitly in terms of $u_{T_{n-1}}$ (or $u_{T_{n-2}}$), where T_m denotes the Chebyshev polynomial of degree m on the appropriate interval (Theorems 3.1, 3.2). It will follow, in particular, that on $[-\omega, \omega]$ and $[0, \omega]$, $\omega > 0$, the condition grows exponentially with n , and that the minimum growth occurs precisely when $\omega = 1$, in which case $\text{cond}_\infty M_n$ grows like $(1 + \sqrt{2})^n$ on $[-1, 1]$ and like $(1 + \sqrt{2})^{2n}$ on $[0, 1]$. This ought to be contrasted with the linear growth $\sqrt{2}n$ for the condition on $[-1, 1]$ of polynomials represented in terms of Chebyshev polynomials [1].

For asymmetric intervals $[a, b]$ with, say, $a < 0 < b$, $|a| < b$, the problem appears to be considerably more complex, and we are no longer able to ascertain the exact growth rate of (1.1). Instead, we obtain two upper bounds for $\text{cond}_\infty M_n$, one being asymptotically sharp in the extreme case $|a| = b$, the other in the extreme case $a = 0$ (Theorem 4.1).

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2. Preliminaries on the Coefficients of Chebyshev Polynomials. In the following we need estimates for the largest coefficients in $T_n(x/\omega)$ and $T_n^*(x/\omega)$, where T_n is the Chebyshev polynomial of the first kind and T_n^* the "shifted" Chebyshev polynomial $T_n^*(x) = T_n(2x - 1)$.

It is well known that

$$(2.1) \quad T_n\left(\frac{x}{\omega}\right) = \sum_{k=0}^{\lfloor n/2 \rfloor} c_k x^{n-2k},$$

where

$$c_k = (-1)^k \frac{n}{2} \frac{(n-k-1)!}{k!(n-2k)!} \left(\frac{2}{\omega}\right)^{n-2k}, \quad 0 \leq k \leq \lfloor n/2 \rfloor.$$

For fixed t , with $0 < t < 1/2$, we put $k = tn$, and let $n \rightarrow \infty$. Using Stirling's formula, we find

$$|c_{tn}| \sim \frac{n^{-1/2}}{2\sqrt{2\pi}} \frac{1}{\sqrt{t(1-t)}(1-2t)} \left(\frac{2}{\omega}\right)^n e^{ng(t)}, \quad n \rightarrow \infty,$$

where

$$g(t) = (1-t) \ln(1-t) - t \ln t - (1-2t) \ln(1-2t) - 2t \ln(2/\omega), \quad 0 < t < 1/2.$$

From $g(0) = 0$, $g(1/2) = -\ln(2/\omega)$, $g'(t) = \ln[(1-2t)^2 \omega^2 / 4t(1-t)]$, it is seen that $g(t)$ has a unique maximum on $[0, 1/2]$, assumed at

$$t = t_0 = \frac{1}{2} \left(1 - \frac{1}{\sqrt{1 + \omega^2}} \right).$$

Since

$$g(t_0) = \ln \frac{1-t_0}{1-2t_0} = \ln [1/2(1 + \sqrt{1 + \omega^2})], \quad \sqrt{t_0(1-t_0)}(1-2t_0) = 1/2 \omega (1 + \omega^2)^{-3/4},$$

we thus find for the maximum coefficient of $T_n(x/\omega)$ the asymptotic approximation

$$(2.2) \quad \|u_{T_n(x/\omega)}\|_\infty \sim \frac{1}{\sqrt{2\pi}} \frac{(1 + \omega^2)^{3/4}}{\omega} n^{-1/2} \left(\frac{1 + \sqrt{1 + \omega^2}}{\omega} \right)^n, \quad n \rightarrow \infty.$$

For $\omega = 1$, this gives

$$(2.2') \quad \|u_{T_n}\|_\infty \sim \frac{2^{1/4}}{\sqrt{\pi}} n^{-1/2} (1 + \sqrt{2})^n, \quad n \rightarrow \infty \quad (\omega = 1),$$

which agrees with a result attributed to an (anonymous) referee in J. R. Rice [3, p. 304].

Since $T_n^*(x^2) = T_{2n}(x)$, the analogous result for $T_n^*(x/\omega)$ is readily obtained from (2.2) by replacing n by $2n$ and ω by $\sqrt{\omega}$,

$$(2.3) \quad \|u_{T_n^*(x/\omega)}\|_\infty \sim \frac{1}{2\sqrt{\pi}} \frac{(1 + \omega)^{3/4}}{\sqrt{\omega}} n^{-1/2} \left(\frac{2 + \omega + 2\sqrt{1 + \omega}}{\omega} \right)^n, \quad n \rightarrow \infty.$$

For $\omega = 1$, this gives

$$(2.3') \quad \|u_{T_n^*}\|_\infty \sim \frac{2^{-1/4}}{\sqrt{\pi}} n^{-1/2} (3 + 2\sqrt{2})^n, \quad n \rightarrow \infty \quad (\omega = 1).$$

In Table 2.1 we compare the true values of $\|u_{T_n(x/\omega)}\|_\infty$ with their asymptotic approximations in (2.2) for selected values of n and ω .

ω	$n = 5$		$n = 10$		$n = 20$		$n = 40$	
	true	(2.2)	true	(2.2)	true	(2.2)	true	(2.2)
10	5.00(-1)	9.36(-1)	1.00	1.09	2.00	2.09	1.06(1)	1.09(1)
5	1.00	1.11	2.00	2.12	1.06(1)	1.09(1)	4.02(2)	4.11(2)
1	2.00(1)	2.46(1)	1.28(3)	1.43(3)	6.55(6)	6.79(6)	2.12(14)	2.17(14)
.2	5.00(4)	9.65(4)	5.00(9)	7.17(9)	5.00(19)	5.59(19)	5.00(39)	4.82(39)
.1	1.60(6)	5.82(6)	5.12(12)	1.33(13)	5.24(25)	9.91(25)	5.50(51)	7.72(51)

TABLE 2.1. The quality of the asymptotic formula (2.2)

We also note that

$$(2.4) \quad \|u_{T_n(x/\omega)}\|_\infty \geq \|u_{T_{n-1}(x/\omega)}\|_\infty, \quad n = 1, 2, 3, \dots, \omega \leq 1,$$

where equality holds only for $n = 1$, $\omega = 1$. This follows easily from the three-term recurrence relation for Chebyshev polynomials and from the alternating character of the coefficients c_k in (2.1). The inequality in (2.4) holds for all $\omega \leq 2$, if n is restricted to $n \geq 2$, and it indeed holds for any fixed ω , if n is sufficiently large, as is seen from (2.2).

3. The Condition of M_n for Symmetric Intervals and for Intervals on One Side of the Origin. We shall always assume (without loss of generality) that our basic interval $[a, b]$ is centered to the right of the origin, so that $0 \leq |a| \leq b$. The Chebyshev polynomial T_m , adjusted to the interval $[a, b]$, will be denoted by $T_m[a, b]$,

$$T_m[a, b](x) = T_m\left(\frac{2x - a - b}{b - a}\right), \quad a \leq x \leq b.$$

Relative to any such interval $[a, b]$, the norm of the map M_n is easily seen to be

$$(3.1) \quad \|M_n\|_\infty = \sum_{k=1}^n b^{k-1} = \begin{cases} \frac{b^n - 1}{b - 1}, & b \neq 1, \\ n, & b = 1. \end{cases}$$

More delicate is the determination of $\|M_n^{-1}\|_\infty$, as this amounts to finding the norms of the linear functionals $\lambda_k: p \mapsto p^{(k-1)}(0)/(k-1)!$, $p \in \mathbf{P}_{n-1}[a, b]$, $k = 1, 2, \dots, n$.

Indeed,

$$(3.2) \quad \|M_n^{-1}\|_\infty = \max_{1 \leq k \leq n} \|\lambda_k\|_\infty.$$

While it is known [5, Satz 6.11] that, for $2 \leq k \leq n$, the extremal in $\mathbf{P}_{n-1}[a, b]$ for

the functional λ_k is a Zolotarev polynomial of degree $n - 1$, it appears difficult, in the case of a general interval $[a, b]$, to pinpoint the parameter involved in the Zolotarev polynomial, and there may correspond different Zolotarev polynomials to different values of k . For these reasons the case of an arbitrary interval will be dealt with by other (less sophisticated and cruder) methods in Section 4.

For symmetric intervals $[-\omega, \omega]$, $\omega > 0$, on the other hand, the appropriate Zolotarev polynomials are known to be the Chebyshev polynomials T_{n-1} or T_{n-2} ; indeed, $\|\lambda_k\|_\infty = |T_{n-1}^{(k-1)}[-\omega, \omega](0) + T_{n-2}^{(k-1)}[-\omega, \omega](0)|/(k-1)!$, $k = 1, 2, \dots, n$, $n \geq 2$ [5, p. 167], and therefore,

$$\max_{1 \leq k \leq n} \|\lambda_k\|_\infty = \|u_{T_{n-1}[-\omega, \omega] + T_{n-2}[-\omega, \omega]}\|_\infty.$$

Since $T_n[-\omega, \omega](x) = T_n(x/\omega)$, and T_m is an even or odd polynomial, depending on the parity of m , we thus have, in view of (3.1), (3.2):

THEOREM 3.1. *The condition number (1.1) on $[-\omega, \omega]$ is given by*

$$(3.3) \quad \text{cond}_\infty M_n = \frac{\omega^n - 1}{\omega - 1} \max \{ \|u_{T_{n-1}(x/\omega)}\|_\infty, \|u_{T_{n-2}(x/\omega)}\|_\infty \},$$

where $(\omega^n - 1)/(\omega - 1)$ (here and in the sequel) is to be interpreted as having the value n if $\omega = 1$.

It follows from (2.2) that for $\omega > 1$, $\omega = 1$, $0 < \omega < 1$, the condition of M_n for large n grows, respectively, like $(1 + \sqrt{1 + \omega^2})^n$, $(1 + \sqrt{2})^n$, $[(1 + \sqrt{1 + \omega^2})/\omega]^n$ (disregarding a factor $n^{\pm 1/2}$ and constant factors), so that the growth is smallest, asymptotically, when $\omega = 1$. Selected numerical values of $\text{cond } M_n$ are shown in Table 3.1.

ω	$n = 5$	$n = 10$	$n = 20$	$n = 40$
10	1.11(4)	1.11(9)	2.11(19)	1.10(40)
5	7.81(2)	4.39(6)	2.17(14)	7.74(29)
1	4.00(1)	5.76(3)	5.45(7)	3.51(15)
.2	6.25(3)	6.25(8)	6.25(18)	6.25(38)
.1	8.89(4)	2.84(11)	2.91(24)	3.05(50)

TABLE 3.1. The condition of M_n on $[-\omega, \omega]$

Another special case which can be disposed of similarly is the case of an interval $[a, b]$ with $0 \leq a < b$. Here (see, e.g., [4, p. 93]) $\|\lambda_k\|_\infty = |T_{n-1}^{(k-1)}[a, b](0)|/(k-1)!$, and we can state

THEOREM 3.2. *The condition number (1.1) on $[a, b]$, where $0 \leq a < b$, is given by*

$$(3.4) \quad \text{cond}_\infty M_n = \frac{b^n - 1}{b - 1} \|u_{T_{n-1}[a, b]}\|_\infty.$$

We note that the expression on the right of (3.4), even for an arbitrary interval $[a, b]$, is always a lower bound for $\text{cond}_\infty M_n$, since

$$(3.5) \quad \|M_n^{-1}\|_\infty = \sup_{p \in \mathcal{P}_{n-1}[a, b]} \frac{\|M_n^{-1}p\|_\infty}{\|p\|_\infty} \geq \|u_{T_{n-1}[a, b]}\|_\infty.$$

To illustrate Theorem 3.2, we consider the interval $[0, \omega]$, $\omega > 0$. Here, $T_{n-1}[0, \omega](x) = T_{n-1}^*(x/\omega)$, and depending on whether $\omega > 1$, $\omega = 1$, or $0 < \omega < 1$, Eq. (2.3) shows that the condition grows, respectively, like $(2 + \omega + 2\sqrt{1 + \omega})^n$, $(3 + 2\sqrt{2})^n$ and $[(2 + \omega + 2\sqrt{1 + \omega})/\omega]^n$, thus again slowest, asymptotically, when $\omega = 1$. Selected numerical values are shown in Table 3.2.

ω	$n = 5$	$n = 10$	$n = 20$	$n = 40$
10	3.56(4)	4.93(10)	1.80(23)	3.27(48)
5	5.00(3)	8.91(8)	3.67(19)	8.47(40)
1	1.28(3)	1.12(7)	7.34(14)	2.16(30)
.2	1.00(5)	3.20(11)	6.23(24)	3.02(51)
.1	1.42(6)	1.46(14)	1.53(30)	3.27(62)

TABLE 3.2. The condition of M_n on $[0, \omega]$

4. The Condition of M_n on an Arbitrary Interval. We now wish to make some progress towards the more difficult problem of estimating $\text{cond}_\infty M_n$ for an arbitrary right-centered interval $[a, b]$, $0 \leq |a| \leq b$. We content ourselves with establishing upper bounds for $\text{cond}_\infty M_n$. (A trivial, but not very useful, lower bound can be had from (3.1) and (3.5).)

Our main tool is the following simple observation.

LEMMA 4.1. Let $s^T = [s_1, s_2, \dots, s_n]$ be any vector of n distinct nodes in $[a, b]$ and $V_n(s)$ the corresponding Vandermonde matrix

$$(4.1) \quad V_n(s) = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ s_1 & s_2 & \cdots & s_n \\ \vdots & \vdots & \ddots & \vdots \\ s_1^{n-1} & s_2^{n-1} & \cdots & s_n^{n-1} \end{bmatrix} \quad (a \leq s_\nu \leq b, \nu = 1, 2, \dots, n).$$

Then

$$(4.2) \quad \|M_n^{-1}\|_\infty \leq n \|V_n^{-1}(s)\|_\infty.$$

Proof. Let

$$p(x) = \sum_{k=1}^n u_k x^{k-1}, \quad a \leq x \leq b,$$

be an arbitrary polynomial of degree $\leq n - 1$. From

$$\sum_{k=1}^n s_\nu^{k-1} u_k = p(s_\nu), \quad \nu = 1, 2, \dots, n,$$

or, equivalently,

$$V_n^T(s)u = \pi, \quad u^T = [u_1, u_2, \dots, u_n], \quad \pi^T = [p(s_1), p(s_2), \dots, p(s_n)],$$

one gets immediately

$$\|u\|_\infty \leq \|u\|_1 \leq \| [V_n^{-1}(s)]^T \|_1 \|\pi\|_1 \leq n \|V_n^{-1}(s)\|_\infty \|\pi\|_\infty \leq n \|V_n^{-1}(s)\|_\infty \|p\|_\infty,$$

hence (4.2). $\square$

It is tempting to optimize the bound in (4.2) by minimizing $\|V_n^{-1}(s)\|_\infty$ over all admissible node vectors s . Unfortunately, the corresponding optimal nodes are not known explicitly. We expect, however, the Chebyshev points on $[a, b]$ to provide a reasonably good alternative. In order to carry out the necessary computations, we need the following properties of Vandermonde matrices.

LEMMA 4.2 (SHIFT PROPERTY). *Let $t = [t_1, t_2, \dots, t_n]^T$ and $t - \mu = [t_1 - \mu, t_2 - \mu, \dots, t_n - \mu]^T$. Then*

$$(4.3) \quad V_n^{-1}(t - \mu) = V_n^{-1}(t)(D_n^{-1}P_nD_n)^T,$$

where $D_n = \text{diag}(1, \mu, \mu^2, \dots, \mu^{n-1})$ and P_n is the initial $(n \times n)$ -segment of the Pascal triangle, that is

$$(4.4) \quad D_n^{-1}P_nD_n = \begin{bmatrix} 1 & \mu & \mu^2 & \mu^3 & \dots \\ 0 & 1 & \binom{2}{1}\mu & \binom{3}{2}\mu^2 & \dots \\ 0 & 0 & 1 & \binom{3}{1}\mu & \dots \\ 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}_{(n \times n)}.$$

Proof. It is well known (see, e.g., [2]) that $V_n^{-1}(t) = [u_{\kappa\lambda}]$, where

$$\prod_{\substack{\nu=1 \\ \nu \neq \kappa}}^n \frac{x - t_\nu}{t_\kappa - t_\nu} \equiv \sum_{\lambda=1}^n u_{\kappa\lambda} x^{\lambda-1}.$$

The elements $u'_{\kappa\lambda}$ of $V_n^{-1}(t - \mu)$, therefore, are the coefficients of the polynomial

$$\begin{aligned} \prod_{\nu \neq \kappa} \frac{x + \mu - t_\nu}{t_\kappa - t_\nu} &= \sum_{\rho=1}^n u_{\kappa\rho} (x + \mu)^{\rho-1} = \sum_{\rho=1}^n u_{\kappa\rho} \sum_{\lambda=1}^{\rho} \binom{\rho-1}{\lambda-1} x^{\lambda-1} \mu^{\rho-\lambda} \\ &= \sum_{\lambda=1}^n x^{\lambda-1} \sum_{\rho=\lambda}^n u_{\kappa\rho} \binom{\rho-1}{\lambda-1} \mu^{\rho-\lambda}, \end{aligned}$$

that is,

$$u'_{\kappa\lambda} = \sum_{\rho=\lambda}^n u_{\kappa\rho} \binom{\rho-1}{\lambda-1} \mu^{\rho-\lambda}.$$

This, written in matrix form, is precisely (4.3). $\square$

In the following two lemmas,

$$\cos \theta_\nu, \quad \theta_\nu = \frac{2\nu-1}{2n} \pi, \quad \nu = 1, 2, \dots, n,$$

denote the Chebyshev points on $[-1, 1]$.

LEMMA 4.3. *If $t_\nu = \tau \cos \theta_\nu$, $\nu = 1, 2, \dots, n$, $\tau > 0$, then*

$$(4.5) \quad n \|V_n^{-1}(t)\|_\infty \leq \frac{3^{3/4}}{4(\sqrt{2}-1)} (\tau+1) \left| T_n\left(\frac{i}{\tau}\right) \right| \quad (i = \sqrt{-1}).$$

Proof. From [2, Theorem 5.2]** one has

$$n \|V_n^{-1}(t)\|_\infty \leq \frac{(\tau+1)n}{2(\sqrt{2}-1)} \left| \frac{T_n(i/\tau)}{T_n(i)} \right| \left\| V_n^{-1}\left(\frac{1}{\tau} t\right) \right\|_\infty,$$

and from [2, Example 6.2]

$$n \left\| V_n^{-1}\left(\frac{1}{\tau} t\right) \right\|_\infty \leq \frac{3^{3/4}}{2} |T_n(i)|.$$

LEMMA 4.4. *If $t_\nu = \tau(1 + \cos \theta_\nu)$, $\nu = 1, 2, \dots, n$, $\tau > 0$, then*

$$(4.6) \quad n \|V_n^{-1}(t)\|_\infty \leq \frac{\tau}{\sqrt{1+2\tau}} T_n\left(\frac{1}{\tau} + 1\right).$$

Proof. From [2, Eq. (4.1')] one obtains

$$(4.7) \quad n \|V_n^{-1}(t)\|_\infty \leq \frac{T_n(1/\tau + 1)}{\min_{1 \leq \nu \leq n} \left\{ \frac{1/\tau + 1 + \cos \theta_\nu}{\sin \theta_\nu} \right\}},$$

having used $|T'_n(\cos \theta_\nu)| = n/\sin \theta_\nu$. An elementary calculation will show that

$$f(\theta) = \frac{1/\tau + 1 + \cos \theta}{\sin \theta}$$

has a unique minimum on $0 < \theta < \pi$ at $\theta = \theta_0$, where $\cos \theta_0 = -\tau/(\tau+1)$. Thus

$$\min_{0 < \theta < \pi} f(\theta) = \frac{1/\tau + 1 - \tau/(\tau+1)}{\sqrt{1 - \tau^2/(\tau+1)^2}} = \frac{1}{\tau} \sqrt{1+2\tau},$$

from which (4.6) follows by virtue of (4.7). $\square$

Now the Chebyshev points on $[a, b]$ are given by

$$(4.8) \quad s_\nu = \frac{a+b}{2} + \frac{b-a}{2} \cos \theta_\nu = a + \frac{b-a}{2} (1 + \cos \theta_\nu), \quad \nu = 1, 2, \dots, n.$$

Each of these two representations suggests an application of the shift property in Lemma 4.2, the first with $t_\nu = \tau \cos \theta_\nu$, $\mu = -(a+b)/2$, the second with $t_\nu = \tau(1 + \cos \theta_\nu)$, $\mu = -a$, where $\tau = (b-a)/2$ in both. Observing also that

$$\|V_n^{-1}(t - \mu)\|_\infty \leq \|V_n^{-1}(t)\|_\infty \|D_n^{-1} P_n D_n\|_1 = (1 + |\mu|)^{n-1} \|V_n^{-1}(t)\|_\infty,$$

**Theorem 5.2 in [2] is stated for n even; the same theorem, however, also holds if n is odd.

and using Lemmas 4.3 and 4.4 to estimate $\|V_n^{-1}(t)\|_\infty$, we can easily estimate $\|V_n^{-1}(s)\|_\infty$ for the nodes in (4.8), hence $\|M_n^{-1}\|_\infty$ by Lemma 4.1, and finally $\text{cond}_\infty M_n$, using (3.1). The result is stated as

THEOREM 4.1. *The condition number (1.1) on $[a, b]$, where $0 \leq |a| \leq b$, satisfies the inequality*

$$(4.9) \quad \text{cond}_\infty M_n \leq \frac{3^{3/4}}{4(\sqrt{2}-1)} \frac{2+b-a}{2+b+a} \frac{b^n-1}{b-1} \left(1 + \frac{b+a}{2}\right)^n \left|T_n\left(\frac{2i}{b-a}\right)\right|,$$

as well as the inequality

$$(4.10) \quad \text{cond}_\infty M_n \leq \frac{b-a}{2(1+|a|\sqrt{1+b-a})} \frac{b^n-1}{b-1} (1+|a|)^n T_n\left(\frac{2}{b-a} + 1\right).$$

Theorem 4.1 holds for arbitrary intervals $[a, b]$, subject to $|a| \leq b$, but is of interest only in the case $a \leq 0 < b$ of an interval containing the origin. It will be useful to characterize such an interval by its "degree of asymmetry"

$$\alpha = (b+a)/(b-a), \quad 0 \leq \alpha \leq 1,$$

and its half-width

$$\tau = (b-a)/2,$$

in terms of which $b = (1+\alpha)\tau$, $a = -(1-\alpha)\tau$.

We first examine the extreme cases $\alpha = 0$ (perfect symmetry) and $\alpha = 1$ (perfect asymmetry), typified by the intervals $[-\omega, \omega]$ and $[0, \omega]$, $\omega > 0$. In the first case, by virtue of

$$2 \left|T_n\left(\frac{i}{\omega}\right)\right| = \left(\frac{1+\sqrt{1+\omega^2}}{\omega}\right)^n + \left(\frac{1-\sqrt{1+\omega^2}}{\omega}\right)^n \sim \left(\frac{1+\sqrt{1+\omega^2}}{\omega}\right)^n, \quad n \rightarrow \infty,$$

we find that the bound in (4.9) has the correct exponential growth rate as $n \rightarrow \infty$, which can be obtained from (3.3) and (2.2), while the bound in (4.10) grows at a larger exponential rate. (We say here that a sequence $\{c_n\}$ has exponential growth rate γ if $|c_{n+1}/c_n| \sim \gamma$ as $n \rightarrow \infty$.) The reverse is true in the second case, as can be seen from

$$\begin{aligned} 2T_n\left(\frac{2}{\omega} + 1\right) &= \left(\frac{2+\omega+2\sqrt{1+\omega}}{\omega}\right)^n + \left(\frac{2+\omega-2\sqrt{1+\omega}}{\omega}\right)^n \\ &\sim \left(\frac{2+\omega+2\sqrt{1+\omega}}{\omega}\right)^n, \quad n \rightarrow \infty, \end{aligned}$$

and comparison with (3.4), (2.3). We, therefore, expect (4.9) to be sharper than (4.10) if the interval $[a, b]$ is more nearly symmetric (i.e., α small), and (4.10) better than (4.9) for more asymmetric intervals (α close to 1). That this is indeed the case can be seen by forming the ratio ρ of the exponential growth rates in (4.9) and (4.10), and expressing the result in terms of α and τ ,

$$\rho = \frac{1+\alpha\tau}{1+(1-\alpha)\tau} \lambda(\tau), \quad \lambda(\tau) = \frac{1+\sqrt{1+\tau^2}}{1+\tau+\sqrt{1+2\tau}}.$$

One verifies that $\lambda(\tau) < 1$ for all τ , with $\lambda(0) = \lambda(\infty) = 1$, so that $\rho < 1$ certainly if $1 + \alpha\tau < 1 + (1 - \alpha)\tau$, i.e., $\alpha < \frac{1}{2}$. Thus, (4.9) is *asymptotically sharper than* (4.10) *whenever* $\alpha < \frac{1}{2}$. The condition on α is best possible for $\tau \rightarrow \infty$, but too stringent for specific finite values of τ . If $\tau = 1$, e.g., one finds (4.9) better than (4.10) whenever $\alpha < .8216 \dots$, and as $\tau \rightarrow 0$, (4.9) is always better.

We illustrate Theorem 4.1 in Figure 4.1, where we plot the exponential growth rates of the bounds in (4.9) and (4.10) for intervals of fixed half-width $\tau = 1$, and asymmetries α varying from 0 to 1. (The growth rates are $(1 + \alpha)^2(1 + \sqrt{2})$ and $(1 + \alpha)(2 - \alpha)(2 + \sqrt{3})$, respectively.) The true asymptotic growth rate presumably interpolates somehow between the boundary values $1 + \sqrt{2}$ and $2(2 + \sqrt{3})$ (cf. the dashed line in Figure 4.1).

FIGURE 4.1. The asymptotic growth rates of the bounds in (4.9) and (4.10) for $a = -1 + \alpha$, $b = 1 + \alpha$, $0 \leq \alpha \leq 1$.

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Department of Computer Sciences
Purdue University
West Lafayette, Indiana 47907

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