The ν -Zeros of $J_{-\nu}(x)$

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Abstract. We compute the positive ν -zeros of $J_{-\nu}(x)$, regarded as a function of ν . $J_{\nu}(x)$ stands for the Bessel function of first kind of order ν . Some related inequalities are verified and algorithms the computer uses are described briefly.

1. Introduction. In a recent paper D. Naylor [5] has considered an integral transform adapted to the solution of certain boundary value problems connected with the Helmholtz equation in cylindrical or spherical polar coordinates, where the radial variable r varies over some infinite interval $0 \leq a \leq r < \infty$. At infinity a radiation type boundary condition is imposed.

Precisely, suppose that $f(r)$ is twice continuously differentiable for $r \geq a$,

$$r^{-1/2}(rf_{rr} + f_r + k^2rf) \in L(a, \infty), \quad \lim_{r \rightarrow \infty} r^{1/2}f(r)e^{-ikr}$$

exists and $\lim_{r \rightarrow \infty} r^{1/2}(f_r - ikf) = 0$, where k is real and positive. Let

$$(1.1) \quad G(u) = \int_a^\infty [J_u(kr)H_u^{(1)}(ka) - J_u(ka)H_u^{(1)}(kr)] f(r) \frac{dr}{r},$$

where $a > 0$, $k > 0$; then

$$(1.2) \quad f(r) = \frac{1}{2} \int_L \frac{uJ_{-u}(kr)G(u)du}{J_{-u}(ka)} + i\pi \sum_{u=u'_n} \frac{uJ_{-u}(kr)G(u)}{\left(\frac{\partial}{\partial u}\right)J_{-u}(ka)},$$

where L denotes the imaginary axis of the complex u -plane and the summation is extended over all the positive zeros u'_n of the function $J_{-u}(ka)$, regarded as a function of u .

Thus, to apply this interesting transform we need the positive zeros u'_n of the function $J_{-u}(ka)$ [3], [5] and [6] regarded as a function of u . As such zeros are not easily available, in this paper we give ten ν -zeros (for a particular value of the argument) to eight decimal places.

2. Tables. Tables of numerical values of the ν -zeros of $J_{-\nu}(A)$ are to be found in the microfiche supplement of this issue. The ranges for the parameter A are

Received March 2, 1978; revised May 30, 1978.

AMS (MOS) subject classifications (1970). Primary 33A40, 65D20; Secondary 44A20.

Key words and phrases. ν -zeros, series, Bessel functions, asymptotic formula, numerical.

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0025-5718/79/0000-0032/\$02.00

$$\begin{aligned}
A = & 0.001, \quad 0.002, \quad 0.005 \quad (0.005) \quad 0.020, \\
& 0.030 \quad (0.010) \quad 0.100, \quad 0.120, \quad 0.150 \\
& 0.200 \quad (0.050) \quad 1.000, \quad 1.100 \quad (0.100) \quad 4.900, \\
& 5.000 \quad (0.250) \quad 9.750 \quad 10.00 \quad (0.50) \quad 29.50 \\
& 30.00 \quad (1.00) \quad 99.00, \quad 100.0 \quad (5.0) \quad 495.0, \\
& 500.0 \quad (10.0) \quad 990.0, \quad 1000 \quad (100) \quad 1900, \\
& 2000 \quad (200) \quad 4800, \quad 5000 \quad (500) \quad 9500, \\
& 10^4 \quad (10^3) \quad 9 \times 10^4, \quad 10^5 \quad (2 \times 10^4) \quad 2.8 \times 10^5, \\
& 3 \times 10^5 \quad (25 \times 10^3) \quad 10^6.
\end{aligned}$$

We consider the applicability of formulas on p. 506 of [6] or see [2], when x is large $x \gg |\nu|$. Thus, large zeros of $J_{-\nu}(x)$ are

$$(2.1) \quad x \sim \varphi - \frac{4\nu^2 - 1}{8\varphi} - \frac{(4\nu^2 - 1)(28\nu^2 - 31)}{384\varphi^3} - \dots,$$

where $\varphi = (n - (2\nu + 1)/4)\pi$ and n is any integer.

Taking some values of ν from our tables, we calculate approximate x using this formula.

For example:

- (i) $(A = 10) \quad n = 4, \nu = 1.10315023,$
 $\varphi = 10.04814812, x = 10.04814812 - 0.04811536$
 $- 0.00003052, x = 10.00000224 \simeq 10.0000.$
- (ii) $(A = 11) \quad n = 4, \nu = 0.49726122, \varphi = 10.99987635$
 $x = 10.99987635 - 0.00012415 - 0.00000051$
 $x = 10.99999999 \simeq 11.0000.$
- (iii) $(A = 80) \quad n = 26, \nu = 0.57011967, \varphi = 80.00046897,$
 $x = 80.00046897 - 0.00046897 - (-0.00000003),$
 $x = 80.00000003 \simeq 80.0000.$
- (iv) $(A = 1000) \quad n = 321, \nu = 4.87274935,$
 $\varphi = 1000.011747, x = 1000.011747 - 0.01174671$
 $- 0.00000016, x = 999.9999998 \simeq 1000.0000.$

These results reflect a complete agreement with the numerical values of our tables.

Now,

$$(2.2) \quad J_{-\nu}(x) = J_{\nu}(x) \cos \nu\pi - Y_{\nu}(x) \sin \nu\pi.$$

Hence, zeros of $J_{-\nu}(x)$ are given by the above formula. This formula is interesting in

that the zeros of $J_{-\nu}(x)$ are those of $J_{\nu}(x)$, when ν is a positive integer and the zeros of $J_{-\nu}(x)$ are those of $Y_{\nu}(x)$ when ν is half an odd integer.

It has been verified that there can be at most one ν -zero between two consecutive natural numbers. We observe that for large x , two consecutive ν -zeros differ by two, and the large ν -zeros are asymptotic to the positive integers [5].

Also, if ν is a zero of $J_{-\nu}(x)$, then from (2.2) it follows that

$$(2.3) \quad J_{\nu}(x) = Y_{\nu}(x) \tan \nu\pi.$$

3. Naylor's Inequalities. Naylor [5] has proved the following inequalities, when ν is a zero of $J_{-\nu}(ka)$,

$$(3.1) \quad |J_{\nu}(ka)J_{-\nu}(kr)| \leq \left| \frac{\sin \nu\pi}{\nu\pi} \right| \left(\frac{r}{a} \right)^{\nu}, \quad r > a,$$

and

$$(3.2) \quad \left| J_{\nu}(ka) \frac{\partial}{\partial \nu} J_{-\nu}(ka) \right| \geq \frac{|\sin \nu\pi|}{2\pi} (2/ka)^{2\nu} [\Gamma(\nu)]^2.$$

Using the ν -zeros from our tables, we present here some numerical values for the results, (2.3), (3.1) and (3.2). In the following table the left-hand sides of these results are denoted by F_1 , G_1 , and H_1 , where as the right-hand sides are denoted by F_2 , G_2 and H_2 , respectively. Further, $ka = A$ and $kr = R$.

A	NU	F1	F2	R	G1	G2	H1	H2
0.030	0.99977503	0.01501	0.01501	0.040	0.00713	0.00030	1.00011	0.49912
0.120	0.99640645	0.06059	0.06059	0.160	0.00299	0.00540	1.00177	0.49115
1.500	0.53778118	0.64890	0.64890	2.400	0.26138	0.75675	0.79934	0.58720
1.800	4.99991440	0.00429	0.00429	2.800	0.00013	0.00016	0.21536	0.07068
5.000	2.06369324	0.07387	0.07387	10.000	0.01875	0.12817	0.05448	0.00076
7.000	10.99941628	0.00834	0.00834	20.000	0.00051	5.49313	0.11902	0.00412
20.000	2.65903622	-0.15728	-0.15728	30.000	0.01137	0.30884	0.04799	0.00000
23.000	15.42914542	-0.18808	-0.18808	45.000	0.00969	632.43842	0.08346	0.00000
30.000	4.21532712	-0.09165	-0.09165	32.000	0.01190	0.06205	0.02207	0.00000

As one would expect, $F_1 = F_2$, $G_1 \leq G_2$ ($R > A$) and $H_1 \geq H_2$.

4. Algorithms the Computer Uses. We have used our own routine to calculate $J_{\nu}(x)$, and then the false position method (Regula falsi or secant method) is used to determine the ν -zeros. The function $J_{\nu}(x)$ is calculated by two different methods, according to its argument.

For $x < 10$, the basic series representation

$$(4.1) \quad J_{\nu}(x) = \sum_{r=1}^m \frac{(-1)^r (x/2)^{\nu+2r}}{\Gamma(r+1)\Gamma(\nu+r+1)}$$

is used, and the value of m is chosen in such a way that the truncation error comes out to be of order 0.5×10^{-12} . When ν is a negative integer, the relation $J_{-n}(x) =$

$(-1)^n J_n(x)$ is used to avoid the poles of $\Gamma(\nu + r + 1)$.

For $x \geq 10$, we use the following asymptotic formula

$$(4.2) \quad J_\nu(x) = \sqrt{\frac{2}{\pi x}} \left[\cos \left(x - \frac{\nu\pi}{2} - \frac{\pi}{4} \right) \sum_{r=0}^m \frac{(-1)^r \Gamma(\nu, 2r)}{(2x)^{2r}} \right. \\ \left. - \sin \left(x - \frac{\nu\pi}{2} - \frac{\pi}{4} \right) \sum_{r=0}^m \frac{(-1)^r \Gamma(\nu, 2r+1)}{(2x)^{2r+1}} \right]$$

where

$$(4.3) \quad (\nu, k) = \frac{(4\nu^2 - 1)(4\nu^2 - 3^2) \cdots (4\nu^2 - (2k-1)^2)}{2^{2k} k!};$$

and m is determined by the relation $m = 35/\ln|x|$, which leads to a truncation error of order 10^{-30} in both series.

In the calculations ν is considered to be a rational number of the form $u = N/ND$. For this particular case, we set $ND = 10^8$, and determine N with a precision of one unit, which guarantees us at precision of 10^{-8} in the tabulation of ν in the whole interval $0 \leq \nu \leq 25$ (that is, up to the first ten positive zeros).

Suppose A is given. Then by the method of false position $J_{-\nu}(A)$ is calculated to determine ν such that $J_{-\nu}(A) = 0$. For the given value of A , $J_{-\nu}(A)$ is evaluated for $\nu + 0.1 \cdot 10^{-8}$ and $\nu - 0.1 \cdot 10^{-8}$ and a change in sign for $J_{-\nu}(A)$ is noted and by linear interpolation the value of ν is confirmed.

Uniform asymptotic expansions are useful, if both ν and x are large [4].

Acknowledgement. The authors are very grateful to Professor Y. L. Luke for his many suggestions and comments.

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