MATHEMATICS OF COMPUTATION, VOLUME 33, NUMBER 147
JULY 1979, PAGES 891–918

Finite Element Collocation Methods for First Order Systems

By P. Lesaint and P. A. Raviart

Abstract. Finite element methods and the associate collocation methods are considered for solving first-order hyperbolic systems, positive in the sense of Friedrichs. Applied in the case when the meshes are rectangle, those methods lead for example to the successfully used box scheme for the heat equation or D.S.N. scheme for the neutron transport equation. Generalizations of these methods are described here for nonrectangle meshes and (or) noncylindrical domains; stability results and error estimates are derived.

1. Introduction. Let Ω be a bounded domain in the (x, y) plane with boundary Γ. We denote by $n = (n_x, n_y)$ the outward unit vector normal to Γ.

We consider the following problem: Given a vector-valued function $f = (f_1, \ldots, f_p) \in (L^2(\Omega))^p$, find a vector-valued function $u = (u_1, \ldots, u_p)$: $\Omega \longrightarrow \mathbf{R}^p$, which is a solution of the first-order system

$$Lu \equiv A \frac{\partial u}{\partial x} + B \frac{\partial u}{\partial y} + Cu = f \quad \text{in } \Omega, \tag{1.1}$$

with the boundary condition

$$(An_x + Bn_y - M)u = 0 \quad \text{on } \Gamma. \tag{1.2}$$

In (1.1), (1.2), A, B, C and M are $p \times p$ matrix-valued functions. We assume that

(i) the matrices A, B are symmetric,

(ii) the functions $(x, y) \longrightarrow A(x, y), B(x, y)$ belong to $W^{1,\infty}(\Omega; L(\mathbf{R}^p))$, i.e., are Lipschitz-continuous in Ω.

(iii) the function $(x, y) \longrightarrow C(x, y)$ belongs to $L^\infty(\Omega; L(\mathbf{R}^p))$.

Sufficient conditions for the problem (1.1), (1.2) to have a unique strong solution have been obtained by Friedrichs [4]. *In particular*, we shall use the property

$$M + M^* \geqslant 0, \tag{1.3}$$

(where M^* denotes the adjoint of M). In fact, in the sequel, we shall consider only specific examples of problem (1.1), (1.2).

Finite element methods for solving first order systems, symmetric and positive, have been considered by the first author in [9], [10]. He has introduced continuous finite element methods in which the space of trial functions coincides with the space of the test functions and is a finite-dimensional subspace of $(H^1(\Omega))^p$.

Received March 14, 1978.
AMS (MOS) subject classifications (1970). Primary 65N30.

Similarly, discontinuous finite element methods have been introduced in [9], [12]; again the same finite-dimensional space is used for the trial functions and the test functions, but these functions are now discontinuous at the interelement boundaries.

On the other hand, it has been noticed [9], [11] that classical finite-difference methods for the neutron transport equation could be interpreted as finite element methods in which the space of test functions differs from that of trial functions, or equivalently as finite element collocation methods. The same is true of a finite-difference scheme for the heat equation introduced by H. B. Keller [7] in view of boundary layer computations: *the box scheme.*

The purpose of this paper is to provide a fairly general analysis of such methods which are successfully used in practice. An outline of the paper is as follows. Section 2 is devoted to the description of the finite element method and the associated collocation method. In Section 3, we analyze a simple case where the matrices A, B, C have *constant* coefficients and the domain Ω is a *rectangle* but the operator L is otherwise general. The obtained results are applied in Section 4 to the case of conforming and nonconforming finite elements, and the neutron transport equation is considered as a particular example. Some technical preliminary results are given in Section 5 for the case where the matrices A, B and C have *nonconstant* coefficients and these results are applied in Section 6 to the heat equation in a noncylindrical domain, written as a first order system. Stability results and error estimates are derived, hence generalizing the results of Keller [7].

Throughout this paper, we shall make a constant use of the classical Sobolev spaces $H^m(\Omega)$ provided with the norms

$$\|v\|_{m,\Omega} = \left(\sum_{|\alpha|\leqslant m} \int_\Omega |\partial^\alpha v|^2\, dx\right)^{1/2},$$

and the seminorms

$$|v|_{m,\Omega} = \left(\sum_{|\alpha|=m} \int_\Omega |\partial^\alpha v|^2\, dx\right)^{1/2},$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi-index such that $\alpha_i \geqslant 0$, $|\alpha| = \Sigma_{i=1}^n \alpha_i$, and where $\partial^\alpha = (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_n)^{\alpha_n}$.

2. The Finite Element Schemes. Let us define a general (nonconforming) finite element method of approximation of problem (1.1), (1.2). Let $\overline{\Omega} = \bigcup_{K\in T_h} K$ be a decomposition of $\overline{\Omega}$ into closed subsets K with diameters $\leqslant h$ such that:

(i) $\overset{\circ}{K} \neq \varnothing$ for all $K \in T_h$,

(ii) $\overset{\circ}{K}_1 \cap \overset{\circ}{K}_2 = \varnothing$ for any pair of elements $K_1, K_2 \in T_h$.

With each element $K \in T_h$, we associate two finite-dimensional spaces X_K and Y_K of smooth scalar functions defined on K. Next, we are given a finite-dimensional space X_h of functions φ_h whose restrictions $\varphi_{h|K}$ to any element $K \in T_h$ belong to X_K. In the conforming case, we have the inclusion $X_h \subset H^1(\Omega)$, while in the nonconforming case this inclusion does not hold but is replaced by some weaker continuity requirement (see condition (3.3)). We set

(2.1) $$Y_h = \{\varphi_h \in L^2(\Omega);\ \forall K \in \mathcal{T}_h,\ \varphi_{h|K} \in Y_K\}.$$

Let V_h be a subspace of the product space X_h^p which consists of functions $v_h = (v_{h,1}, \ldots, v_{h,p})$ which satisfy the boundary condition (1.2) in some approximate sense. We assume that

(2.2) $$\dim V_h = \dim Y_h^p \quad \left(= \sum_{K \in \mathcal{T}_h} \dim Y_K^p\right);$$

and we consider the following form of the *weighted residual method*: Find a function $u_h \in V_h$ such that

(2.3) $$\forall K \in \mathcal{T}_h,\ \forall v \in Y_K^p,\quad \int_K (Lu_h - f, v)\, dx\, dy = 0,$$

where $(\cdot\,,\cdot)$ denotes the Euclidean inner product of $\mathbf{R}^p$ and $|\cdot|$ the corresponding Euclidean norm.

In all the sequel, we assume that the spaces X_h and Y_h are constructed by means of *quadrilateral* finite elements. In fact, all the examples that we have in mind require the use of such elements. Therefore, we suppose that, for any $K \in \mathcal{T}_h$, there exists a C^1-diffeomorphism F_K from the reference square $\hat{K} = [-1, +1]^2$ in the (ξ, η) plane onto K. We shall make a constant use of the one-to-one correspondences

$$\hat{\varphi} \longmapsto \varphi = \hat{\varphi} \circ F_K^{-1}, \qquad \varphi \longmapsto \hat{\varphi} = \varphi \circ F_K$$

between the functions $\hat{\varphi}$ defined on $\hat{K}$ and the functions φ defined on K.

Let $\hat{X}$ and $\hat{Y}$ be two finite-dimensional spaces such that

$$\hat{X} \subset H^1(\hat{K}), \qquad \hat{Y} \subset L^2(\hat{K});$$

we set for all $K \in \mathcal{T}_h$

(2.4) $$X_K \text{ (resp. } Y_K) = \{\varphi = \hat{\varphi} \circ F_K^{-1},\ \hat{\varphi} \in \hat{X} \text{ (resp. } \hat{Y})\}.$$

In order to be *specific*, we assume in the following that

(2.5) $$P_m \subset \hat{X} \subset Q_m, \qquad \hat{Y} = Q_{m-1}$$

for some integer $m \geqslant 1$, where P_m denotes the set of all polynomials of degree $\leqslant m$ in the two variables ξ, η and Q_m is the set of all polynomials of the form

$$p(\xi, \eta) = \sum_{0 \leqslant i,j \leqslant m} c_{ij} \xi^i \eta^j.$$

Now, for computing the integral of (2.3), we use the mapping $F_K\colon (\xi, \eta) \in \hat{K} \longmapsto (x, y) = F_K(\xi, \eta) \in K$. Setting $\hat{u} = \widehat{u_{h|K}}$, we obtain by a simple calculation

(2.6) $$\begin{cases} \displaystyle\int_K (Lu_h - f, v)\, dx\, dy \\ \displaystyle = \int_{\hat{K}} \left(\left(\frac{\partial y}{\partial \eta} \hat{A} - \frac{\partial x}{\partial \eta} \hat{B} \right) \frac{\partial \hat{u}}{\partial \xi} + \left(-\frac{\partial y}{\partial \xi} \hat{A} + \frac{\partial x}{\partial \xi} \hat{B} \right) \frac{\partial \hat{u}}{\partial \eta} + J_K(\hat{C}\hat{u} - \hat{f}), \hat{v} \right) d\xi\, d\eta, \end{cases}$$

where J_K denotes the Jacobian determinant of F_K that we may assume > 0.

Next, to get a more computational form of the weighted residual method, we replace in (2.3) the integral $\int_K$ by a quadrature formula. Let $\hat{\omega}_i$ and $\hat{g}_i$, $1 \leqslant i \leqslant m$, be the Gauss-Legendre weights and abscissae for $[-1, +1]$; we set

$$\hat{\omega}_{ij} = \hat{\omega}_i\hat{\omega}_j, \qquad \hat{g}_{ij} = (\hat{g}_i, \hat{g}_j),$$
$$\omega_{ij}^K = J_K(\hat{g}_{ij})\hat{\omega}_{ij}, \quad g_{ij}^K = F_K(\hat{g}_{ij}), \qquad 1 \leqslant i, j \leqslant m, K \in \mathcal{T}_h.$$

Then

$$\text{(2.7)} \qquad \int_{\hat{K}} \hat{\varphi}(\xi, \eta)\, d\xi\, d\eta \quad \textit{is approximated by} \quad \sum_{i,j=1}^{m} \hat{\omega}_{ij}\hat{\varphi}(\hat{g}_{ij})$$

or equivalently

$$\text{(2.8)} \qquad \int_K \varphi(x, y)\, dx\, dy \quad \textit{is approximated by} \quad \sum_{i,j=1}^{m} \omega_{ij}^K \varphi(g_{ij}^K).$$

Now, using (2.8), the weighted residual method becomes

$$\text{(2.9)} \qquad \forall K \in \mathcal{T}_h,\ \forall v \in Y_K^p,\ \sum_{i,j=1}^{m} \omega_{ij}^K (Lu_h - f, v)(g_{ij}^K) = 0.$$

Let us introduce the basis φ_{ij}, $1 \leqslant i, j \leqslant m$, of the space Y_K defined by

$$\varphi_{ij}(g_{kl}^K) = \delta_{ik}\delta_{jl}, \qquad 1 \leqslant i, j, k, l \leqslant m.$$

By replacing successively in (2.9) v by $(0, \ldots, 0, \varphi_{ij}, 0, \ldots, 0)$, we find

$$\text{(2.10)} \qquad \forall K \in \mathcal{T}_h,\ (Lu_h - f)(g_{ij}^K) = 0, \qquad 1 \leqslant i, j \leqslant m,$$

so that $u_h \in V_h$ collocates to (1.1) at the points g_{ij}^K. We thus obtain a *collocation method* at the Gauss-Legendre points of each quadrilateral $K \in \mathcal{T}_h$.

Note that (2.10) can be also written in the form

$$\text{(2.11)} \quad \left\{\left(\frac{\partial y}{\partial \eta}\hat{A} - \frac{\partial x}{\partial \eta}\hat{B}\right)\frac{\partial \hat{u}}{\partial \xi} + \left(-\frac{\partial y}{\partial \xi}\hat{A} + \frac{\partial x}{\partial \xi}\hat{B}\right)\frac{\partial \hat{u}}{\partial \eta} + J_K(\hat{C}\hat{u} - \hat{f})\right\}(\hat{g}_{ij}) = 0,$$
$$1 \leqslant i, j \leqslant m.$$

Remark 1. The finite element methods introduced here may be viewed as generalizations to P.D.E's of the one-step Galerkin methods for O.D.E's which are described in [5], [6]. □

3. A Model Problem.

We first consider the following simple situation:

(i) The set Ω and all the elements K of $\mathcal{T}_h$ are rectangles whose sides are parallel to the (x, y) axes. Hence, for all $K \in \mathcal{T}_h$, F_K is an affine mapping.

(ii) The matrices A and B have constant coefficients and the coefficients of the matrix C are elementwise constant.

In that case, we are able to derive in an easy way stability and convergence results for the weighted residual method (2.3). This is done in Sections 3 and 4 where we assume without mentioning it again that the conditions (i) and (ii) hold. In Sections 5 and 6 we shall extend the corresponding analysis to the collocation method (2.10) in a more complicated situation.

Let us define the operator π_h to be the orthogonal projector from $L^2(\Omega)$ onto

Y_h and, for all $K \in \mathcal{T}_h$, the operator π_K to be the orthogonal projector from $L^2(K)$ onto Y_K. We set $\pi_{\hat{K}} = \hat{\pi}$. Clearly we have

$$\forall \varphi \in L^2(\Omega), \quad \pi_h \varphi_{|K} = \pi_K(\varphi_{|K}),$$

$$\forall \varphi \in L^2(K), \quad \widehat{\pi_K \varphi} = \hat{\pi}\hat{\varphi}.$$

In order to prove a stability result, we want to evaluate

$$\sum_{K \in \mathcal{T}_h} \int_K (Lv, \pi_K v)\, dx\, dy, \qquad v \in V_h.$$

We then denote by $\hat{\pi}_\xi$ (respectively $\hat{\pi}_\eta$) the orthogonal projector with respect to the variable ξ (respectively to the variable η) from $L^2(-1, +1)$ onto the space of all polynomials of degree $\leqslant m - 1$. We shall often use the following property

$$\hat{\pi} = \hat{\pi}_\xi \hat{\pi}_\eta = \hat{\pi}_\eta \hat{\pi}_\xi. \tag{3.1}$$

LEMMA 1. *We have for all symmetric $p \times p$ matrices $\hat{E}$ with constant coefficients and for all $\hat{v} \in \hat{X}^p$*

$$\begin{cases} \int_{\hat{K}} \left(\hat{E} \dfrac{\partial \hat{v}}{\partial \xi}, \hat{\pi}\hat{v}\right) d\xi\, d\eta \\ = \dfrac{1}{2} \int_{-1}^{+1} \{(\hat{E}\hat{\pi}_\eta \hat{v}, \hat{\pi}_\eta \hat{v})(1, \eta) - (\hat{E}\hat{\pi}_\eta \hat{v}, \hat{\pi}_\eta \hat{v})(-1, \eta)\}\, d\eta. \end{cases} \tag{3.2}$$

Proof. We notice that, for all η, each component of the function $\xi \longrightarrow (\hat{E}(\partial \hat{v}/\partial \xi)(\xi, \eta))$ is a polynomial of degree $\leqslant m - 1$. Hence, using (3.1), we may write

$$\int_{\hat{K}} \left(\hat{E} \frac{\partial \hat{v}}{\partial \xi}, \hat{\pi} v\right) d\xi\, d\eta = \int_{\hat{K}} \left(\hat{E} \frac{\partial \hat{v}}{\partial \xi}, \hat{\pi}_\xi \hat{\pi}_\eta \hat{v}\right) d\xi\, d\eta$$

$$= \int_{\hat{K}} \left(\hat{E} \frac{\partial \hat{v}}{\partial \xi}, \hat{\pi}_\eta \hat{v}\right) d\xi\, d\eta = \int_{\hat{K}} \left(\hat{E} \frac{\partial}{\partial \xi}(\hat{\pi}_\eta \hat{v}), \hat{\pi}_\eta \hat{v}\right) d\xi\, d\eta.$$

Next, by the symmetry of the matrix $\hat{E}$, we obtain

$$\int_{\hat{K}} \left(\hat{E} \frac{\partial \hat{v}}{\partial \xi}, \hat{\pi}\hat{v}\right) d\xi\, d\eta = \frac{1}{2} \int_{\hat{K}} \frac{\partial}{\partial \xi} (\hat{E}\hat{\pi}_\eta \hat{v}, \hat{\pi}_\eta \hat{v})\, d\xi\, d\eta,$$

from which (3.2) follows at once. □

Let us now define the operator π_h^S to be the orthogonal projector from $L^2(\Gamma)$ onto the space $Y_{h|\Gamma}$ of the traces over Γ of all functions of Y_h. Similarly, for all $K \in \mathcal{T}_h$, we denote by π_K^S the orthogonal projector from $L^2(\partial K)$ onto $Y_{K|\partial K}$, where ∂K is the boundary of K.

Concerning the continuity properties of the functions of X_h, we assume that, for any pair of adjacent elements K_1, K_2 of $\mathcal{T}_h$ and for any function $\varphi \in X_h$, we have

$$\pi_{K_1}^S \varphi = \pi_{K_2}^S \varphi \quad \text{on the edge } K' = K_1 \cap K_2. \tag{3.3}$$

Since $\pi_K^S \varphi = \varphi$ at the m Gauss-Legendre points of each side of ∂K, the assumption (3.3) exactly means that *a function $\varphi \in X_h$ is necessarily continuous at the m Gauss-Legendre points of each edge K' of $\mathcal{T}_h$.*

We also assume that the functions of V_h satisfy the boundary condition (1.2) in the following sense:

$$\forall v_h \in V_h, \quad (An_x + Bn_y - M)\pi_h^S v_h = 0 \quad \text{on } \Gamma. \tag{3.4}$$

Let us now state a *weak-stability* result.

THEOREM 1. *Assume that the hypotheses* (3.3), (3.4) *hold. Assume in addition that there exists a constant* $\alpha > 0$ *such that*

$$C + C^* \geqslant \alpha I. \tag{3.5}$$

Then we have for all $v_h \in V_h$

$$2\sum_{K\in T_h}\int_K (Lv_h, \pi_h v_h)\,dx\,dy \geqslant \int_\Gamma (M\pi_h^S v_h, \pi_h^S v_h)\,dS + \alpha\|\pi_h v_h\|_{0,\Omega}^2. \tag{3.6}$$

Proof. Let $K \in T_h$ and $v \in X_K^p$. Since K is a rectangle whose sides are parallel to the (x, y) axes, we may write

$$\int_K \left(A\frac{\partial v}{\partial x} + B\frac{\partial v}{\partial y}, \pi_K v\right) dx\,dy$$
$$= \frac{\partial y}{\partial \eta}\int_{\hat K}\left(\hat A\frac{\partial \hat v}{\partial \xi}, \hat\pi\hat v\right) d\xi\,d\eta + \frac{\partial x}{\partial \xi}\int_{\hat K}\left(\hat B\frac{\partial \hat v}{\partial \eta}, \hat\pi\hat v\right) d\xi\,d\eta$$

and by Lemma 1

$$\int_K \left(A\frac{\partial v}{\partial x} + B\frac{\partial v}{\partial y}, \pi_K v\right) dx\,dy$$
$$= \frac{1}{2}\frac{\partial y}{\partial \eta}\int_{-1}^{+1}\{(\hat A\hat\pi_\eta\hat v, \pi_\eta\hat v)(1, \eta) - (\hat A\hat\pi_\eta\hat v, \hat\pi_\eta\hat v)(-1, \eta)\}\,d\eta$$
$$+ \frac{1}{2}\frac{\partial x}{\partial \xi}\int_{-1}^{+1}\{(\hat B\hat\pi_\xi\hat v, \hat\pi_\xi\hat v)(\xi, 1) - (\hat B\hat\pi_\xi\hat v, \hat\pi_\xi\hat v)(\xi, -1)\}\,d\xi.$$

Hence we get

$$\int_K \left(A\frac{\partial v}{\partial x} + B\frac{\partial v}{\partial y}, \pi_K v\right) dx\,dy = \frac{1}{2}\int_{\partial K}((An_x + Bn_y)\pi_K^S v, \pi_K^S v)\,dS, \tag{3.7}$$

where $n = (n_x, n_y)$ is the unit outward normal along ∂K.

Next, let v_h be in V_h. Using (3.7) and the hypothesis (3.3), we obtain

$$\sum_{K\in T_h}\int_K \left(A\frac{\partial v_h}{\partial x} + B\frac{\partial v_h}{\partial y}, \pi_h v_h\right) dx\,dy$$
$$= \frac{1}{2}\sum_{K\in T_h}\int_{\partial K}((An_x + Bn_y)\pi_K^S v_h, \pi_K^S v_h)\,dS$$
$$= \frac{1}{2}\int_\Gamma ((An_x + Bn_y)\pi_h^S v_h, \pi_h^S v_h)\,dS,$$

and by (3.4)

$$(3.8)\qquad \sum_{K\in T_h}\int_K\left(A\frac{\partial v_h}{\partial x}+B\frac{\partial v_h}{\partial y},\pi_h v_h\right)dx\,dy=\frac{1}{2}\int_\Gamma(M\pi_h^S v_h,\pi_h^S v_h)\,dS.$$

On the other hand, since the coefficients of the matrix C are elementwise constant, we have

$$\int_\Omega (Cv_h,\pi_h v_h)\,dx\,dy=\int_\Omega(C\pi_h v_h,\pi_h v_h)\,dx\,dy$$

and by (3.5)

$$(3.9)\qquad \int_\Omega(Cv_h,\pi_h v_h)\,dx\,dy\geqslant\frac{\alpha}{2}\|\pi_h v_h\|_{0,\Omega}^2.$$

Therefore, the desired inequality (3.6) follows from (3.8) and (3.9). □

Observe that Theorem 1 does not necessarily imply the existence and uniqueness of the solution $u_h\in V_h$ of problem (2.3). In fact, if $f=0$, using (1.3) and (3.6), we only obtain $\pi_h u_h=0$. Therefore, we introduce the following hypothesis

$$(3.10)\qquad \left\{v_h\in V_h;\ \forall K\in T_h,\ \forall w\in Y_K^p,\ \int_K(Lv_h,w)\,dx\,dy=0\right\}=\{0\}.$$

As a consequence of Theorem 1, we get

THEOREM 2. *Assume that the conditions* (3.3), (3.4), (3.5) *and* (3.10) *are satisfied. Then problem* (2.3) *has a unique solution* $u_h\in V_h$ *and we have the error bound*

$$\left(\int_\Gamma(M\pi_h^S(u_h-u),\pi_h^S(u_h-u))\,dS\right)^{1/2}+\|\pi_h(u_h-u)\|_{0,\Omega}$$

$$(3.11)\qquad \leqslant\inf_{v_h\in V_h}\left\{C\sup_{w_h\in Y_h^p}\frac{\sum_{K\in T_h}\int_K(L(u-v_h),w_h)\,dx\,dy}{\|w_h\|_{0,\Omega}}+\left(\int_\Gamma(M\pi_h^S(u-v_h),\pi_h^S(u-v_h))\,dS\right)^{1/2}+\|\pi_h(u-v_h)\|_{0,\Omega}\right\},$$

for some constant $C>0$ *independent of* h.

Proof. First, we note that the hypotheses (2.2) and (3.10) imply the existence and uniqueness of the solution $u_h\in V_h$ of problem (2.3). Next, let v_h be in V_h and let w_h be in Y_h^p; we may write

$$\sum_{K\in T_h}\int_K(L(u_h-v_h),w_h)\,dx\,dy$$

$$=\sum_{K\in T_h}\int_K(f-Lv_h,w_h)\,dx\,dy=\sum_{K\in T_h}\int_K(L(u-v_h),w_h)\,dx\,dy.$$

By using Theorem 1, we have

$$\int_\Gamma (M\pi_h^S(u_h - v_h), \pi_h^S(u_h - v_h))\, dS + \alpha \|\pi_h(u_h - v_h)\|_{0,\Omega}^2$$

$$\leqslant 2 \sum_{K \in \mathcal{T}_h} \int_K (L(u - v_h), \pi_h(u_h - v_h))\, dx\, dy$$

$$\leqslant 2 \left(\sup_{w_h \in Y_h^p} \frac{\sum_{K \in \mathcal{T}_h} \int_K (L(u - v_h), w_h)\, dx\, dy}{\|w_h\|_{0,\Omega}} \right) \|\pi_h(u_h - v_h)\|_{0,\Omega}.$$

Hence, we get by the Cauchy-Schwarz inequality

$$\left(\int_\Gamma (M\pi_h^S(u_h - v_h), \pi_h^S(u_h - v_h))\, dS \right)^{1/2} + \|\pi_h(u_h - v_h)\|_{0,\Omega}$$

$$\leqslant C \sup_{w_h \in Y_h^p} \frac{\sum_{K \in \mathcal{T}_h} \int_K (L(u - v_h), w_h)\, dx\, dy}{\|w_h\|_{0,\Omega}},$$

so that (3.11) follows by the triangle inequality. □

Remark 2. Assume that in (3.11) the right-hand side tends to zero as h tends to zero. Then Theorem 2 gives

$$\lim_{h \to 0} \|\pi_h(u_h - u)\|_{0,\Omega} = 0,$$

and this does not imply the convergence of u_h to u. However, since $\pi_h u_h = u_h$ at the Gauss-Legendre points g_{ij}^K, $1 \leqslant i, j \leqslant m$, $K \in \mathcal{T}_h$, we obtain the convergence of the approximate solution u_h "at these Gauss points". □

4. Applications. We now turn to some examples.

Example 1. *A conforming method.* We first choose

$$\hat{X} = Q_m, \tag{4.1}$$

$$X_h = \{\varphi_h \in C^0(\overline{\Omega}); \forall K \in \mathcal{T}_h, \varphi_{h|K} \in X_K \ (= Q_m)\}, \tag{4.2}$$

so that we define a conforming finite element method. The degrees of freedom of a function $\varphi_h \in X_h$ are then determined in a standard way.

Let us evaluate the right-hand side of the inequality (3.11). Clearly, assuming classical regularity hypotheses and taking as function v_h some interpolate of u, we get for the last two terms

$$\left(\int_\Gamma (M\pi_h^S(u - v_h), \pi_h^S(u - v_h))\, dS \right)^{1/2} = O(h^{m+1}), \qquad \|\pi_h(u - v_h)\|_{0,\Omega} = O(h^{m+1}).$$

We now prove that we can choose the function v_h in such way that the same bound holds for the first term.

Denote by $-1 = \hat{l}_0 < \hat{l}_1 < \cdots < \hat{l}_m = 1$ the $(m + 1)$ Gauss-Lobatto abscissae for $[-1, +1]$. Given a function $\hat{\varphi} \in C^0(\hat{K})$, we define $\hat{r}\hat{\varphi}$ as the unique polynomial of Q_m which satisfies

$$\hat{r}\hat{\varphi}(\hat{l}_{ij}) = \hat{\varphi}(\hat{l}_{ij}), \quad \hat{l}_{ij} = (\hat{l}_i, \hat{l}_j), \quad 0 \leqslant i, j \leqslant m. \tag{4.3}$$

Next, given a quadrilateral K and a function $\varphi \in C^0(K)$, we define $r_K\varphi \in X_K$ by

$$\widehat{r_K\varphi} = \hat{r}\hat{\varphi}. \tag{4.4}$$

Finally, with any function $\varphi \in C^0(\overline{\Omega})$, we associate $r_h\varphi \in X_h$ by

$$r_h\varphi_{|K} = r_K(\varphi_{|K}). \tag{4.5}$$

We then consider a regular family $(\mathcal{T}_h)$ of quadrangulations of $\overline{\Omega}$ in the sense that there exists a constant $\sigma > 0$ independent of h such that

$$h_K \leqslant \sigma\rho_K, \qquad K \in \mathcal{T}_h,$$

where h_K is the diameter of K and ρ_K is the diameter of the inscribed circle of K.

LEMMA 2. *Assume that $(\mathcal{T}_h)$ is a regular family of quadrangulations of $\overline{\Omega}$. Then, if $v \in (H^{m+2}(\Omega))^p$, we have for all $w_h \in Y_h^p$*

$$\left|\int_\Omega (L(v - r_hv), w_h)\,dx\,dy\right| \leqslant Ch^{m+1}\|v\|_{m+2,\Omega}\|w_h\|_{0,\Omega}.^* \tag{4.6}$$

Proof. Let $\hat{\psi} \in Q_{m-1}$; we consider the continuous linear functional on $H^{m+2}(\hat{K})$

$$\hat{\varphi} \longmapsto \int_{\hat{K}}\left(\frac{\partial}{\partial\xi}(\hat{\varphi} - \hat{r}\hat{\varphi})\right)\hat{\psi}\,d\xi\,d\eta.$$

Since $\hat{\varphi} = \hat{r}\hat{\varphi}$ for all $\hat{\varphi} \in Q_m$, this linear functional vanishes over Q_m. Let us show that it also vanishes over the space P_{m+1}. If $\hat{\varphi} = \xi^{m+1}$, $\hat{\varphi} - \hat{r}\hat{\varphi}$ is a polynomial of degree $\leqslant m + 1$ in the variable ξ which vanishes at the $m + 1$ Gauss-Lobatto points $\hat{l}_i$, $0 \leqslant i \leqslant m$. Hence $\partial(\hat{\varphi} - \hat{r}\hat{\varphi})/\partial\xi$ is a polynomial of degree $\leqslant m$ which vanishes at the m Gauss-Legendre points $\hat{g}_i$, $1 \leqslant i \leqslant m$. As a consequence, $(\partial(\hat{\varphi} - \hat{r}\hat{\varphi})/\partial\xi)\hat{\psi} \in Q_{2m-1}$ vanishes at the points $\hat{g}_{ij}$, $1 \leqslant i, j \leqslant m$, so that $\int_{\hat{K}}(\partial(\hat{\varphi} - \hat{r}\hat{\varphi})/\partial\xi)\hat{\psi}\,d\xi\,d\eta = 0$.

On the other hand, if $\hat{\varphi} = \eta^{m+1}$, the function $\hat{\varphi} - \hat{r}\hat{\varphi}$ does not depend on ξ so that the previous integral vanishes again. Thus, the linear functional considered above vanishes over P_{m+1}. Using the Bramble-Hilbert lemma (cf. for instance [2, Lemma 6]), we get for all $\hat{\varphi} \in H^{m+2}(\hat{K})$ and all $\hat{\psi} \in Q_{m-1}$

$$\left|\int_{\hat{K}}\left(\frac{\partial}{\partial\xi}(\hat{\varphi} - \hat{r}\hat{\varphi})\right)\hat{\psi}\,d\xi\,d\eta\right| \leqslant c|\hat{\varphi}|_{m+2,\hat{K}}\|\hat{\psi}\|_{0,\hat{K}}. \tag{4.7}$$

Now let $K \in \mathcal{T}_h$, $v \in H^{m+2}(K)$ and $w \in Y_K$. We have

$$\int_K\left(A\frac{\partial}{\partial x}(v - r_Kv), w\right)dx\,dy = \frac{\partial y}{\partial\eta}\int_{\hat{K}}\left(\hat{A}\frac{\partial}{\partial\xi}(\hat{v} - \hat{r}\hat{v}), \hat{w}\right)d\xi\,d\eta$$

and by (4.7)

$$\left|\int_K\left(A\frac{\partial}{\partial x}(v - r_Kv), w\right)dx\,dy\right| \leqslant C\left|\frac{\partial y}{\partial\eta}\right||\hat{v}|_{m+2,\hat{K}}\|\hat{w}\|_{0,\hat{K}}.$$

---

*Here and in all the sequel, C will denote a generic positive constant independent of h.

Since (T_h) is a regular family of quadrangulations, we get

$$|\hat{v}|_{m+2,\hat{K}} \leqslant cJ_K^{-1/2}h_K^{m+2}|v|_{m+2,K} \leqslant ch_K^{m+1}|v|_{m+2,K},$$

$$\|\hat{w}\|_{0,\hat{K}} = J_K^{-1/2}\|w\|_{0,K} \leqslant ch_K^{-1}\|w\|_{0,K},$$

$$|\partial y/\partial\eta| \leqslant h_K/2,$$

so that

$$\left|\int_K \left(A\frac{\partial}{\partial x}(v - r_K v), w\right) dx\, dy\right| \leqslant ch_K^{m+1}|v|_{m+2,K}\|w\|_{0,K}.$$

Similarly, we obtain

$$\left|\int_K \left(B\frac{\partial}{\partial y}(v - r_K v), w\right) dx\, dy\right| \leqslant ch_K^{m+1}|v|_{m+2,K}\|w\|_{0,K}$$

and

$$\left|\int_K (C(v - r_K v), w)\, dx\, dy\right| \leqslant ch_K^{m+1}|v|_{m+1,K}\|w\|_{0,K}.$$

Therefore, we get

$$\left|\int_K (L(v - r_K v), w)\, dx\, dy\right| \leqslant ch_K^{m+1}\|w\|_{m+2,K}\|w\|_{0,K},$$

from which the inequality (4.6) follows immediately. □

We are now able to prove

THEOREM 3. *Assume that the hypotheses* (3.4), (3.5), (3.10), (4.1), (4.2) *hold and that* (T_h) *is a regular family of quadrangulations of* $\overline{\Omega}$. *Assume in addition that* $u \in (H^{m+2}(\Omega))^p$ *and* $r_h u \in V_h$. *Then problem* (2.3) *has a unique solution* $u_h \in V_h$, *and we have the error estimate*

$$(4.8)\quad \left(\int_\Gamma (M\pi_h^S(u_h - u), \pi_h^S(u_h - u))\, dS\right)^{1/2} + \|\pi_h(u_h - u)\|_{0,\Omega} \leqslant Ch^{m+1}\|u\|_{m+2,\Omega}.$$

Proof. Clearly, the hypotheses (4.1) and (4.2) imply (3.1) and (3.3), respectively. Since $r_h u$ is assumed to belong to V_h, we may apply Theorem 2 with v_h replaced by $r_h u$ in (3.11). Standard approximation results give

$$\left(\int_\Gamma (M\pi_h^S(u - r_h u), \pi_h^S(u - r_h u))\, dS\right)^{1/2} \leqslant ch^{m+1}|u|_{m+1,\Gamma} \leqslant ch^{m+1}\|u\|_{m+2,\Omega},$$

$$\|\pi_h(u - r_h u)\|_{0,\Omega} \leqslant \|u - r_h u\|_{0,\Omega} \leqslant ch^{m+1}|u|_{m+1,\Omega}.$$

Hence, the desired inequality (4.8) follows from Lemma 2. □

Example 2. *A nonconforming method.* We next study a nonconforming finite element method which appears to be more effective than the previous conforming one in some practical problems. We shall use here the techniques of [14].

Let $\hat{\Sigma}_1$ be the set of Gauss-Legendre points of $\partial\hat{K}$ of the form $(\hat{g}_i, \pm 1)$, $(\pm 1, \hat{g}_j)$, $1 \leqslant i, j \leqslant m$. We denote these points by $\hat{b}_i$, $1 \leqslant i \leqslant 4m$, and we number them counterclockwise. Let $\hat{\Sigma}_2$ be a Q_{m-2}-unisolvent subset** of $\mathring{\hat{K}}$, $m \geqslant 2$. We set

$$\hat{\Sigma} = \hat{\Sigma}_1 \cup \hat{\Sigma}_2.$$

LEMMA 3. *The space of functions $\hat{\varphi} \in Q_m$ which vanish on $\hat{\Sigma}$ is one-dimensional.*

Proof. Let $\hat{\varphi} \in Q_m$ vanish on $\hat{\Sigma}$. By using the symmetry properties of the set $\hat{\Sigma}_1$, we get

$$\alpha = \hat{\varphi}(1, 1) = (-1)^m \hat{\varphi}(-1, 1) = \hat{\varphi}(-1, -1) = (-1)^m \varphi(1, -1).$$

Let us next introduce the function $\hat{\psi}$ defined by

$$\hat{\psi}(\xi, \eta) = \hat{\varphi}(\xi, \eta) - \lambda \prod_{i,j=1}^{m} (\xi - \hat{g}_i)(\eta - \hat{g}_j),$$

where λ is determined so that $\hat{\psi}(1, 1) = 0$. Then, we have $\hat{\psi}(1, 1) = \hat{\psi}(-1, 1) = \hat{\psi}(-1, -1) = \hat{\psi}(1, -1) = 0$, and $\hat{\psi}$ vanishes at $m + 2$ points of each side of $\partial\hat{K}$. Therefore, $\hat{\psi}$ vanishes on $\partial\hat{K}$.

For $m = 1$, we get $\hat{\psi} = 0$ so that $\hat{\varphi}(\xi, \eta) = \alpha\xi\eta$. For $m \geqslant 2$, we obtain

$$\hat{\psi}(\xi, \eta) = \lambda(\xi^2 - 1)(\eta^2 - 1)\hat{\chi}(\xi, \eta), \qquad \hat{\chi} \in Q_{m-2}.$$

Since $\hat{\varphi}$ vanishes on $\hat{\Sigma}_2$, the function $\hat{\chi}$ takes given values on $\hat{\Sigma}_2$ and is uniquely determined. Hence, we have

$$\hat{\varphi}(\xi, \eta) = \lambda\left[\prod_{i,j=1}^{m} (\xi - \hat{g}_i)(\eta - \hat{g}_j) + (\xi^2 - 1)(\eta^2 - 1)\hat{\chi}(\xi, \eta)\right]$$

and the function $\hat{\varphi}$ is unique up to a multiplicative constant. □

We denote by $\hat{\varphi}_0$ the polynomial of Q_m which vanishes on $\hat{\Sigma}$ and satisfies the condition $\hat{\varphi}_0(1, 1) = 1$. We have

$$\varphi_0(\xi, \eta) = \begin{cases} \xi\eta, \ m = 1, \ \hat{\Sigma}_2 = \varnothing, \\ 2\xi^2\eta^2 - \tfrac{1}{2}(\xi^2 + \eta^2), \ m = 2, \ \hat{\Sigma}_2 = \{0\}. \end{cases}$$

Now, let Q'_m be the space of polynomials spanned by $\xi^i\eta^j$, $0 \leqslant i, j \leqslant m$, $i + j < 2m$. Note that $Q_m = Q'_m \oplus \{\xi^m\eta^m\}$. Then we choose

$$\hat{X} = Q'_m, \tag{4.9}$$

$$X_h = \{\varphi_h;\ \forall K \in \mathcal{T}_h,\ \varphi_{h|K} \in X_K \text{ and } \varphi_h \text{ is continuous at the } m \text{ Gauss-Legendre points of each edge } K' \text{ of } \mathcal{T}_h\}. \tag{4.10}$$

Let us determine the degrees of freedom of a function $\varphi \in X_K$. Denote by $\hat{T}_m$ the space of the traces over $\partial\hat{K}$ of all functions of Q_m. It follows from the first part of the proof of Lemma 3 that the space of functions $\hat{\varphi} \in \hat{T}_m$ which vanish on $\hat{\Sigma}$ is one-dimensional. Therefore, there exists a set $\{\alpha_i\}_{i=1}^{4m}$ of nonsimultaneously zero

**Let us recall that a set $\Sigma = \{a_i\}_{i=1}^N$ is P-unisolvent if for any set of scalars α_i, $1 \leqslant i \leqslant N$, there exists a unique function $p \in P$ such that $p(a_i) = \alpha_i$, $1 \leqslant i \leqslant N$.

scalars such that

$$\forall \hat{\varphi} \in \hat{T}_m, \quad \sum_{i=1}^{4m} \alpha_i \hat{\varphi}(\hat{b}_i) = 0.$$

Then, if on the one hand we have $\alpha_{i_0} \neq 0$ and if on the other hand the set $\hat{\Sigma}_2$ is so chosen that $\hat{\varphi}_0 \notin \hat{X}$, we set

$$\hat{\Sigma}'_2 = \{\hat{b}_i\}_{1 \leqslant i \leqslant 4m; i \neq i_0}, \quad \hat{\Sigma}' = \hat{\Sigma}'_1 \cup \hat{\Sigma}_2.$$

Hence, using Lemma 3, we find that a function $\hat{\varphi} \in \hat{X}$ is uniquely determined by its values at the points of $\hat{\Sigma}'$.

One can easily check that, for $m = 1, 2$, we have $\alpha_i = (-1)^i$ so that we can select the index i_0 arbitrarily and take

$$\hat{\Sigma}' = \begin{cases} \hat{\Sigma}'_1, & m = 1, \\ \hat{\Sigma}'_1 \cup \{0\}, & m = 2. \end{cases}$$

Next, setting $\Sigma_K = F_K(\hat{\Sigma})$, $\Sigma'_K = F_K(\hat{\Sigma}')$, we obtain that the degrees of freedom of a function $\varphi \in X_K$ may be chosen as its values at the points of Σ'_K. Moreover, using (4.10), a function $\varphi_h \in X_h$ is easily constructed by means of its values at the points of $\Sigma_h = \bigcup_{K \in T_h} \Sigma_K$.

Finally, we want to prove the analogue of Lemma 2. Given a function $\hat{\varphi} \in C^0(\hat{K})$, we define $\hat{r}'\hat{\varphi}$ to be unique polynomial of Q'_m which coincides with $\hat{r}\hat{\varphi}$ on $\hat{\Sigma}$. We then introduce the functions $r'_K\varphi$ and $r'_h\varphi$ as in (4.4), (4.5).

LEMMA 4. *Assume that (T_h) is a regular family of quadrangulations of $\bar{\Omega}$. Then, if $v \in (H^{m+2}(\Omega))^p$, we have for all $w_h \in Y_h^p$*

$$\left| \sum_{K \in T_h} \int_K (L(v - r'_h v), w_h)\, dx\, dy \right| \leqslant C h^{m+1} \|v\|_{m+2,\Omega} \|w_h\|_{0,\Omega}. \tag{4.11}$$

Proof. Let $\hat{\psi} \in Q_{m-1}$; we want to show that the continuous linear functional on $H^{m+2}(\hat{K})$

$$\hat{\varphi} \longmapsto \int_{\hat{K}} \left(\frac{\partial}{\partial \xi} (\hat{\varphi} - \hat{r}'\hat{\varphi}) \right) \hat{\psi}\, d\xi\, d\eta \tag{4.12}$$

vanishes over the space P_{m+1}. First of all, since $\hat{\varphi} = \hat{r}\hat{\varphi} = \hat{r}'\hat{\varphi}$ for all $\hat{\varphi} \in Q'_m$, the linear functional (4.12) vanishes over Q'_m. On the other hand, if $\hat{\varphi} = \xi^{m+1}$ (respectively, $\hat{\varphi} = \eta^{m+1}$), the function $\hat{r}\hat{\varphi}$ depends only on ξ (respectively, on η), and we have $\hat{r}\hat{\varphi} = \hat{r}'\hat{\varphi}$. Hence, using the first part of the proof of Lemma 2, we get

$$\int_{\hat{K}} \left(\frac{\partial}{\partial \xi} (\hat{\varphi} - \hat{r}'\hat{\varphi}) \right) \hat{\psi}\, d\xi\, d\eta = 0, \qquad \hat{\varphi} = \xi^{m+1}, \eta^{m+1}.$$

Since $P_{m+1} \subset Q'_m \oplus \{\xi^{m+1}\} \oplus \{\eta^{m+1}\}$ for $m \geqslant 2$, it remains only to consider the case $m = 1$, $\hat{\varphi} = \xi\eta$. Clearly, we have

$$\int_{\hat{K}} \frac{\partial \hat{\varphi}}{\partial \xi}\, d\xi\, d\eta = \int_{\hat{K}} \eta\, d\xi\, d\eta = 0,$$

and an explicit calculation shows that $\hat{r}'\hat{\varphi} = 0$ in that case. Hence, our assertion is proved for all $m \geqslant 1$ and the desired estimate (4.11) follows just as in the proof of Lemma 2. □

THEOREM 4. *Assume that the hypotheses* (3.4), (3.5), (3.10), (4.9), (4.10) *hold and that* $(\mathcal{T}_h)$ *is a regular family of quadrangulations of* $\overline{\Omega}$. *Assume in addition that* $u \in (H^{m+2}(\Omega))^p$ *and* $r'_h u \in V_h$. *Then problem* (2.3) *has a unique solution* $u_h \in V_h$, *and we have the estimate* (4.8).

Proof. The hypotheses (4.9) and (4.10) imply (3.1) and (3.3), respectively so that Theorem 2 applies. Using Lemma 4, the proof parallels that of Theorem 3. □

As a model application of Theorems 3 and 4, we consider the simple hyperbolic equation

$$\mu \frac{\partial u}{\partial x} + \nu \frac{\partial u}{\partial y} + \sigma u = f \quad \text{in } \Omega \tag{4.13}$$

with the boundary condition

$$u = 0 \quad \text{on } \Gamma_- = \{(x, y) \in \Gamma; (\mu n_x + \nu n_y)(x, y) < 0\}. \tag{4.14}$$

Equation (4.13) arises in neutron transport theory. μ and ν are parameters such that $\mu^2 + \nu^2 \leqslant 1$ and the function $u = u(x, y, \mu, \nu)$ represents a flux of neutrons at the point (x, y) in the angular direction (μ, ν); the quantity σ denotes the nuclear cross section which satisfies $\sigma(x, y) \geqslant \alpha > 0$ and $f = f(x, y, \mu, \nu)$ stands for the scattering, the fission and the source terms.

Note that the boundary condition (4.14) is of the form (1.2) with $M = |\mu n_x + \nu n_y|$.

Assume for convenience that $\mu, \nu \neq 0$. We define

$$V_h = \{v_h \in X_h; v_h = 0 \text{ on } \Gamma_-\} \quad \text{in the case of Example 1} \tag{4.15}$$

and

$$V_h = \{v_h \in X_h; v_h = 0 \text{ at the Gauss-Legendre points located on } \Gamma_-\}, \tag{4.16}$$

in the case of Example 2. One can check that in each case

$$\dim V_h = \dim Y_h = m^2 \operatorname{card}(\mathcal{T}_h).$$

In order to check the hypothesis (3.10), we state

LEMMA 5. *We have*

$$\begin{aligned} \{\hat{\varphi} \in Q_m; \hat{\pi}\hat{\varphi} = 0 \textit{ on } \hat{K}, \hat{\varphi}(-1, \eta) = \hat{\varphi}(\xi, -1) = 0, \\ \hat{\pi}_\eta \hat{\varphi}(1, \eta) = \hat{\pi}_\xi \hat{\varphi}(\xi, 1) = 0\} = \{0\} \end{aligned} \tag{4.17}$$

and

$$\{\hat{\varphi} \in Q'_m; \hat{\pi}\hat{\varphi} = 0 \textit{ on } \hat{K}, \hat{\pi}_\eta \hat{\varphi}(\pm 1, \eta) = \hat{\pi}_\xi \hat{\varphi}(\xi, \pm 1) = 0\} = \{0\}. \tag{4.18}$$

Proof. Let $\hat{\varphi}$ be in Q_m. Assume that $\hat{\pi}\hat{\varphi} = 0$, $\pi_\xi \hat{\varphi}(\xi, \pm 1) = \pi_\eta \hat{\varphi}(\eta, \pm 1) = 0$. We have

$$\hat{\varphi}(\hat{g}_i, \hat{g}_j) = \hat{\varphi}(\pm 1, \hat{g}_j) = \hat{\varphi}(\hat{g}_i, \pm 1) = 0, \qquad 1 \leqslant i, j \leqslant m.$$

Hence, using Lemma 3, $\hat{\varphi}$ is necessarily of the form

$$\varphi = \lambda \prod_{i,j=1}^{m} (\xi - \hat{g}_i)(\eta - \hat{g}_j).$$

Assume in addition that $\hat{\varphi}(1, 1) = 0$, we get $\lambda = 0$ so that (4.17) holds. Likewise, if we assume that $\hat{\varphi} \in Q'_m$, we also get $\lambda = 0$ which implies (4.18). □

THEOREM 5. *Assume that either the hypotheses* (4.1), (4.2), (4.15) *hold or the hypotheses* (4.9), (4.10), (4.16) *hold. Assume in addition that* (T_h) *is a regular family of quadrangulations of* $\overline{\Omega}$ *and that the solution* u *of* (4.13), (4.14) *belongs to* $H^{m+2}(\Omega)$. *Then each of the corresponding problems* (2.3) *has a unique solution* $u_h \in V_h$, *and we have the error estimate*

$$\| |\mu n_x + \nu n_y|^{1/2} \pi_h^S(u_h - u)\|_{0,\Gamma} + \|\pi_h(u_h - u)\|_{0,\Omega} \leqslant ch^{m+1} \|u\|_{m+2,\Omega}. \tag{4.19}$$

Proof. Let us check the hypothesis (3.10). Assume that a function $v \in X_K$ satisfies

$$\forall w \in Y_K, \quad \int_K (Lv, w)\, dx\, dy = 0. \tag{4.20}$$

Then using (3.5), (3.7) and (3.9) (Theorem 1), we get

$$\alpha \|\pi_K v\|_{0,K}^2 + \int_{\partial K} (\mu n_x + \nu n_y)(\pi_K^S v)^2\, ds = 0.$$

Assume that $v = 0$ on $\partial\_K = \{(x, y) \in \partial K, (\mu n_x + \nu n_y)(x, y) < 0\}$. We have

$$\pi_K v = 0 \quad \text{on } K, \qquad \pi_K^S v = 0 \quad \text{on } \partial K, \qquad v = 0 \quad \text{on } \partial\_K.$$

Then by applying (4.17), we get $v = 0$ on K. Now by sweeping through the mesh, starting from the part $\Gamma\_$ of the boundary we find that (3.10) holds in the case of Example 1.

Likewise, for the second example, if we assume that (4.20) holds and that $\pi_K^S v = 0$ on ∂_K, we get $v = 0$ on K as a consequence of (4.18). Using the same argument as for Example 1, we find that (3.10) holds in the case of Example 2.

The desired result (4.19) is then a direct consequence of Theorems 3 and 4. □

The conforming and nonconforming finite element methods of approximation of Eq. (4.13) appear to be higher order extensions of the SNG method and DSN method of Carlson (cf. [8]), respectively. The convergence result (4.19) generalizes the analysis of Madsen [13]. Although the two above finite element methods have the same rate of convergence, numerical experiments show that the nonconforming method is more effective in practice than the conforming one, particularly when using low order methods with a moderate number of elements.

5. Some Technical Preliminaries. We next consider the general situation where the quadrilaterals of T_h are not necessarily rectangles, and the differential operator L has variable coefficients. We want to extend the analysis of Section 3 to the collocation method (2.10). In order to prove weak-stability results of the form (3.6), we now need to estimate the following expression

$$\sum_{K \in \mathcal{T}_h} \sum_{i,j=1}^{m} \omega_{ij}^K (Lv_h, v_h)(g_{ij}^K), \qquad v_h \in V_h,$$

which occurs when evaluating $\Sigma_{K \in \mathcal{T}_h} \int_K (Lv_h, \pi_h v_h)\, dx\, dy$ by means of the quadrature formula (2.8). Unfortunately, such stability results are much more complicated to establish in that case and we first derive some technical preliminaries.

Since

$$(5.1)\qquad \begin{cases} \displaystyle\sum_{i,j=1}^{m} \omega_{ij}^K (Lv, v)(g_{ij}^K) \\ \displaystyle = \sum_{i,j=1}^{m} \hat{\omega}_{ij} \left(\left(\frac{\partial y}{\partial \eta} \hat{A} - \frac{\partial x}{\partial \eta} \hat{B} \right) \frac{\partial \hat{v}}{\partial \xi} + \left(-\frac{\partial y}{\partial \xi} \hat{A} + \frac{\partial x}{\partial \xi} \hat{B} \right) \frac{\partial \hat{v}}{\partial \eta} + J_K \hat{C} \hat{v}, \hat{v} \right) (\hat{g}_{ij}), \end{cases}$$

we have to evaluate expressions of the form

$$\sum_{i,j=1}^{m} \hat{\omega}_{ij} \left(\hat{E} \frac{\partial \hat{v}}{\partial \xi}, \hat{v} \right) (\hat{g}_{ij}), \qquad \hat{v} \in \hat{X}^p,$$

where

(5.2) the $p \times p$ matrix $\hat{E}$ is symmetric;

(5.3) the function $(\xi, \eta) \longrightarrow \hat{E}(\xi, \eta)$ belongs to $W^{1,\infty}(\hat{K}; L(\mathbf{R}^p))$.

We begin with the following result which is a mere restatement of Lemma 1.

LEMMA 6. *We have for all symmetric $p \times p$ matrix $\hat{E}_0$ with constant coefficients and for all $\hat{v} \in \hat{X}^p$*

$$(5.4)\qquad \sum_{i,j=1}^{m} \hat{\omega}_{ij} \left(\hat{E}_0 \frac{\partial \hat{v}}{\partial \xi}, \hat{v} \right) (\hat{g}_{ij}) = \frac{1}{2} \sum_{j=1}^{m} \hat{\omega}_j \{ (\hat{E}_0 \hat{v}, \hat{v})(1, \hat{g}_j) - (\hat{E}_0 \hat{v}, \hat{v})(-1, \hat{g}_j) \}.$$

Proof. Since $\hat{\pi}\hat{v}(\hat{g}_{ij}) = \hat{v}(\hat{g}_{ij})$, $1 \leqslant i, j \leqslant m$, and the quadrature formula (2.7) is exact for all polynomials $\hat{\varphi} \in Q_{2m-1}$, we have by using Lemma 1:

$$\sum_{i,j=1}^{m} \hat{\omega}_{ij} \left(\hat{E}_0 \frac{\partial \hat{v}}{\partial \xi}, \hat{v} \right) (\hat{g}_{ij}) = \int_{\hat{K}} \left(\hat{E}_0 \frac{\partial \hat{v}}{\partial \xi}, \hat{\pi}\hat{v} \right) d\xi\, d\eta$$

$$= \frac{1}{2} \int_{\hat{K}} \{ (\hat{E}_0 \hat{\pi}_\eta \hat{v}, \hat{\pi}_\eta \hat{v})(1, \eta) - (\hat{E}_0 \hat{\pi}_\eta \hat{v}, \hat{\pi}_\eta \hat{v})(-1, \eta) \}\, d\eta.$$

The result follows by noticing that $\hat{\pi}_\eta \hat{v}(\pm 1, \hat{g}_j) = \hat{v}(\pm 1, \hat{g}_j)$, $1 \leqslant j \leqslant m$. $\square$

When the matrix $\hat{E}$ has variable coefficients, we set

$$(5.5)\qquad |\hat{E}|_{1,\infty} = \sup_{(\xi,\eta) \in \hat{K}} \left\{ \left| \frac{\partial \hat{E}}{\partial \xi}(\xi, \eta) \right| + \left| \frac{\partial \hat{E}}{\partial \eta}(\xi, \eta) \right| \right\},$$

where, in (5.5), $|\cdot|$ denotes the matrix spectral norm.

LEMMA 7. *Assume that the hypotheses* (5.2) *and* (5.3) *hold. Then, for all* $\epsilon > 0$, *there exists a constant* $C(\epsilon) > 0$ *such that for all* $\hat{v} \in \hat{X}^p$

$$(5.6)\quad \begin{cases} \displaystyle\sum_{i,j=1}^{m} \hat{\omega}_{ij}\left(\hat{E}\frac{\partial\hat{v}}{\partial\xi}, \hat{v}\right)(\hat{g}_{ij}) \geqslant \frac{1}{2}\sum_{j=1}^{m} \hat{\omega}_j\{(\hat{E}\hat{v}, \hat{v})(1, \hat{g}_j) - (\hat{E}\hat{v}, \hat{v})(-1, \hat{g}_j)\} \\ \displaystyle - |\hat{E}|_{1,\infty}\left[\left(\frac{1}{2}+\epsilon\right)\sum_{j=1}^{m} \hat{\omega}_j\{|\hat{v}(1, \hat{g}_j)|^2 + |\hat{v}(-1, \hat{g}_j)|^2\}\right. \\ \displaystyle \qquad\qquad\qquad\qquad \left. + C(\epsilon)\sum_{i,j=1}^{m} \hat{\omega}_{ij}|\hat{v}(\hat{g}_{ij})|^2\right]. \end{cases}$$

Proof. Let $\hat{E}_0$ be a symmetric $p \times p$ matrix with constant coefficients. Using Lemma 6, we get

$$\sum_{i,j=1}^{m} \hat{\omega}_{ij}\left(\hat{E}\frac{\partial\hat{v}}{\partial\xi}, \hat{v}\right)(\hat{g}_{ij}) = \sum_{i,j=1}^{m} \hat{\omega}_{ij}\left\{\left(\hat{E}_0\frac{\partial\hat{v}}{\partial\xi}, \hat{v}\right) + \left((\hat{E} - \hat{E}_0)\frac{\partial\hat{v}}{\partial\xi}, \hat{v}\right)\right\}(\hat{g}_{ij})$$

$$= \frac{1}{2}\sum_{j=1}^{m} \hat{\omega}_j\{(\hat{E}_0\hat{v}, \hat{v})(1, \hat{g}_j) - (\hat{E}_0\hat{v}, \hat{v})(-1, \hat{g}_j)\}$$

$$+ \sum_{i,j=1}^{m} \hat{\omega}_{ij}\left((\hat{E} - \hat{E}_0)\frac{\partial\hat{v}}{\partial\xi}, \hat{v}\right)(\hat{g}_{ij});$$

and therefore,

$$(5.7)\quad \begin{cases} \displaystyle\sum_{i,j=1}^{m} \hat{\omega}_{ij}\left(\hat{E}\frac{\partial\hat{v}}{\partial\xi}, \hat{v}\right)(\hat{g}_{ij}) = \frac{1}{2}\sum_{j=1}^{m} \hat{\omega}_j\{(\hat{E}\hat{v}, \hat{v})(1, \hat{g}_j) - (\hat{E}\hat{v}, \hat{v})(-1, \hat{g}_j)\} \\ \displaystyle + \frac{1}{2}\sum_{j=1}^{m} \hat{\omega}_j\{((\hat{E}_0 - \hat{E})\hat{v}, \hat{v})(1, \hat{g}_j) - ((\hat{E}_0 - \hat{E})\hat{v}, \hat{v})(-1, \hat{g}_j)\} \\ \displaystyle + \sum_{i,j=1}^{m} \hat{\omega}_{ij}\left((\hat{E} - \hat{E}_0)\frac{\partial\hat{v}}{\partial\xi}, \hat{v}\right)(\hat{g}_{ij}). \end{cases}$$

Now we choose the matrix $\hat{E}_0$ so that

$$(5.8)\quad \sup_{(\xi,\eta)\in\hat{K}} |\hat{E}(\xi, \eta) - \hat{E}_0| \leqslant |\hat{E}|_{1,\infty},$$

(take for instance $\hat{E}_0 = \hat{E}(0)$). Then we have

$$(5.9)\quad \begin{cases} \displaystyle\left|\sum_{j=1}^{m} \hat{\omega}_j\{((\hat{E}_0 - \hat{E})\hat{v}, \hat{v})(1, \hat{g}_j) - ((\hat{E}_0 - \hat{E})\hat{v}, \hat{v})(-1, \hat{g}_j)\}\right| \\ \displaystyle \leqslant |\hat{E}|_{1,\infty}\sum_{j=1}^{m} \hat{\omega}_j\{|\hat{v}(1, \hat{g}_j)|^2 + |\hat{v}(-1, \hat{g}_j)|^2\}. \end{cases}$$

Next, we notice that the mapping

$$\varphi \longrightarrow \left\{ \varphi(1)^2 + \varphi(-1)^2 + \sum_{i=1}^{m} \hat{\omega}_i \varphi(\hat{g}_i)^2 \right\}^{1/2}$$

is a norm over the space p_m of all polynomials of degree $\leqslant m$ in one variable ξ. Hence we get for some constant $c_1 > 0$ and for all $\varphi \in p_m$

$$\sum_{i=1}^{m} \hat{\omega}_i \left| \frac{d\varphi}{d\xi}(\hat{g}_i) \right|^2 \leqslant c_1 \left\{ \varphi(1)^2 + \varphi(-1)^2 + \sum_{i=1}^{m} \hat{\omega}_i \varphi(\hat{g}_i)^2 \right\}.$$

Applying this inequality to each component of the function $\xi \longrightarrow \hat{v}(\xi, \hat{g}_j)$, multiplying by $\hat{\omega}_j$ and summing from $j = 1$ to $j = m$, we find

$$\sum_{i,j=1}^{m} \hat{\omega}_{ij} \left| \frac{\partial \hat{v}}{\partial \xi}(\hat{g}_{ij}) \right|^2$$

$$\leqslant C_1 \left\{ \sum_{j=1}^{m} \hat{\omega}_j \{ |\hat{v}(1, \hat{g}_j)|^2 + |\hat{v}(-1, \hat{g}_j)|^2 \} + \sum_{i,j=1}^{m} \hat{\omega}_{ij} |\hat{v}(\hat{g}_{ij})|^2 \right\}.$$

Therefore, there exists a constant $c_2 > 0$ such that

$$\left| \sum_{i,j=1}^{m} \hat{\omega}_{ij} \left((\hat{E} - \hat{E}_0) \frac{\partial \hat{v}}{\partial \xi}, \hat{v} \right)(\hat{g}_{ij}) \right|$$

$$\leqslant C_2 |\hat{E}|_{1,\infty} \left[\left(\sum_{j=1}^{m} \hat{\omega}_j \{ |\hat{v}(1, \hat{g}_j)|^2 + |\hat{v}(-1, \hat{g}_j)|^2 \} \right)^{1/2} + \left(\sum_{i,j=1}^{m} \hat{\omega}_{ij} |\hat{v}(\hat{g}_{ij})|^2 \right)^{1/2} \right]$$

$$\times \left(\sum_{i,j=1}^{m} \hat{\omega}_{ij} |\hat{v}(\hat{g}_{ij})|^2 \right)^{1/2}.$$

Using the inequality $c_2 ab \leqslant \epsilon a^2 + c_2^2 b^2/4\epsilon$, we obtain

$$(5.10) \quad \begin{cases} \left| \sum_{i,j=1}^{m} \hat{\omega}_{ij} \left((\hat{E} - \hat{E}_0) \frac{\partial \hat{v}}{\partial \xi}, \hat{v} \right)(\hat{g}_{ij}) \right| \\ \leqslant |\hat{E}|_{1,\infty} \left\{ \epsilon \sum_{j=1}^{m} \hat{\omega}_j \{ |\hat{v}(1, \hat{g}_j)|^2 + |\hat{v}(-1, \hat{g}_j)|^2 \} + C(\epsilon) \sum_{i,j=1}^{m} \hat{\omega}_{ij} |\hat{v}(\hat{g}_{ij})|^2 \right\}, \end{cases}$$

with $C(\epsilon) = C_2 + C_2^2/4\epsilon$.

The inequality (5.6) follows from (5.7), (5.9) and (5.10). □

In fact, under an additional assumption, one can improve the previous estimate.

LEMMA 8. *Assume that the hypotheses* (5.2) *and* (5.3) *hold. Assume in addition that there exists a symmetric* $p \times p$ *matrix* $\hat{E}_0$ *with constant coefficients which satisfy the conditions* (5.8) *and*

(5.11) $\hat{E}_0 - \hat{E}(1, \hat{g}_j) \geqslant \beta I, \quad \hat{E}(-1, \hat{g}_j) - \hat{E}_0 \geqslant \beta I, \quad 1 \leqslant j \leqslant m, \quad \beta > 0.$

Then, there exists a constant $C > 0$ *such that*

$$(5.12)\quad \begin{cases} \sum_{i,j=1}^{m} \hat{\omega}_{ij}\left(\hat{E}\frac{\partial \hat{v}}{\partial \xi}, \hat{v}\right)(\hat{g}_{ij}) \geqslant \frac{1}{2}\sum_{j=1}^{m} \hat{\omega}_j\{(\hat{E}, \hat{v}, \hat{v})(1, \hat{g}_j) - (\hat{E}\hat{v}, \hat{v})(-1, \hat{g}_j)\} \\ \qquad\qquad - c\,|\hat{E}|_{1,\infty} \sum_{i,j=1}^{m} \hat{\omega}_{ij}|\hat{v}(\hat{g}_{ij})|^2. \end{cases}$$

Proof. We start again from (5.7). Using the assumption (5.11), we have

$$(5.13)\quad \begin{cases} \sum_{j=1}^{m} \hat{\omega}_j\{((\hat{E}_0 - \hat{E})\hat{v}, \hat{v})(1, \hat{g}_j) - ((\hat{E}_0 - \hat{E})\hat{v}, \hat{v})(-1, \hat{g}_j)\} \\ \qquad \geqslant \beta \sum_{j=1}^{m} \hat{\omega}_j\{|\hat{v}(1, \hat{g}_j)|^2 + |\hat{v}(-1, \hat{g}_j)|^2\}. \end{cases}$$

Combining (5.7), (5.10) and (5.13), we obtain

$$\begin{aligned} \sum_{i,j=1} \hat{\omega}_{ij}\left(\hat{E}\frac{\partial v}{\partial \xi}, \hat{v}\right)(\hat{g}_{ij}) &\geqslant \frac{1}{2}\sum_{j=1}^{m} \hat{\omega}_j\{(\hat{E}\hat{v}, \hat{v})(1, \hat{g}_j) - (\hat{E}\hat{v}, \hat{v})(-1, \hat{g}_j)\} \\ &\quad + (\beta - \epsilon|\hat{E}|_{1,\infty})\sum_{j=1}^{m} \hat{\omega}_j\{|\hat{v}(1, \hat{g}_j)|^2 + |\hat{v}(-1, \hat{g}_j)|^2\} \\ &\quad - C(\epsilon)|\hat{E}|_{1,\infty} \sum_{i,j=1}^{m} \hat{\omega}_{ij}|\hat{v}(\hat{g}_{ij})|^2; \end{aligned}$$

and (5.12) follows by choosing $\epsilon = \beta^{-1}|\hat{E}|_{1,\infty}$. □

In the sequel, we shall use Lemma 8 with $\hat{E}_0 = \hat{E}(0)$.

6. A Collocation Method for Parabolic Equations. Let us now study the stability and convergence properties of the collocation method (2.10). For the sake of simplicity, we shall restrict ourselves to first order systems associated with parabolic problems. However, the analysis can be essentially carried out for other first order systems with only technical modifications.

Let us assume that the domain Ω is of the form

(6.1) $$\Omega = \{(x, y) \in \mathbf{R}^2; g_1(x) < y < g_2(x), 0 < x < R\}.$$

We then consider the parabolic equation with variable coefficients

(6.2) $$\frac{\partial u_1}{\partial x} - \frac{\partial}{\partial y}\left(a(x, y)\frac{\partial u_1}{\partial y}\right) + c(x, y)u_1 = f_1 \quad \text{in } \Omega$$

with the boundary conditions

(6.3) $$u_1(0, y) = u_1^0(y), \qquad g_1(0) < y < g_2(0),$$

$$(6.4) \qquad u_1(x, g_1(x)) = u_1(x, g_2(x)) = 0, \qquad 0 < x < R.$$

In (6.2), the functions a, c and f_1 are assumed to be continuous in $\overline{\Omega}$ with $a_1 \geqslant a(x, y) \geqslant a_0 > 0$.

Let us reformulate the equation (6.2) in terms of a first order system. We set

$$(6.5) \qquad u_2 = a(x, y)\frac{\partial u_1}{\partial y}, \quad u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad f = \begin{pmatrix} f_1 \\ 0 \end{pmatrix}.$$

Then (6.2) is equivalent to Eq. (1.1) with

$$(6.6) \qquad A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} c & 0 \\ 0 & \frac{1}{a} \end{pmatrix}.$$

By changing u_1 in $u_1 \exp(\lambda x)$, we may always assume $c(x, y) \geqslant c_0 > 0$ with c_0 arbitrarily large.

We next define the collocation method. Let $0 = x_0 < x_1 < x_i < \cdots < x_I = R$ be a subdivision of the interval $[0, R]$; we assume that each strip

$$B_{i+1/2} = \{(x, y) \in \overline{\Omega}; x_i \leqslant x \leqslant x_{i+1}\}, \qquad 0 \leqslant i \leqslant I - 1,$$

is partitioned into J quadrilaterals K as in Figure 1. We also assume *for convenience* that the lateral boundaries of Ω are polygonal lines so that every quadrilateral $K \in \mathcal{T}_h$ has straight sides.

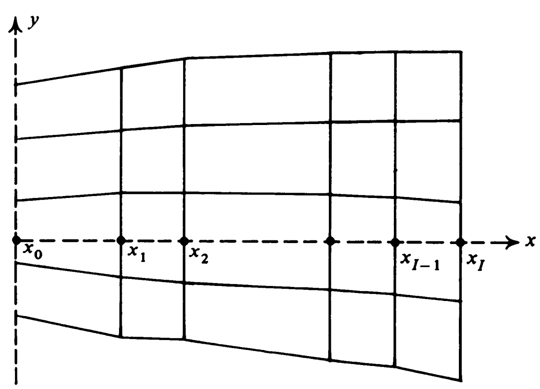

FIGURE 1

We set

$$(6.7) \qquad \hat{X} = Q_m,$$

$$(6.8) \qquad X_h = \{\varphi_h \in C^0(\overline{\Omega});\ \forall K \in \mathcal{T}_h,\ \varphi_{h|K} \in X_K\},$$

and we denote by X_h^0 the space of the traces over $x = 0$ of all functions of X_h. Notice that the functions of X_h^0 are continuous and piecewise polynomials of degree $\leqslant m$.

Now, given a function $u_h^0 = (u_{h,1}^0, u_{h,2}^0) \in (X_h^0)^2$, we want to find a function $u_h = (u_{h,1}, u_{h,2}) \in X_h^2$ which satisfies

(6.9) $$(Lu_h - f)(g_{ij}^K) = 0, \qquad 1 \leqslant i, j \leqslant m, K \in \mathcal{T}_h,$$

and the boundary conditions

(6.10) $$u_h(0, y) = u_h^0(y), \qquad g_1(0) \leqslant y \leqslant g_2(0),$$

(6.11) $$\begin{cases} u_{h,1} = 0 \text{ at the } m \text{ Gauss-Legendre points of each edge} \\ \text{of the lateral boundaries } y = g_i(x), i = 1, 2. \end{cases}$$

One can easily check that the problem (6.9)–(6.11) is equivalent to a linear system of $2m^2IJ$ equations in $2m^2IJ$ unknowns.

In order to study the stability properties of the collocation method (6.9)–(6.11), we first consider a quadrilateral K of $\mathcal{T}_h$ whose vertices are denoted by $S_i = (s_i, t_i)$, $1 \leqslant i \leqslant 4$ (see Figure 2). Let F_K: $(\xi, \eta) \longrightarrow (x, y) = F_K(\xi, \eta)$ be the transformation which maps $\hat{K}$ onto K. We set

(6.12) $$\gamma_K = \frac{\partial^2 y}{\partial \xi \partial \eta} = \frac{1}{4}(t_1 - t_2 + t_3 - t_4).$$

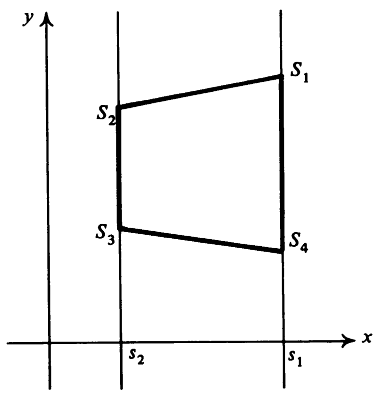

FIGURE 2

LEMMA 9. *Assume $\gamma_K \geqslant 0$. Then there exist two constants c_1 and $c_2 > 0$ independent of K such that for all $v \in X_K^2$*

(6.13)
$$\begin{aligned}
&\sum_{i,j=1}^{m} \omega_{ij}^K (Lv, v)(g_{ij}^K) \\
&\quad \geqslant \frac{1}{2} \sum_{j=1}^{m} \hat{\omega}_j \left\{ \left(\frac{\partial y}{\partial \eta} - c_1 \gamma_K \right) \hat{v}_1^2(1, \hat{g}_j) - \left(\frac{\partial y}{\partial \eta} + c_1 \gamma_K \right) \hat{v}_1^2(-1, \hat{g}_j) \right\} \\
&\quad - \frac{1}{2} \sum_{i=1}^{m} \hat{\omega}_i \left\{ \left(\frac{\partial y}{\partial \xi} \hat{v}_1^2 + 2 \frac{\partial x}{\partial \xi} \hat{v}_1 \hat{v}_2 \right)(\hat{g}_i, 1) - \left(\frac{\partial y}{\partial \xi} \hat{v}_1^2 + 2 \frac{\partial x}{\partial \xi} \hat{v}_1 \hat{v}_2 \right)(\hat{g}_i, -1) \right\} \\
&\quad + \sum_{i,j=1}^{m} \hat{\omega}_{ij} \left((c_0 J_K - c_2 \gamma_K) \hat{v}_1^2 + \frac{J_K}{a_1} \hat{v}_2^2 \right)(\hat{g}_{ij}).
\end{aligned}$$

Proof. We start from the expression (5.1), and we notice that $\partial x/\partial\eta = 0$. Applying Lemma 7 with $\hat{E} = \partial y\hat{A}/\partial\eta$ and $|\hat{E}|_{1,\infty} = \gamma_K$, we obtain for all $\epsilon > 0$

$$\begin{aligned}(6.14)\quad &\sum_{i,j=1}^{m} \hat{\omega}_{ij}\left(\left(\frac{\partial y}{\partial\eta}\hat{A} - \frac{\partial x}{\partial\eta}\hat{B}\right)\frac{\partial\hat{v}}{\partial\xi}, \hat{v}\right)(\hat{g}_{ij}) \\ &\geqslant \frac{1}{2}\sum_{j=1}^{m}\hat{\omega}_j\left\{\left(\left(\frac{\partial y}{\partial\eta} - (1+2\epsilon)\gamma_K\right)\hat{v}_1^2\right)(1,\hat{g}_j) - \left(\left(\frac{\partial y}{\partial\eta} + (1+2\epsilon)\gamma_K\right)\hat{v}_1^2\right)(-1,\hat{g}_j)\right\} \\ &\quad - C(\epsilon)\gamma_K\sum_{i,j=1}^{m}\hat{\omega}_{ij}\hat{v}_1^2(\hat{g}_{ij}).\end{aligned}$$

On the other hand, we have

$$-\sum_{i,j=1}^{m}\hat{\omega}_{ij}\left(\frac{\partial y}{\partial\xi}\hat{A}\frac{\partial\hat{v}}{\partial\eta}, \hat{v}\right)(\hat{g}_{ij}) = -\sum_{i,j=1}^{m}\hat{\omega}_{ij}\left(\frac{\partial y}{\partial\xi}\frac{\partial\hat{v}_1}{\partial\eta}\hat{v}_1\right)(\hat{g}_{ij}).$$

Since

$$-\left(\frac{\partial y}{\partial\xi}(0) - \frac{\partial y}{\partial\xi}(1)\right) = -\left(\frac{\partial y}{\partial\xi}(-1) - \frac{\partial y}{\partial\xi}(0)\right) = \frac{\partial^2 y}{\partial\xi\partial\eta} = \gamma_K \geqslant 0,$$

we may apply Lemma 8 with $\hat{E} = \partial y/\partial\xi$, $\hat{E}_0 = -\partial y(0)/\partial\xi$, $|\hat{E}|_{1,\infty} = \gamma_K$ and ξ replaced by η; we get for some constant $d > 0$

$$\begin{aligned}(6.15)\quad -\sum_{i,j=1}^{m}\hat{\omega}_{ij}\left(\frac{\partial y}{\partial\xi}\hat{A}\frac{\partial\hat{v}}{\partial\eta}, \hat{v}\right)(\hat{g}_{ij}) &\geqslant -\frac{1}{2}\sum_{i=1}^{m}\hat{\omega}_i\left\{\left(\frac{\partial y}{\partial\xi}\hat{v}_1^2\right)(\hat{g}_i, 1) - \left(\frac{\partial y}{\partial\xi}\hat{v}_1^2\right)(\hat{g}_i, -1)\right\} \\ &\quad - d\gamma_K\sum_{i,j=1}^{m}\hat{\omega}_{ij}\hat{v}_1^2(\hat{g}_{ij}).\end{aligned}$$

Next, since $\partial x/\partial\xi = s_2 - s_1$ and the matrix $\hat{B}$ has constant coefficients, we obtain by applying Lemma 6

$$\begin{aligned}(6.16)\quad &\sum_{i,j=1}^{m}\hat{\omega}_{ij}\left(\frac{\partial x}{\partial\xi}\hat{B}\frac{\partial\hat{v}}{\partial\eta}, \hat{v}\right)(\hat{g}_{ij}) \\ &= -\sum_{i=1}^{m}\hat{\omega}_i\left\{\left(\frac{\partial x}{\partial\xi}\hat{v}_1\hat{v}_2\right)(\hat{g}_i, 1) - \left(\frac{\partial x}{\partial\xi}\hat{v}_1\hat{v}_2\right)(\hat{g}_i, -1)\right\}.\end{aligned}$$

Finally, since $c(x, y) \geqslant c_0 > 0$, $a^{-1}(x, y) \geqslant a_1^{-1} > 0$, we find

$$(6.17)\quad \sum_{i,j=1}^{m}\hat{\omega}_{ij}(J_K C\hat{v}, \hat{v})(\hat{g}_{ij}) \geqslant \sum_{i,j=1}^{m}\hat{\omega}_{ij}\left(c_0 J_K\hat{v}_1^2 + \frac{1}{a_1}J_K\hat{v}_2^2\right)(\hat{g}_{ij}).$$

By combining the inequalities (6.14)–(6.17), we get the desired estimate (6.13) with $c_1 = 1 + 2\epsilon$, $c_2 = c(\epsilon) + d$. $\square$

Now, in order to get a weak stability result, we assume that there exists a constant $\delta > 0$ such that for all $K \in \mathcal{T}_h$

(6.18) $$0 \leqslant \gamma_K \leqslant \delta(s_1 - s_2)(t_2 - t_3).$$

The first inequality of (6.18) implies a very strong restriction on the domain Ω and on the geometry of the mesh. The function g_1 (respectively, g_2) must be increasing (respectively, decreasing) on the interval $[0, R]$, and we must have $|S_1S_4| \geqslant |S_2S_3|$ for all $K \in \mathcal{T}_h$. On the other hand, the second inequality of (6.18) means that each quadrilateral $K \in \mathcal{T}_h$ is an $O(h^2)$ perturbation of a parallelogram.

Next, we set

(6.19) $$\Omega_l = \{(x, y) \in \Omega; 0 < x < x_l\}, \quad \Sigma_l = \{(x, y) \in \overline{\Omega}; x = x_l\}, \quad 0 \leqslant l \leqslant I,$$

(6.20) $$h_{l+1/2} = x_{l+1} - x_l, \quad h_l = \tfrac{1}{2}(h_{l+1/2} + h_{l-1/2}) = \tfrac{1}{2}(x_{l+1} - x_{l-1}), \quad 1 \leqslant l \leqslant I - 1,$$

and we introduce the space W_h of functions $v_h \in X_h^2$ which vanish at the m Gauss-Legendre points of each edge of the lateral boundaries $y = g_i(x)$, $i = 1, 2$.

THEOREM 6. *Assume that* $(\mathcal{T}_h)$ *is a regular family of quadrangulations of* $\overline{\Omega}$ *and that the condition* (6.18) *holds. Then there exist two constants C and* $\alpha > 0$ *independent of h such that for all* $v \in W_h$ *and all* $l = 1, \dots, I$,

(6.21) $$\begin{aligned} 2 \sum_{K \subset \overline{\Omega}_l} \sum_{i,j=1}^{m} \omega_{ij}^K (Lv, v)(g_{ij}^K) &\geqslant (1 - ch_{l-1/2}) \|\pi_h^S v_1\|_{0,\Sigma_l}^2 - (1 + ch_{1/2}) \|\pi_h^S v_1\|_{0,\Sigma_0}^2 \\ &\quad - 2c \sum_{k=1}^{l-1} h_k \|\pi_h^S v_1\|_{0,\Sigma_k}^2 + \alpha \|\pi_h v\|_{0,\Omega_l}^2. \end{aligned}$$

Proof. Let K be a quadrilateral of $\mathcal{T}_h$ with vertices S_i, $1 \leqslant i \leqslant 4$. We start from the inequality (6.13). Using (6.18), we get

$$\begin{aligned} \sum_{j=1}^{m} \hat{\omega}_j \left(\frac{\partial y}{\partial \eta} - c_1 \gamma_K \right) \hat{v}_1^2(1, \hat{g}_j) &\geqslant (1 - c_3(s_1 - s_2)) \sum_{j=1}^{m} \hat{\omega}_j \left(\frac{\partial y}{\partial \eta} \hat{v}_1^2 \right)(1, \hat{g}_j) \\ &\geqslant (1 - c_3(s_1 - s_2)) \int_{t_4}^{t_1} (\pi_K^S v_1)^2 (s_1, y)\, dy, \end{aligned}$$

with $c_3 = c_1 \delta$. Similarly, we have

$$\sum_{j=1}^{m} \hat{\omega}_j \left(\frac{\partial y}{\partial \eta} + c_1 \gamma_K \right) \hat{v}_1^2(-1, \hat{g}_j) \geqslant (1 + c_3(s_1 - s_2)) \int_{t_3}^{t_2} (\pi_K^S v_1)^2 (s_2, y)\, dy.$$

On the other hand, using the notations of Figure 2, we obtain

$$\sum_{i=1}^{m} \hat{\omega}_i \left(\frac{\partial y}{\partial \xi} \hat{v}_1^2 + 2 \frac{\partial x}{\partial \xi} \hat{v}_1 \hat{v}_2 \right)(\hat{g}_i, 1) = \int_{S_1}^{S_2} ((\pi_K^S v_1)^2 + 2(\pi_K^S v_1)(\pi_K^S v_2)) n_y\, dS$$

and

$$\sum_{i=1}^{m} \hat{\omega}_i \left(\frac{\partial y}{\partial \xi} \hat{v}_1^2 + 2 \frac{\partial x}{\partial \xi} \hat{v}_1\hat{v}_2\right)(\hat{g}_i, -1) = \int_{S_3}^{S_4} ((\pi_K^S v_1)^2 + 2(\pi_K^S v_1)(\pi_K^S v_2))n_y \, dS.$$

Next, since (T_h) is a regular family of quadrangulations of $\overline{\Omega}$ and by (6.18), there exists a constant $c_4 > 0$ such that

$$\gamma_K \leqslant c_4 J_K.$$

The constant c_0 being assumed arbitrarily large, we have for some constant $\alpha > 0$

$$\sum_{i,j=1}^{m} \hat{\omega}_{ij} \left((c_0 J_K - c_2 \gamma_K)\hat{v}_1^2 + \frac{J_K}{a_1} \hat{v}_2^2\right)(\hat{g}_{ij})$$

$$\geqslant \sum_{i,j=1}^{m} \hat{\omega}_{ij} J_K \left((c_0 - c_2 c_4)\hat{v}_1^2 + \frac{1}{a_1} \hat{v}_2^2\right)(\hat{g}_{ij}) \geqslant \frac{\alpha}{2} \|\pi_K v\|_{0,K}^2.$$

Thus, summing (6.13) all over the quadrilaterals K of $\overline{\Omega}_l$ and using the above estimates, we get, since $\pi_h^S v_1$ vanishes on the lateral boundaries $y = g_i(x)$, $i = 1, 2$,

$$\sum_{K \subset \overline{\Omega}_l} \sum_{i,j=1}^{m} \omega_{ij}^K (Lv, v)(g_{ij}^K)$$

$$\geqslant \frac{1}{2} \sum_{k=1}^{l} \{(1 - c_3 h_{k-1/2})\|\pi_h^S v_1\|_{0,\Sigma_k}^2 - (1 + c_3 h_{k-1/2})\|\pi_h^S v_1\|_{0,\Sigma_{k-1}}^2\}$$

$$+ \frac{\alpha}{2} \|\pi_h v\|_{0,\Omega_l}^2,$$

from which the desired inequality (6.21) follows with $c = c_3$. □

Let us now state the following analogue of Gronwall's lemma whose elementary proof is left to the reader.

LEMMA 10. *Let $(\varphi_l)_{l \geqslant 0}$ be a sequence of nonnegative numbers such that for all $l \geqslant 1$*

$$(6.22) \qquad (1 - ch_{l-1/2})\varphi_l \leqslant (1 + ch_{1/2})\varphi_0 + 2c \sum_{k=1}^{l-1} h_k \varphi_k + \sum_{k=0}^{l-1} \psi_{k+1/2}$$

with $c > 0$, $h_{l-1/2} < 1/c$ and $\psi_{k+1/2} \in \mathbf{R}$. Then we have the estimate

$$(6.23) \qquad \begin{aligned} \varphi_l \leqslant & \left(\prod_{j=0}^{l-1} \frac{1 + ch_{j+1/2}}{1 - ch_{j+1/2}}\right)\varphi_0 \\ & + \sum_{k=0}^{l-1} \frac{1}{1 - ch_{k+1/2}} \left(\prod_{j=k+1}^{l-1} \frac{1 + ch_{j+1/2}}{1 - ch_{j+1/2}}\right)\psi_{k+1/2}. \end{aligned}$$

If $h_{l-1/2} \leqslant h^0 < 1/c$ for all $l \geqslant 1$, the inequality (6.23) implies that, for some constant $\lambda = \lambda(h^0) > 0$, we have

$$\varphi_l \leqslant \varphi_0 \exp(\lambda x_l) + \sum_{k=0} \psi_{k+1/2} \exp(\lambda(x_k - x_l)). \tag{6.24}$$

Let us prove the following result:

THEOREM 7. *Assume the hypotheses of Theorem* 6. *Then, if* u_h *is a solution of problem* (6.9)–(6.11), *we have the error bound for all* $l = 1, \ldots, L$,

$$\|\pi_h^S(u_{h,1} - u_1)\|_{0,\Sigma_l} + \|\pi_h(u_h - u)\|_{0,\Omega_l}$$

$$\leqslant \inf_{\substack{v_h \in W_h \\ v_{h,1} = u^0_{h,1} \text{ on } \Sigma_0}} \left\{ C\left(\sum_{K \subset \overline{\Omega}_l} \left| \sup_{w \in Y_K^2} \frac{\sum_{i,j=1}^m \omega_{ij}^K (L(u - v_h), w)(g_{ij}^K)}{\|\pi_K w\|_{0,K}} \right|^2 \right)^{1/2} + \|\pi_h^S(u_1 - v_{h,1})\|_{0,\Sigma_l} + \|\pi_h(u - v_h)\|_{0,\Omega_l} \right\}. \tag{6.25}$$

Proof. Let v_h be in W_h; using (6.9), we have

$$\sum_{K \subset \Omega_l} \sum_{i,j=1}^m \omega_{ij}^K (L(u_h - v_h), u_h - v_h)(g_{ij}^K)$$

$$= \sum_{K \subset \overline{\Omega}_l} \sum_{i,j=1}^m \omega_{ij}^K (L(u - v_h), u_h - v_h)(g_{ij}^K)$$

$$\leqslant \left(\sum_{K \subset \overline{\Omega}_l} \epsilon_K^2 \right)^{1/2} \|\pi_h(u_h - v_h)\|_{0,\Omega_l} \leqslant \frac{1}{\alpha} \sum_{K \subset \overline{\Omega}_l} \epsilon_K^2 + \frac{\alpha}{4} \|\pi_h(u_h - v_h)\|_{0,\Omega_l}^2,$$

with

$$\epsilon_K = \sup_{w \in Y_K^2} \frac{\sum_{i,j=1}^m \omega_{ij}^K (L(u - v_h), w)(g_{ij}^K)}{\|\pi_K w\|_{0,K}}.$$

Hence, by Theorem 6, we obtain for all $l = 1, \ldots, I$,

$$(1 - ch_{l-1/2})\|\pi_h^S(u_{h,1} - v_{h,1})\|_{0,\Sigma_l}^2 + \frac{\alpha}{2} \|\pi_h(u_h - v_h)\|_{0,\Omega_l}^2$$

$$\leqslant (1 + ch_{1/2})\|\pi_h^S(u_{h,1} - v_{h,1})\|_{0,\Sigma_0}^2$$

$$+ 2c \sum_{k=1}^{l-1} h_k \|\pi_h^S(u_{h,1} - v_{h,1})\|_{0,\Sigma_k}^2 + \frac{2}{\alpha} \sum_{K \subset \overline{\Omega}_l} \epsilon_K^2.$$

Next, we use (6.24) with

$$\varphi_l = \|\pi_h^S(u_{h,1} - v_{h,1})\|_{0,\Sigma_l}^2,$$

$$\psi_{k+1/2} = \frac{2}{\alpha} \sum_{K \subset B_{k+1/2}} \epsilon_K^2 - \frac{\alpha}{2} \|\pi_h(u_h - v_h)\|_{0,B_{k+1/2}}^2.$$

We get, if $v_{h,1} = u^0_{h,1}$ on Σ_0, and for $h_{l-1/2} \leqslant h^0$ small enough,

$$\|\pi^S_h(u_{h,1} - v_{h,1})\|_{0,\Sigma_l} + \|\pi_h(u_h - v_h)\|_{0,\Omega_l} \leqslant C\left(\sum_{K\subset\overline{\Omega}_l} \epsilon_K^2\right)^{1/2}$$

for some constant $C = C(\lambda, R)$ and the inequality (6.25) follows by the triangle inequality. □

In order to get a more precise error bound, we need to prove some analogue of Lemma 2. Let the interpolation operator r_h be defined as in (4.5). We have

LEMMA 11. *Assume the hypotheses of Theorem* 6. *Then we get for all* $v \in (H^{m+2}(\Omega))^2$ *and all* $l = 1, \dots, I$,

$$\text{(6.26)}\quad \left(\sum_{K\subset\overline{\Omega}_l}\left|\sup_{w\in Y_K^2}\frac{\sum_{i,j=1}^m \omega_{ij}^K(L(v - r_h v), w)(g_{ij}^K)}{\|\pi_K w\|_{0,K}}\right|^2\right)^{1/2} \leqslant Ch^{m+1}\|v\|_{m+2,\Omega_l}.$$

Proof. We only sketch the proof (for details, we refer to [9]). Consider the continuous linear functional on $H^{m+2}(\hat K)$

$$\hat\varphi \longrightarrow \frac{\partial}{\partial\xi}(\hat\varphi - \hat r\hat\varphi)(\hat g_{ij}).$$

As in the proof of Lemma 2, it vanishes over the spaces Q_m and P_{m+1}. Now, one can check that the seminorm

$$\hat\varphi \mapsto [\hat\varphi]_{m+2,\hat K} + \left[\frac{\partial^2\hat\varphi}{\partial\xi\partial\eta}\right]_{m,\hat K}, \qquad [\hat\varphi]_{l,\hat K} = \left\|\frac{\partial^l\hat\varphi}{\partial\xi^l}\right\|_{0,\hat K} + \left\|\frac{\partial^l\hat\varphi}{\partial\eta^l}\right\|_{0,\hat K},$$

is a norm on the quotient space $H^{m+2}(\hat K)/(Q_m \oplus P_{m+1})$ which is equivalent to the quotient norm. Therefore, we obtain

$$\left|\frac{\partial}{\partial\xi}(\hat\varphi - \hat r\hat\varphi)(\hat g_{ij})\right| \leqslant C\left([\hat\varphi]_{m+2,\hat K} + \left[\frac{\partial^2\hat\varphi}{\partial\xi\partial\eta}\right]_{m,\hat K}\right).$$

A similar estimate holds for the term $\partial(\hat\varphi - \hat r\hat\varphi)(\hat g_{ij})/\partial\eta$.

Next, let K be a quadrilateral of T_h and φ be in $H^{m+2}(K)$. Since (T_h) is a regular family of triangulations of $\overline{\Omega}$ and the condition (6.18) holds, we get by using the techniques of [3]

$$[\hat\varphi]_{m+2,\hat K} \leqslant C\left(\inf_{(\xi,\eta)\in\hat K} J_K(\xi,\eta)\right)^{-1/2} h_K^{m+2}|\varphi|_{m+2,K},$$

$$\left[\frac{\partial^2\hat\varphi}{\partial\xi\partial\eta}\right]_{m,\hat K} \leqslant C\left(\inf_{(\xi,\eta)\in\hat K} J_K(\xi,\eta)\right)^{-1/2}(h_K^m\gamma_K|\varphi|_{m+1,K} + h_K^{m+2}|\varphi|_{m+2,K})$$

$$\leqslant C\left(\inf_{(\xi,\eta)\in\hat K} J_K(\xi,\eta)\right)^{-1/2} h_K^{m+2}\|\varphi\|_{m+2,K},$$

so that

$$\left|\frac{\partial}{\partial \xi}(\hat{\varphi} - \hat{r}\hat{\varphi})(\hat{g}_{ij})\right| \leqslant C\left(\inf_{(\xi,\eta)\in\hat{K}} J_K(\xi, \eta)\right)^{-1/2} h_K^{m+2}\|\varphi\|_{m+2,K}.$$

Hence, we obtain for all $v \in (H^{m+2}(K))^2$ and all $w \in Y_K^2$

$$\sum_{i,j=1}^{m} \hat{\omega}_{ij}\left(\left(\frac{\partial y}{\partial \eta}\hat{A} - \frac{\partial x}{\partial \eta}\hat{B}\right)\frac{\partial}{\partial \xi}(\hat{v} - \hat{r}\hat{v}), \hat{w}\right)(\hat{g}_{ij})$$

$$\leqslant \left(\sum_{i,j=1}^{m} \hat{\omega}_{ij}\left|\left(\frac{\partial y}{\partial \eta}\hat{A} - \frac{\partial x}{\partial \eta}\hat{B}\right)\frac{\partial}{\partial \xi}(\hat{v} - \hat{r}\hat{v})(\hat{g}_{ij})\right|^2\right)^{1/2} \left(\sum_{i,j=1}^{m} \hat{\omega}_{ij}|\hat{w}(\hat{g}_{ij})|^2\right)^{1/2}$$

$$\leqslant ch_K^{m+1}\|v\|_{m+2,K}\|\pi_K w\|_{0,K}.$$

Likewise, we have

$$\sum_{i,j=1}^{m} \hat{\omega}_{ij}\left(\left(-\frac{\partial y}{\partial \xi}\hat{A} + \frac{\partial x}{\partial \xi}\hat{B}\right)\frac{\partial}{\partial \eta}(\hat{v} - \hat{r}\hat{v}), \hat{w}\right)(\hat{g}_{ij}) \leqslant Ch_K^{m+1}\|v\|_{m+2,K}\|\pi_K w\|_{0,K}$$

and

$$\sum_{i,j=1}^{m} \hat{\omega}_{ij}(J_K\hat{C}(\hat{v} - \hat{r}\hat{v}), \hat{w})(\hat{g}_{ij}) \leqslant Ch_K^{m+1}\|v\|_{m+1,K}\|\pi_K w\|_{0,K}.$$

Thus, using (5.1), we get

$$\left|\sum_{i,j=1}^{m} \omega_{ij}^K(L(v - r_K v), w)(g_{ij}^K)\right| \leqslant Ch_K^{m+1}\|v\|_{m+2,K}\|\pi_K w\|_{0,K}$$

and the inequality (6.26) follows easily. □

We are now able to prove the final result.

THEOREM 8. *Let (T_h) be a regular family of quadrangulations of $\overline{\Omega}$ which satisfy the condition* (6.18). *Then problem* (6.9) *has a unique solution $u_h \in X_h^2$. Moreover, assuming that $a \in W^{m+2,\infty}(\Omega)$ and $u_1 \in H^{m+3}(\Omega)$, we get the error estimate*

(6.27) $$\|\pi_h^S(u_{h,1} - u_1)\|_{0,\Sigma_l} + \|\pi_h(u_h - u)\|_{0,\Omega_l} \leqslant Ch^{m+1}\|u_1\|_{m+3,\Omega}, \quad l = 1, \dots, I.$$

Proof. Let us first prove the existence and uniqueness of the approximate solution u_h. Hence assume that $f_1 = 0$, $u_h^0 = 0$; we have to check that $u_h = 0$. Applying Theorem 6 with $l = 1$ and $v = u_h$, we get

$$\pi_h^S u_{h,1} = 0 \text{ on } \Sigma_1, \quad \pi_h u_h = 0 \text{ on } \Omega_1.$$

Therefore, $u_{h,1}$ vanishes at the m^2 Gauss-Legendre points of each quadrilateral $K \subset \overline{\Omega}_1$. Now, using the boundary condition (6.11), we obtain that $u_{h,1}$ vanishes also at the m Gauss-Legendre points of each lateral side of $K \subset \overline{\Omega}_1$. Since $u_{h,1} = 0$ on Σ_0, we get immediately $u_{h,1} = 0$ in Ω_1.

Let us next show that $u_{h,2} = 0$ in Ω_1. Using (6.9) with $f_1 = 0$, we have for all $K \subset \overline{\Omega}_1$,

$$\frac{\partial u_{h,2}}{\partial y}(g_{ij}^K) = \left(\frac{\partial u_{h,1}}{\partial x} + cu_{h,1}\right)(g_{ij}^K) = 0, \qquad 1 \leqslant i, j \leqslant m.$$

Since

$$u_{h,2}(g_{ij}^K) = \frac{\partial u_{h,2}}{\partial y}(g_{ij}^K) = 0, \qquad 1 \leqslant i, j \leqslant m,$$

we find again that $u_{h,2}$ vanishes at the m Gauss-Legendre points of each lateral side of $K \subset \overline{\Omega}_1$. Since $u_{h,2} = 0$ on Σ_0, we also get $u_{h,2} = 0$ in Ω_1.

Thus, we have proved that $u_h = 0$ in Ω_1. Using a recurrence argument, we obtain $u_h = 0$ in Ω.

Finally, let us assume that a belongs to $W^{m+2,\infty}(\Omega)$ and u_1 belongs to $H^{m+3}(\Omega)$. Then we have $u_2 \in H^{m+2}(\Omega)$. By applying Theorem 7 and Lemma 11 and by using the estimates

$$\|\pi_h(u - r_h u)\| \leqslant Ch^{m+1} \|u_1\|_{m+2,\Omega_l},$$

$$\|\pi_h^S(u_1 - r_h u_1)\|_{0,\Sigma_l} \leqslant Ch^{m+1} \|u_1\|_{m+2,\Omega_l},$$

we obtain the desired error bound (6.27). □

Remark 4. The previous results can be easily extended to the nonconforming elements introduced in Section 4.

Remark 5. The above results clearly generalize those of Keller [7]; we have derived higher order analogues of the box-scheme. Let us also notice that the convergence of the box-scheme can be analyzed in a completely different way by using the techniques of Baker [1].

Laboratoire de Calcul
Faculté des Sciences et des Techniques
Route de Gray
25030 Besançon, Cedex, France

Analyse Numérique
Université P. & M. Curie
4 Place Jussieu
75230 Paris, Cedex 05, France

1. G. BAKER, "A finite element method for first order hyperbolic equations," *Math. Comp.*, v. 29, 1975, pp. 995–1006.

2. P. G. CIARLET & P. A. RAVIART, "General Lagrange and Hermite interpolation in R^n with applications to finite element methods," *Arch. Rational Mech. Anal.*, v. 46, 1972, pp. 177–199.

3. P. G. CIARLET & P. A. RAVIART, "Interpolation theory over curved elements with applications to finite element methods," *Comput. Methods Appl. Mech. Engrg.*, v. 1, 1972, pp. 217–249.

4. K. O. FRIEDRICHS, "Symmetric positive differential equations," *Comm. Pure Appl. Math.*, v. 11, 1958, pp. 333–418.

5. B. L. HULME, "Discrete Galerkin and related one-step methods for ordinary differential equations," *Math. Comp.*, v. 26, 1972, pp. 881–891.

6. B. L. HULME, "One step piecewise polynomial Galerkin methods for initial value problems," *Math. Comp.*, v. 26, 1972, p. 118.

7. H. B. KELLER, "A new difference scheme for parabolic problems," *Numerical Solution of Partial Differential Equations*—II (B. Hubbard, Ed.), Academic Press, New York, 1971, 327–350.

8. K. D. LATHROP & B. G. CARLSON, "Transport theory. The method of discrete ordinates," *Computing Methods in Reactor Physics* (Greenspan, Kelerb, Okrent, Eds.), Gordon and Breach, New York, 1968, pp. 165–266.

9. P. LESAINT, *Sur la Résolution des Systèmes Hyperboliques du Premier Ordre par des Méthodes d'Éléments Finis*, Doctoral thesis, Paris, 1975.

10. P. LESAINT, "Finite element methods for symmetric hyperbolic equations," *Numer. Math.*, v. 21, 1973, pp. 244–255.

11. P. LESAINT, "Finite element methods for the transport equation," *R.A.I.R.O. Sér. Math. R2*, 1974, pp. 67–94.

12. P. LESAINT & P. A. RAVIART, "On a finite element method for solving the neutron transport equation," *Mathematical Aspects of Finite Elements in Partial Differential Equations* (C. de Boor, Ed.), Academic Press, New York, 1974, pp. 89–123.

13. N. K. MADSEN, "Convergent centered difference schemes for the discrete ordinate neutron transport equations," *SIAM J. Numer. Anal.*, v. 12, 1975, pp. 164–176.

14. P. A. RAVIART & J. M. THOMAS, "Primal hybrid finite element methods for second order elliptic equations," *Math. Comp.*, v. 31, 1977, pp. 391–413.