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Semidiscretization in Time for Parabolic Problems

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Abstract. We study the error to the discretization in time of a parabolic evolution equation by a single-step method or by a multistep method when the initial condition is not regular.

Introduction. The problem we are considering is the parabolic evolution equation

$$(*) \qquad \begin{cases} u'(t) + Au(t) = 0, & 0 < t \leqslant T, \\ u(0) = u_0. \end{cases}$$

Here, A is a linear operator, unbounded on Hilbert space H, of domain $D(A)$ dense in H; the initial value u_0 is assumed to be only in H.

In the first part, we study the error due to the discretization in time of the problem $(*)$ by a single-step method. The scheme is defined by the choice of a rational approximation $r(z)$ to the exponential e^{-z} for complex variable z. For the case of A selfadjoint, these methods are analyzed in [1] and [2]. Also in the special case of one space dimension, similar results can be found in [9]. For the case of A nonselfadjoint, the result for the special case $r(z) = 1/(1 + z)$ was obtained by Blair [3] and by Fujita and Mizutani [6]. Using the technique in [1], we generalize these results when the method is strongly $A(\theta)$-stable $(0 < \theta \leqslant \pi/2)$. Concerning examples, a class of rational approximations $\{r_p(z)\}$ to e^{-z} which are strongly $A(0)$-stable with $p \geqslant 3$ is documented in [8] and [2]. It is shown in [8] that for $p \geqslant 3$, r_p is in fact strongly $A(\theta_p)$-stable for some $0 < \theta_p < \pi/2$. For small p, θ_p is close to $\pi/2$ and in the special cases $p = 3, 4$, r_p is A-stable. Examples of rational approximations to e^{-z} which are strongly $A(\theta)$-stable with $r(\infty) = 0$ are provided by the family $r_\nu(z)$ developed in [2].

In the second part, we investigate error estimates when the discretization in time is carried out by means of a multistep method. Zlamal gives an error bound under the assumption that the operator A is selfadjoint and the method strongly $A(0)$-stable. Here, error estimates are obtained if the operator A is maximal sectorial and the method strongly $A(\theta)$-stable $(0 \leqslant \theta \leqslant \pi/2)$.

I. Semidiscretization in Time by a Single-Step Method.

1. *Introduction.* Let A be a linear operator, unbounded on Hilbert space H,

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of domain $D(A)$ dense in H. A is supposed to be maximal sectorial [7]; and we consider the problem

$$(1)\qquad \begin{cases} u'(t) + Au(t) = 0, & 0 < t \leqslant T, \\ u(0) = u_0, \end{cases}$$

where u_0 belongs to H. The approximate values u_n of u at the time level $t_n = n\Delta t$, (Δt denotes the time increment) are determined by

$$(2)\qquad u_{n+1} = r(\Delta tA)u_n, \qquad n \geqslant 0,$$

where $r(z)$ is a rational function of the complex variable z defining the single-step method. The error at the time t_n is given by

$$|u_n - u(t_n)|_H = |(e^{-t_nA} - r^n(\Delta tA))u_0|_H;$$

hence, we deduce

$$(3)\qquad |u_n - u(t_n)|_H \leqslant \|e^{-t_nA} - r^n(\Delta tA)\|_{\mathfrak{L}(H,H)}|u_0|_H.$$

2. *Assumptions on the Method.* We assume that the single-step method is of the order p ($p \geqslant 1$); then there are two constants $\sigma_1 > 0$ and $C > 0$ such that

$$|e^{-z} - r(z)| \leqslant C|z|^{p+1}, \quad \forall z \in \mathbf{C}, \ |z| \leqslant \sigma_1.$$

LEMMA 1.1. *Let the single-step method be of the order p, then, for any $\theta \in [0, \pi/2]$, there are constants σ, β and $c > 0$ depending only on r and θ such that for $z \in \mathbf{C}$, $|z| \leqslant \sigma$, $-\theta \leqslant \operatorname{Arg} z \leqslant +\theta$,*

$$(4)\qquad |r^n(z) - e^{-nz}| \leqslant Cn|z|^{p+1}e^{-\beta n \operatorname{Re} z}.$$

Proof. We have the equality

$$|r^n(z) - e^{-nz}| = |r(z) - e^{-z}| \left| \sum_{j=0}^{n-1} r^j(z)e^{-(n-1-j)z} \right|.$$

Since the method is assumed of the order p, we have

$$|r(z) - e^{-z}| \leqslant C|z|^{p+1}$$

and,

$$|r(z)| \leqslant e^{-\operatorname{Re} z}(1 + C|z|^{p+1}), \quad \forall z \in \mathbf{C}, \ |z| \leqslant \sigma_1.$$

Let $z \in \mathbf{C}$ such that $-\theta \leqslant \operatorname{Arg} z \leqslant +\theta$, $\theta \in [0, \pi/2]$, then, $\operatorname{Re} z \geqslant |z| \cos \theta$; hence

$$|r(z)| \leqslant e^{-\frac{1}{2}\operatorname{Re} z}(e^{-\frac{1}{2}|z|\cos\theta + C|z|^{p+1}})$$

and there is a constant $\sigma \leqslant \sigma_1$ such that

$$C|z|^{p+1} \leqslant \tfrac{1}{2}|z|\cos\theta, \qquad |z| \leqslant \sigma.$$

Therefore,

$$|r(z)| \leqslant e^{-\frac{1}{2}\operatorname{Re} z} \quad \text{for } |z| \leqslant \sigma, -\theta \leqslant \operatorname{Arg} z \leqslant +\theta$$

and

$$|r^n(z) - e^{-nz}| \leqslant C|z|^{p+1} \sum_{j=0}^{n-1} e^{-j\,\mathrm{Re}\,z/2} e^{-(n-1-j)\mathrm{Re}\,z}.$$

Hence,

$$|r^n(z) - e^{-nz}| \leqslant Cn|z|^{p+1} e^{-n\,\mathrm{Re}\,z/2}.$$

This concludes the proof.

We also assume that the method is strongly $A(\theta)$-stable, i.e. if θ is not zero, $\forall z \in S_\theta$, $|r(z)| \leqslant 1$ and $|r(\infty)| < 1$, where S_θ is the sector $\{z \in \overline{\mathbf{C}}/z = \infty$ or 0 or $-\theta \leqslant \operatorname{Arg} z \leqslant \theta\}$; if $\theta = 0, \forall x > 0$, $|r(x)| < 1$ and $|r(\infty)| < 1$.

3. *Error Estimates in the Case of a Selfadjoint Positive Operator.*

THEOREM 1.1. *Let A be a selfadjoint positive operator and let the single-step method be strongly $A(0)$-stable and of the order p. Then, there is a constant C depending only on the method such that,*

$$|u_n - u(t_n)|_H \leqslant C\frac{\Delta t^p}{t_n^p} |u_0|_H \quad \textit{for } n \geqslant 1. \tag{5}$$

Proof. Since the operator A is selfadjoint, we have

$$\|e^{-t_nA} - r^n(\Delta tA)\|_{\mathfrak{L}(H,H)} = \sup_{z\in S_p(A)} |e^{-t_nz} - r^n(\Delta tz)| \leqslant \sup_{x\geqslant 0} |e^{-nx} - r^n(x)|.$$

Let $x \in \mathbf{R}_+$ such that $x \leqslant \sigma$ (σ has the value defined in the Lemma 1.1); we get

$$|e^{-nx} - r^n(x)| \leqslant Cnx^{p+1}e^{-\beta nx} \leqslant \frac{C}{n^p} = C\frac{\Delta t^p}{t_n^p}.$$

Let $x \in \mathbf{R}_+$ such that $x \geqslant \sigma$; then

$$|e^{-nx} - r^n(x)| \leqslant e^{-n\sigma} + \sup_{x\geqslant\sigma} |r(x)|^n.$$

Since the method is strongly $A(0)$-stable,

$$\sup_{x\geqslant\sigma} |r(x)| = r < 1;$$

hence,

$$|e^{-nx} - r^n(x)| \leqslant e^{-n\sigma} + r^n \leqslant \frac{C}{n^p} = C\frac{\Delta t^p}{t_n^p}.$$

Then we get

$$\|e^{-t_nA} - r^n(\Delta tA)\|_{\mathfrak{L}(H,H)} \leqslant C\frac{\Delta t^p}{t_n^p},$$

and the result follows from (3).

4. *Error Estimates When A is Not a Selfadjoint Operator.* In this case, we shall need the following lemma:

LEMMA 1.2. *Let A be a maximal positive operator for which there is some constant θ_0 $(0 \leqslant \theta_0 \leqslant \pi/2)$ such that*

$$\forall u \in D(A), \quad (Au, u) \in S_{\theta_0}.$$

Let φ be a continuous function on the sector S_θ $(\theta_0 < \theta < \pi/2)$ which is holomorphic in the interior of S_θ and satisfies for some constant $R > 0$ and two functions f_1 and f_2 from $\mathbf{R}_+$ to $\mathbf{R}_+$ the following estimates:

$$\forall z \in S_\theta, \ |z| \leqslant R, \quad |\varphi(z)| \leqslant f_1(|z|), \tag{6}$$

$$\forall z \in S_\theta, \ |z| \geqslant R, \quad |\varphi(z) - \varphi(\infty)| \leqslant f_2(|z|). \tag{7}$$

Further, assume that the functions f_1, f_2 satisfy

$$\int_0^R f_1(r)\frac{dr}{r} < +\infty \quad \textit{and} \quad \int_R^{+\infty} f_2(r)\frac{dr}{r} < +\infty.$$

Then there is a constant C such that

$$\begin{aligned} \|\varphi(A)\|_{\mathfrak{L}(H,H)} \leqslant \frac{C}{\theta - \theta_0}&\left\{\int_0^R f_1(r)\frac{dr}{r} + \int_R^\infty f_2(r)\frac{dr}{r} + \left(R + \frac{1}{R}\right)|\varphi(\infty)|\right\} \\ &+ |\varphi(\infty)|. \end{aligned} \tag{8}$$

Proof. We set

$$h(z) = \varphi(z) - \frac{z}{1+z}\varphi(\infty).$$

We have

$$\varphi(A) = h(A) + \varphi(\infty)A(I + A)^{-1} \quad \text{and} \quad \|A(I + A)^{-1}\|_{\mathfrak{L}(H,H)} \leqslant 1.$$

Hence,

$$\|\varphi(A)\|_{\mathfrak{L}(H,H)} \leqslant \|h(A)\|_{\mathfrak{L}(H,H)} + |\varphi(\infty)|.$$

Besides, we have

$$h(A) = \frac{1}{2\pi i}\int_\Gamma h(z)(zI - A)^{-1}\,dz,$$

where Γ is the continuous, positively oriented curve defined by $\operatorname{Arg} z = \pm\theta$. Let $\Gamma_1 = \{z \in \Gamma, |z| \leqslant R\}$ and $\Gamma_2 = \{z \in \Gamma, |z| \geqslant R\}$. For $z \in \Gamma$, the following estimate holds [5]

$$\|(zI - A)^{-1}\|_{\mathfrak{L}(H,H)} \leqslant \frac{C}{\theta - \theta_0}\frac{1}{|z|}.$$

Now, from (6), we get

$$|h(z)| = \left| \varphi(z) - \frac{z}{1+z}\varphi(\infty) \right| \leqslant f_1(|z|) + |z||\varphi(\infty)|, \quad \forall z \in \Gamma_1;$$

hence

$$\left\| \frac{1}{2\pi i} \int_{\Gamma_1} h(z)(zI - A)^{-1}\, dz \right\| \leqslant \frac{C}{\theta - \theta_0} \left(\int_0^R f_1(r)\, \frac{dr}{r} + R|\varphi(\infty)| \right);$$

from (7), we get

$$|h(z)| = \left| \varphi(z) - \varphi(\infty) + \frac{1}{1+z}\varphi(\infty) \right| \leqslant f_2(|z|) + \frac{1}{|z|}|\varphi(\infty)|, \quad \forall z \in \Gamma_2;$$

hence,

$$\frac{1}{2\pi i} \int_{\Gamma_2} h(z)(zI - A)^{-1}\, dz \leqslant \frac{C}{\theta - \theta_0} \left(\int_R^\infty f_2(r)\, \frac{dr}{r} + \frac{1}{R}\, |\varphi(\infty)| \right).$$

The estimate (8) now follows immediately.

THEOREM 1.2. *Let A be a maximal positive operator satisfying for some constant θ_0 $(0 \leqslant \theta_0 < \pi/2)$,*

$$\forall u \in D(A), \quad (Au, u) \in S_{\theta_0}.$$

Further, assume that the single-step method is of the order p and strongly $A(\theta)$-stable $(\theta_0 < \theta < \pi/2)$. Then there is a constant C depending only on the single-step method θ and θ_0 such that

$$|u_n - u(t_n)|_H \leqslant C\, \frac{\Delta t^p}{t_n^p}\, |u_0|_H. \tag{9}$$

Proof. We apply Lemma 1.2 with $\varphi(z) = e^{-nz} - r^n(z)$. Then, from (4), we get

$$\forall z \in S_\theta, \; |z| \leqslant \sigma, \quad |\varphi(z)| \leqslant Cn|z|^{p+1} e^{-\beta n |z| \cos\theta}$$

and

$$\int_0^{+\infty} n r^p e^{-\beta n r \cos\theta} dr \leqslant \frac{C}{n^p} \int_0^\infty x^p e^{-x}\, dx \leqslant C'\, \frac{\Delta t^p}{t_n^p}.$$

Besides, we have

$$\forall z \in S_\theta, \; |z| \geqslant \sigma, \quad |\varphi(z) - \varphi(\infty)| \leqslant e^{-n|z|\cos\theta} + |r^n(z) - r^n(\infty)|$$

and

$$r^n(z) - r^n(\infty) = (r(z) - r(\infty)) \sum_{j=0}^{n-1} r^j(z) r^{n-1-j}(\infty).$$

Since the method is strongly $A(\theta)$-stable, we may set

$$\sup_{z \in S_\theta;|z|\geq\sigma} |r(z)| = e^{-\delta}$$

for some $\delta > 0$; and since r is a rational function, there is a constant C such that

$$|r(z) - r(\infty)| \leq \frac{C}{|z|} \quad \text{for } |z| \geq \sigma.$$

Hence, for $|z| \geq \sigma$

$$|\varphi(z) - \varphi(\infty)| \leq e^{-n|z|\cos\theta} + \frac{C}{|z|} e^{-n\delta}$$

and

$$\int_\sigma^{+\infty} \left(\frac{e^{-nr\cos\theta}}{r} + C\frac{e^{-n\delta}}{r^2} \right) dr \leq C\left(\frac{1}{n^p} + e^{-n\delta} \right) \leq C' \frac{\Delta t^p}{t_n^p}.$$

Besides, $|\varphi(\infty)| = |r^n(\infty)|$ and $|r^n(\infty)| \leq C/n^p$, since $|r(\infty)| < 1$; then using (8), (9) follows.

II. Semidiscretization in Time by a Multistep Method.

1. *Introduction.* We again consider the equation (1). Let ρ and σ be two real polynomials of degree less than or equal to q,

$$\rho(\zeta) = \sum_{i=0}^{q} \alpha_i \zeta^i \quad \text{and} \quad \sigma(\zeta) = \sum_{i=0}^{q} \beta_i \zeta^i \qquad (\alpha_q > 0).$$

The approximate values u_n of u at the time level $t_n = n\Delta t$ are determined by

$$\sum_{i=0}^{q} (\alpha_i + \Delta t \beta_i A) u_{n+i} = 0, \tag{10}$$

assuming the starting values $u_0, u_1, \ldots, u_{q-1}$ to be given (by another method).

2. *Assumptions on the Method.* (a) We assume that the multistep method is of order p; then we have

$$\sum_{i=0}^{q} i^l \alpha_i = l \sum_{i=0}^{q} i^{l-1} \beta_i, \qquad l = 0, 1, \ldots, p. \tag{11}$$

(b) We also assume that the (ρ, σ) method is strongly $A(\theta)$-stable. We set $\overline{\omega}(\zeta; z) = \rho(\zeta) + z\sigma(\zeta)$,

$$S_\theta = \{z \in \overline{\mathbf{C}}/z = \infty \text{ or } z = 0 \text{ or } -\theta \leq \operatorname{Arg} z \leq +\theta\}.$$

The method is strongly $A(\theta)$-stable $(0 < \theta \leq \pi/2)$ if and only if the modulus of all roots of the polynomials $\overline{\omega}(\cdot, z)$ are less than one for any z in the interior of S_θ. If $\theta = 0$, the method is strongly $A(0)$-stable if and only if for any $x > 0$ the modulus of all roots of the polynomials $\overline{\omega}(\cdot, x)$ and σ are less than one; the roots of the polynomial ρ with modulus equal to one, ζ_i, are simple and the growth parameters λ_i satisfy Re $\lambda_i > 0$; these growth parameters λ_i are given by

$$\lambda_i = \sigma(\zeta_i)/\zeta_i \rho'(\zeta_i).$$

We now define

$$\delta_i(z) = \frac{\alpha_i + \beta_i z}{\alpha_q + \beta_q z} \qquad (0 \leqslant i \leqslant q),$$

$$\gamma_l(z) = 0 \quad \text{for } l < 0,$$

$$\gamma_0(z) = 1,$$

$$\sum_{k=0}^{q} \gamma_{l-k}(z)\delta_{q-k}(z) = 0 \quad \text{for } l > 0, l \in \mathbf{Z},$$

$$E_l(z) = \sum_{i=0}^{q} \delta_i(z) e^{-(l+i)z}, \qquad l \geqslant 0.$$

Then, we have

$$\sum_{i=0}^{q} \delta_i(\Delta t A)(u(t_{n+i}) - u_{n+i}) = E_n(\Delta t A)u_0.$$

Hence [5]

$$u(t_{n+q}) - u_{n+q} = \sum_{s=0}^{q-1} \sum_{k=0}^{s} \gamma_{n-k}(\Delta t A)\delta_{s-k}(\Delta t A)[u_s - u(t_s)]$$
$$+ \sum_{l=0}^{n} \gamma_{n-l}(\Delta t A)E_l(\Delta t A)u_0.$$

LEMMA 2.1. *Let the method* (ρ, σ) *be of the order* p *and* $A(\theta)$*-stable; then for any* $z \in \mathbf{C}$ *with* $\operatorname{Re} z \geqslant 0$, *we have*

(12) $$|E_l(z)| \leqslant C|z|^{p+1} e^{-l \operatorname{Re} z}, \qquad l \geqslant 0.$$

Proof. We set

$$v(t) = e^{-tz}, \qquad t \geqslant 0, \operatorname{Re} z \geqslant 0.$$

We have

$$E_l(z) = (\alpha_q + \beta_q z)^{-1} \left\{ \sum_{i=0}^{q} \alpha_i v(l+i) - \sum_{i=0}^{q} \beta_i v'(l+i) \right\}.$$

Since the method is of the order p, we get

$$E_l(z) = (\alpha_q + \beta_q z)^{-1} \left\{ \sum_{i=1}^{q} \alpha_i \int_l^{l+i} \frac{(l+i-t)^p}{p!} v^{(p+1)}(t)\,dt \right.$$
$$\left. - \sum_{i=0}^{q} \beta_i \int_l^{l+i} \frac{(l+i-t)^{p-1}}{(p-1)!} v^{(p+1)}(t)\,dt \right\}.$$

Now, $v^{(p+1)}(t) = (-z)^{p+1} e^{-tz}$ and, since the method is strongly $A(\theta)$-stable, we have $\alpha_q \beta_q > 0$. Hence,

$$|E_l(z)| \leqslant \frac{1}{|\alpha_q|}\left(\sum_{i=1}^{q} \frac{|\alpha_i|}{p!} i^p + \sum_{i=0}^{q} \frac{|\beta_i|}{(p-1)!} i^{p-1}\right)|z|^{p+1} e^{-l \operatorname{Re} z}.$$

LEMMA 2.2. *Let the method (ρ, σ) be strongly $A(\theta)$-stable $(0 \leqslant \theta \leqslant \pi/2)$. Let $z \in \mathbf{C}$ such that $-\theta_1 \leqslant \operatorname{Arg} z \leqslant \theta_1$ $(0 \leqslant \theta_1 < \theta)$ if $\theta > 0$ and $z \in \mathbf{R}$, $z \geqslant 0$, if $\theta = 0$. Then, there are constants R, μ, C, such that*

$$\text{(13)} \qquad \begin{cases} |\gamma_l(z)| \leqslant C e^{-\mu l} & \text{for } |z| \geqslant R, \\ |\gamma_l(z)| \leqslant C e^{-\mu l |z|} & \text{for } |z| \leqslant R. \end{cases}$$

Proof. The first inequality follows from the following result [5]. If the method (ρ, σ) is strongly $A(\theta)$-stable, there are constants $c \in]0, 1[$, $R > 0$ and $c_\infty > 0$ such that

$$\forall z \in S_\theta,\ |z| \geqslant R, \quad |\gamma_l(z)| \leqslant c_\infty c^l.$$

The second inequality follows from the following result [5]. If the method (ρ, σ) is strongly $A(\theta)$-stable $(0 \leqslant \theta \leqslant \pi/2)$, for any $a > 0$, there are two constants μ' and $C > 0$ such that, for any $x \in \mathbf{R}_+$, $x \leqslant a$ and for any $\lambda \in \Gamma_\theta$, ($\Gamma_\theta = \{z \in \overline{\mathbf{C}}/z = \infty$ or $\operatorname{Arg} z = \pm\theta\}$), $|\gamma_l(x + \lambda)| \leqslant Ce^{-\mu' lx}$.

Case (i); $\theta > 0$. Let $z \in \mathbf{C}$ such that $|z| \leqslant R$, $\operatorname{Arg} z = \alpha$, $0 \leqslant \alpha \leqslant \theta_1$; then we have

$$z = |z| \frac{\sin(\theta - \alpha)}{\sin \theta} + |z| \frac{\sin \alpha}{\sin \theta} e^{i\theta},$$

and

$$|z| \frac{\sin(\theta - \alpha)}{\sin \theta} \leqslant R.$$

Hence,

$$|\gamma_l(z)| \leqslant Ce^{-\mu' l |z| \sin(\theta - \alpha)/\sin\theta}$$

that is,

$$|\gamma_l(z)| \leqslant Ce^{-\mu l |z|} \quad \text{with} \quad \mu = \mu' \frac{\sin(\theta - \theta_1)}{\sin \theta}.$$

Case (ii); $\theta = 0$. Let $x \leqslant R$; then there are two constants μ' and C such that $|\gamma_l(x)| \leqslant Ce^{-\mu' lx}$ and (13) follows.

3. *Error Estimates in the Case of a Selfadjoint Positive Operator.*

THEOREM 2.1. *Let A be a selfadjoint positive operator and let the (ρ, σ) method be strongly $A(0)$-stable and of the order p. Further, assume that the starting values are obtained by a single-step weakly $A(0)$-stable method of the order $p - 1$; then there is a constant C depending only on the (ρ, σ) method and on the single-step method such that*

$$\text{(14)} \qquad |u_n - u(t_n)|_H \leqslant C \frac{\Delta t^p}{t_n^p} |u_0|_H.$$

Proof. Since the operator A is selfadjoint and positive, we have

$$\left\| \sum_{l=0}^{n} \gamma_{n-l}(\Delta t A) E_l(\Delta t A) \right\|_{\mathfrak{L}(H,H)} \leqslant \sup_{x \geqslant 0} \left| \sum_{l=0}^{n} \gamma_{n-l}(x) E_l(x) \right| .$$

Let $x \in \mathbf{R}_+$, such that $x \leqslant R$ (R is defined in Lemma 2.2).

From Lemmas 2.1 and 2.2, we get

$$|\gamma_{n-l}(x)| \leqslant Ce^{-(n-l)x}, \qquad 0 \leqslant l \leqslant n,$$

and

$$|E_l(x)| \leqslant Cx^{p+1} e^{-lx}, \qquad l \geqslant 0.$$

Set $\nu = \inf(\mu, 1)$; then

$$\left| \sum_{l=0}^{n} \gamma_{n-l}(x) E_l(x) \right| \leqslant Cne^{-\nu n x} x^{p+1} \leqslant \frac{C}{n^p}.$$

Now, let $x \geqslant R$ and we have the estimates

$$|\gamma_{n-l}(x)| \leqslant Ce^{-\mu(n-l)}, \qquad 0 \leqslant l \leqslant n,$$
$$|E_l(x)| \leqslant Cx^{p+1} e^{-lx}, \qquad 1 \leqslant l \leqslant n,$$
$$|E_0(x)| \leqslant C.$$

Hence,

$$\left| \sum_{l=0}^{n} \gamma_{n-l}(x) E_l(x) \right| \leqslant C \left\{ e^{-\mu n} + \sum_{l=1}^{n} e^{-\mu(n-l)} e^{-lx} x^{p+1} \right\}.$$

Set $\nu = \inf(\mu, R)$; then

$$\left| \sum_{l=0}^{n} \gamma_{n-l}(x) E_l(x) \right| \leqslant C\{e^{-\mu n} + ne^{-\nu n} e^{-(x-\nu)} x^{p+1}\} \leqslant \frac{C}{n^p}.$$

Hence,

$$\left\| \sum_{l=0}^{n} \gamma_{n-l}(\Delta t A) E_l(\Delta t A) \right\|_{\mathfrak{L}(H,H)} \leqslant \frac{C}{n^p}.$$

Since we have assumed that the starting values are obtained by a weakly $A(0)$-stable single-step method of order $p-1$,

$$u_s = r^s(\Delta t A) u_0, \qquad 0 \leqslant s \leqslant q-1,$$

where r is a rational function satisfying $\forall x \geqslant 0$, $|r(x)| \leqslant 1$ and for which there are constants $\sigma > 0$ and $c > 0$ such that

$$|(r(x) - e^{-x}| \leqslant C|x|^p \quad \forall x \leqslant \sigma.$$

Since the operator A is selfadjoint and positive, we have

$$\|\gamma_{n-k}(\Delta t A)\delta_{s-k}(\Delta t A)(u_s - u(t_s))\| \leqslant \sup_{x \geqslant 0} |\gamma_{n-k}(x)\delta_{s-k}(x)(r^s(x) - e^{-sx})| |u_0|_H.$$

Let $x \in \mathbf{R}_+$, $x \leqslant \inf(\sigma, R)$; then

$$|\gamma_{n-k}(x)| \leqslant Ce^{-\mu(n-k)x} \leqslant Ce^{-\mu nx}e^{\mu qx}, \qquad |\delta_{s-k}(x)| \leqslant C,$$

and

$$|r^s(x) - e^{-sx}| \leqslant |r(x) - e^{-x}| \left| \sum_{j=0}^{s-1} r^j(x)e^{-(s-1-j)x} \right| \leqslant Cx^p.$$

Hence,

$$|\gamma_{n-k}(x)\delta_{s-k}(x)(r^s(x) - e^{-sx})| \leqslant Ce^{-\mu nx}x^p \leqslant \frac{C'}{n^p} = C' \frac{\Delta t^p}{t_n^p}.$$

Let $x \geqslant \inf(\sigma, R)$; then

$$|\gamma_{n-k}(x)| \leqslant Ce^{-\mu(n-k)} \leqslant Ce^{\mu q}e^{-\mu n},$$

so that

$$|\gamma_{n-k}(x)\delta_{s-k}(x)(r^s(x) - e^{-sx})| \leqslant Ce^{-\mu n} \leqslant \frac{C'}{n^p} = C' \frac{\Delta t^p}{t_n^p}.$$

Hence,

$$\left\| \sum_{s=0}^{q-1} \sum_{k=0}^{s} \gamma_{n-k}(\Delta tA)\delta_{s-k}(\Delta tA)(u_s - u(t_s)) \right\| \leqslant C \frac{\Delta t^p}{t_n^p} |u_0|_H.$$

4. *Error Estimates When A is Not a Selfadjoint Operator.*

THEOREM 2.2. *Let A be a maximal positive operator for which there is some constant θ_0 $(0 \leqslant \theta_0 < \pi/2)$ such that*

$$\forall u \in D(A), \quad (Au, u) \in S_{\theta_0}.$$

Let the (ρ, σ) method be strongly $A(\theta)$-stable $(\theta_0 < \theta < \pi/2)$ and of the order p. Further, assume that the starting values are obtained by a weakly $A(\theta)$-stable, single-step method of order $p - 1$; then there is a constant C depending only on the (ρ, σ) method, on the single-step method, on θ and θ_0 such that

$$|u_n - u(t_n)|_H \leqslant C \frac{\Delta t^p}{t_n^p} |u_0|_H. \tag{15}$$

Proof. We apply Lemma 1.2 to estimate

$$\left\| \sum_{l=0}^{n} \gamma_{n-l}(\Delta tA)E_l(\Delta tA) \right\|_{\mathfrak{L}(H,H)}.$$

From Lemmas 2.1 and 2.2, for any $z \in S_{\theta_1}$ and $|z| \leqslant R$ $(\theta_1 \in]\theta_0, \theta[)$, we have

$$\sum_{l=0}^{n} \gamma_{n-l}(z)E_l(z) \leqslant C \sum_{l=0}^{n} e^{-\mu(n-l)|z|}|z|^{p+1}e^{-l|z|\cos\theta_1}.$$

Set $\nu = \inf(\mu, \cos\theta_1)$; then

$$\left| \sum_{l=0}^{n} \gamma_{n-l}(z) E_l(z) \right| \leqslant Cne^{-\nu n|z|}|z|^{p+1}$$

and

$$\int_0^\infty ne^{-\nu n r} r^p \, dr = \frac{1}{n^p} \left(\int_0^\infty e^{-\nu\rho} \rho^p \, d\rho \right) \leqslant \frac{C}{n^p}.$$

Besides, for $l \geqslant 1$, $E_l(\infty) = 0$ and from (13) and (14), for $z \in S_{\theta_1}$, $|z| \geqslant R$

$$\left| \sum_{l=1}^{n} \gamma_{n-l}(z) E_l(z) \right| \leqslant C \sum_{l=1}^{n} e^{-\mu(n-l)} |z|^{p+1} e^{-l|z|\cos\theta_1}.$$

Now, set $\nu = \inf(\mu, R\cos\theta_1)$; then

$$e^{-\mu(n-l)} e^{-\nu|z|\cos\theta_1} \leqslant e^{-\nu n} e^{-l(|z|\cos\theta_1 - \nu)}$$

and

$$\left| \sum_{l=1}^{n} \gamma_{n-l}(z) E_l(z) \right| \leqslant Cne^{-\nu n} e^{\nu - |z|\cos\theta_1} |z|^{p+1}$$

and

$$\int_0^\infty ne^{-\nu n} e^{\nu - r\cos\theta_1} r^p \, dr \leqslant \frac{C}{n^p}.$$

For $l = 0$, $E_0(\infty)$ may be not equal to zero,

$$\lim_{|z|\to\infty; z\in S_\theta} E_0(z) = \frac{\beta_0}{\beta_q}.$$

We have

$$\left| \gamma_n(z) E_0(z) - \gamma_n(\infty) \frac{\beta_0}{\beta_q} \right| \leqslant |\gamma_n(z) - \gamma_n(\infty)||E_0(z)| + |E_0(z) - E_0(\infty)||\gamma_n(\infty)|.$$

Now, for $|z| \geqslant R$, we have [5]

$$|\gamma_n(z) - \gamma_n(\infty)| \leqslant C \frac{e^{-\mu n}}{1 + (\mathrm{Log}(1 + |z|))^2},$$

and $|E_0(z)| \leqslant C$. Also,

$$E_0(z) - E_0(\infty) = \frac{\alpha_0\beta_q - \alpha_q\beta_0}{\alpha_q + \beta_q z} + \sum_{i=1}^{q} \delta_i(z) e^{-iz};$$

hence, for $z \in S_{\theta_1}$ and $|z| \geqslant R$,

$$|E_0(z) - E_0(\infty)| \leqslant \frac{C}{|z|} + Ce^{-|z|\cos\theta_1}.$$

Then, since $|\gamma_n(\infty)| \leqslant Ce^{-\mu n}$, we get

$$\left|\gamma_n(z)E_0(z) - \gamma_n(\infty)\frac{\beta_0}{\beta_q}\right| \leqslant Ce^{-\mu n}\left\{\frac{1}{1 + (\mathrm{Log}(1 + |z|))^2} + \frac{1}{|z|} + e^{-|z|\cos\theta_1}\right\}$$

and

$$\int_R^{+\infty} e^{-\mu n}\left(\frac{1}{r(1 + \mathrm{Log}(1 + r))^2} + \frac{1}{r^2} + \frac{e^{-r\cos\theta_1}}{r}\right) dr \leqslant Ce^{-\mu n}.$$

Then, from Lemma 1.2, we get

$$\left\|\sum_{l=0}^{n} \gamma_{n-l}(\Delta tA)E_l(\Delta tA)\right\|_{\mathfrak{L}(H,H)} \leqslant C\left\{\frac{1}{n^p} + e^{-\mu n} + |\gamma_n(\infty)E_0(\infty)|\right\};$$

hence,

$$\left\|\sum_{l=0}^{n} \gamma_{n-l}(\Delta tA)E_l(\Delta tA)\right\|_{\mathfrak{L}(H,H)} \leqslant \frac{C}{n^p}.$$

Now, we assume that the starting values are obtained by a weakly $A(\theta)$-stable single-step method, of the order $p - 1$; then

$$u_s = r^s(\Delta tA)u_0, \qquad 0 \leqslant s \leqslant q - 1,$$

where r is a rational function satisfying $\forall z \in S_\theta$, $|r(z)| \leqslant 1$ and for which there are some constants σ and C such that

$$|r(z) - e^{-z}| \leqslant C|z|^p \quad \forall z \in S_\theta,\ |z| \leqslant \sigma.$$

We again apply Lemma 1.2 to estimate

$$\|\gamma_{n-k}(\Delta tA)\delta_{s-k}(\Delta tA)(r^s(\Delta tA) - e^{-s\Delta tA})\|_{\mathfrak{L}(H,H)}, \qquad 0 \leqslant k \leqslant s,\ 0 \leqslant s \leqslant q - 1.$$

Set $\sigma_1 = \inf(\sigma, R)$. Then for $z \in S_{\theta_1}$, $|z| \leqslant \sigma_1$ and $n > k$,

$$|\gamma_{n-k}(z)\delta_{s-k}(z)(r^s(z) - e^{-sz})| \leqslant Ce^{-\mu n|z|}|z|^p$$

and

$$\int_0^{+\infty} e^{-\mu nr}r^{p-1}\, dr \leqslant \frac{C}{n^p}.$$

Besides, we have

$$\begin{aligned}
&|\gamma_{n-k}(z)\delta_{s-k}(z)(r^s(z) - e^{-sz}) - \gamma_{n-k}(\infty)\delta_{s-k}(\infty)r^s(\infty)| \\
&\quad \leqslant |\gamma_{n-k}(z) - \gamma_{n-k}(\infty)||\delta_{s-k}(z)r^s(z)| + |\gamma_{n-k}(\infty)||\delta_{s-k}(z) - \delta_{s-k}(\infty)||r^s(z)| \\
&\quad + |\gamma_{n-k}(\infty)||\delta_{s-k}(\infty)||r^s(z) - r^s(\infty)|.
\end{aligned}$$

Now, for $|z| \geqslant \sigma_1$ we have

$$|\gamma_{n-k}(z) - \gamma_{n-k}(\infty)| \leqslant C\frac{e^{-\mu n}}{1 + (\mathrm{Log}(1 + |z|))^2};$$

and since r and δ_{s-k} are rational functions,

$$|\delta_{s-k}(z) - \delta_{s-k}(\infty)| \leqslant \frac{C}{|z|}, \qquad |r(z) - r(\infty)| \leqslant \frac{C}{|z|} \quad \text{for } |z| \geqslant \sigma_1.$$

Also,

$$|\gamma_n(\infty)| \leqslant Ce^{-\mu n} \quad \text{and} \quad |\delta_{s-k}(z)| \leqslant C \quad \text{for } |z| \geqslant \sigma_1.$$

Hence,

$$\begin{aligned} &|\gamma_{n-k}(z)\delta_{s-k}(z)(r^s(z) - e^{-sz}) - \gamma_{n-k}(\infty)\delta_{s-k}(\infty)r^s(\infty)| \\ &\qquad \leqslant Ce^{-\mu n}\left(\frac{1}{1 + (\operatorname{Log}(1 + |z|))^2} + \frac{1}{|z|}\right), \end{aligned}$$

and from Lemma 1.2, it follows that

$$\begin{aligned} &\|\gamma_{n-k}(\Delta tA)\delta_{s-k}(\Delta tA)(r^s(\Delta tA) - e^{-s\Delta tA})\| \\ &\qquad \leqslant C\left\{\frac{1}{n^p} + |\gamma_{n-k}(\infty)\delta_{s-k}(\infty)r^s(\infty)|\right\} \leqslant \frac{C''}{n^p}. \end{aligned}$$

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