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A Weak Discrete Maximum Principle and Stability of the Finite Element Method in L_∞ on Plane Polygonal Domains. I

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Abstract. Let Ω be a polygonal domain in the plane and $S_r^h(\Omega)$ denote the finite element space of continuous piecewise polynomials of degree $\leqslant r - 1$ ($r \geqslant 2$) defined on a quasi-uniform triangulation of Ω (with triangles roughly of size h). It is shown that if $u_h \in S_r^h(\Omega)$ is a "discrete harmonic function" then an a priori estimate (a weak maximum principle) of the form

$$\|u_h\|_{L_\infty(\Omega)} \leqslant C\|u_h\|_{L_\infty(\partial\Omega)}$$

holds.

Now let u be a continuous function on $\bar{\Omega}$ and u_h be the usual finite element projection of u into $S_r^h(\Omega)$ (with u_h interpolating u at the boundary nodes). It is shown that for any $\chi \in S_r^h(\Omega)$

$$\|u - u_h\|_{L_\infty(\Omega)} \leqslant c\left(\ln \frac{1}{h}\right)^{\bar{r}} \|u - \chi\|_{L_\infty(\Omega)}, \quad \text{where } \bar{r} = \begin{cases} 1 & \text{if } r = 2, \\ 0 & \text{if } r \geqslant 3. \end{cases}$$

This says that (modulo a logarithm for $r = 2$) the finite element method is bounded in L_∞ on plane polygonal domains.

0. Introduction and Statement of Results. The purpose of this paper is to discuss some estimates for the finite element method on polygonal domains. In particular, we shall consider the validity of (for want of a better terminology) a "discrete weak maximum principle" for discrete harmonic functions and then use this result to discuss the boundedness in L_∞ of the finite element projection. In this part we shall discuss the case of a quasi-uniform mesh. In Part II we shall concern ourselves with meshes which are refined near points. Let us first formulate the problems we wish to consider and state our results. References to other work in the literature which are relevant to our considerations will be given as we go along.

For simplicity let Ω be a simply connected (this is not essential) polygonal domain in $\mathbf{R}^2$ with boundary $\partial\Omega$ and maximal interior angle α, $0 < \alpha < 2\pi$, where we emphasize that in general Ω is not convex. On Ω we define a family of finite element spaces. For simplicity of presentation we shall restrict ourselves to a special but important class of piecewise polynomials. For each $0 < h < 1$, let T_h denote a triangulation of Ω with triangles having straight edges. We shall assume that each triangle τ is contained in a sphere of radius h and contains a sphere of radius γh for some positive constant γ. We shall also assume that the family $\{T_n\}$ of triangulations

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is quasi-uniform, i.e., there exist a constant γ_0 such that

$$\gamma \geqslant \gamma_0 > 0 \tag{0.1}$$

independent of h. Let $S_r^h(\Omega) = S_r^h$, $r \geqslant 2$, an integer, denote the finite dimensional space of continuous functions on $\bar{\Omega}$ whose restriction to each triangle $\tau \in T_h$ is a polynomial of degree $\leqslant r-1$ and $\mathring{S}_r^h(\Omega)$ denote the subspace of $S_r^h(\Omega)$ consisting of those functions which vanish on $\partial\Omega$. Note that $S_r^h(\Omega) \subset W_\infty^1(\Omega)$.

Consider the bilinear form $D(\cdot, \cdot)$ on $W_2^1(\Omega) \times W_2^1(\Omega)$ defined by

$$D(u, v) = \int_\Omega \nabla u \cdot \nabla v \, dx. \tag{0.2}$$

Now if $u \in W_2^1(\Omega)$ is harmonic in Ω then

$$D(u, v) = 0 \quad \text{for all } v \in \mathring{W}_2^1(\Omega).$$

It is well known that if in addition u is continuous on $\bar{\Omega}$ then it satisfies a maximum principle, i.e., its maximum and minimum values are taken on $\partial\Omega$. We shall say that $u_h \in S_r^h$ is discrete harmonic in Ω (relative to $S_r^h(\Omega)$) if

$$D(u_h, \chi) = 0 \quad \text{for all } \chi \in \mathring{S}_r^h(\Omega). \tag{0.3}$$

In [1], Ciarlet and Raviart showed that if $S_r^h(\Omega)$ is taken as the space of piecewise linear functions ($r = 2$), then u_n satisfies a maximum principle if and only if the maximum angle in any triangle $\tau \in T_h$ is $\leqslant \pi/2$. Recently Mittelman [4], has shown that if $S_r^h(\Omega)$ is taken to be piecewise quadratics ($r = 3$) then a maximum principle holds if and only if all the triangles $\tau \in T_h$ are equilateral. Obviously then a discrete maximum principle holds under very restrictive conditions.

More generally we shall show (see Section 3) that the following a priori estimate (a "weak discrete maximum principle") holds for the subspaces $S_r^h(\Omega)$:

THEOREM 1. *Let Ω be as above. Suppose that the $\{T_h\}$ are quasi-uniform (i.e., satisfy* (0.1)) *and $u_h \in S_r^h(\Omega)$ satisfies* (0.3). *Then for h sufficiently small*

$$\|u_h\|_{L_\infty(\Omega)} \leqslant C\|u_h\|_{L_\infty(\partial\Omega)}, \tag{0.4}$$

where in general $C \geqslant 1$ is independent of h and u_h, but may depend on Ω, γ and r. If Ω is convex then C is independent of Ω.

Remark 1. The condition that Ω be simply connected is not essential. An inequality of the form (0.4) is also valid in this case.

Remark 2. The methods used here in proving (0.4) differ entirely from those considered in [1]. Here we shall apply in our situation, techniques developed in Natterer [5], Nitsche [6], [7], Nitsche and Schatz [8], Schatz and Wahlbin [9], [10], [11], R. Scott [12]. Our proof depends in part on a priori estimates given below (see Lemma 1.2) for the problem $-\Delta u = f$ in Ω, $v = 0$ on $\partial\Omega$ on polygonal domains. Thus the inequality (0.4) also holds for the discrete analogue of solutions of homogeneous second order differential equations for which estimates of this type are known. As a simple example of this we may take $u_h \in S_r^h(\Omega)$ satisfying (with $(u, v) = \int_\Omega uv\,dx$)

$$D(u_h, \psi) + \sigma(u_h, \psi) = 0 \quad \text{for all } \psi \in \mathring{S}_r^h(\Omega),$$

where σ is a given constant. Here u_n is the discrete analogue of a solution of $-\Delta u + \sigma u = 0$ in Ω. If $\sigma \geqslant 0$ then a maximum principle holds for u but if $\sigma < 0$ and is not an eigenvalue then it does not hold. However, an inequality of the form

$$\|u\|_{L_\infty(\Omega)} \leqslant C\|u\|_{L_\infty(\partial\Omega)}$$

remains valid and (0.4) also for the corresponding discrete solution u_n.

Remark 3. As is well known, a major feature of the maximum principle for the continuous case is its independence of the domain. Theorem 1 states that for convex Ω, the constant C appearing in (0.4) is independent of Ω, i.e., Ω may be any convex mesh domain. In this case C depends only on γ in (0.1) and the order of piecewise polynomials used. The corresponding result for a general mesh domain is an open question. Our proof for nonconvex regions does not yield this because of several points, for example the use of the a priori estimate (1.5) which is domain dependent.

Remark 4. If we are willing to replace the constant C appearing in (0.4) with a term of the form $C(\ln 1/h)$ or $ch^{-\epsilon}$ for arbitrary $\epsilon > 0$ where in general $C = C(\epsilon, \Omega, \gamma, r)$ then a simpler proof of Theorem 1 may be given than that presented in Section 3. In other words we shall go through some extra difficulties in order to obtain the form (0.4).

Remark 5. Let Ω be fixed and let $G_d \subseteq \Omega$ be the set of points in Ω whose distance from $\partial\Omega$ is greater or equal to d. One can show that if $d = h^{1-\epsilon}$ for any $\epsilon > 0$, then

$$\|u_h\|_{L_\infty(G_d)} \leqslant C^*\|u_h\|_{L_\infty(\partial\Omega)},$$

where $C^* \longrightarrow 1$ as $h \longrightarrow 0$. We conjecture that this is also the case for the constant C occurring in (0.4).

We shall now apply (0.4) to investigate the stability of the finite element projection in L_∞ on a polygonal (not necessarily convex) domain. Let u be a continuous function on $\bar{\Omega}$. For example, u may be thought of as a weak solution of

$$-\Delta u = f \quad \text{in } \Omega,$$

$$u = g \quad \text{on } \partial\Omega,$$

where f and g are prescribed (but not specified here) so that u is continuous on $\bar{\Omega}$. Let us note that for any $\psi \in S_r^h(\Omega)$, $D(u, \psi)$ makes sense by integration by parts. In fact, for continuous u we may define (see Schatz and Wahlbin [9])

$$D(u, \psi) \equiv \sum_{\tau \in T_h} \left(\int_\tau u\Delta\psi\,dx + \int_{\partial\tau} u\,\frac{\partial\psi}{\partial n}\,ds \right),$$

where ds denotes arc length along $\partial\tau$.

Let u_h be the finite element approximation to u determined in the following way: On $\partial\Omega$ let u_h interpolate (see Section 2 for further details) u at the boundary nodes, then $u_h \in S_r^h(\Omega)$ is defined to be the unique solution of

$$D(u_h, \psi) = D(u, \psi) \quad \text{for all } \psi \in \mathring{S}_r^h(\Omega). \tag{0.5}$$

We wish to estimate $\|u - u_h\|_{L_\infty(\Omega)}$. Our approach here will be to compare u_h to another related problem defined on a convex polygon say $\widetilde{\Omega}$ containing Ω. To this end we shall make the following assumption which says that the family $\{T_h\}$ may be extended to a quasi-uniform family of triangulations of $\widetilde{\Omega}$.

A.1. There is a convex polygonal domain $\widetilde{\Omega}$, $\Omega \subset \widetilde{\Omega}$, such that for h sufficiently small, each triangulation T_h of Ω can be extended to a triangulation $\widetilde{T}_h$ of $\widetilde{\Omega}$ and the family $\{\widetilde{T}_h\}$ is quasi-uniform with the same constant γ appearing in (0.1).

In Section 4 we shall show

THEOREM 2. *Let Ω be as above and suppose that $S_r^h(\Omega)$ satisfies* A.1. *Let u be a continuous function on $\bar{\Omega}$ and $u_h \in S_r^h(\Omega)$ satisfy* (0.5) *where on $\partial\Omega$, u_h interpolates u at the boundary nodes. Then there exists a constant C independent of u, u_h and h (for h sufficiently small) such that for any $\chi \in S_r^h(\Omega)$*

$$\|u - u_h\|_{L_\infty(\Omega)} \leqslant C\left(\ln \frac{1}{h}\right)^{\bar{r}} \|u - \chi\|_{L_\infty(\Omega)}. \tag{0.6}$$

If $u \in W_\infty^1(\Omega)$,

$$\|u - u_h\|_{W_\infty^1} \leqslant C\left(\ln \frac{1}{h}\right)^{\bar{r}} \|u - \chi\|_{W_\infty^1(\Omega)}. \tag{0.7}$$

Several remarks are in order.

(1) The inequality (0.6) says that (modulo the usual logarithm for $r = 2$) one can obtain the best rate of convergence in L_∞ that the subspace can provide even when the domain is nonconvex. This is in contrast to the rate of convergence in $L_2(\Omega)$ where for nonconvex domains the finite element method is not bounded in $L_2(\Omega)$. Now if $l \geqslant 0$ is an integer let $C^l(\bar{\Omega})$ denote the space of functions having continuous partial derivatives up to order l which are continuous in $\bar{\Omega}$ with the norm

$$\|u\|_{C^l(\Omega)} = \sum_{|\gamma| \leqslant l} \max_{\bar{\Omega}} |D^\alpha u|.$$

Define the seminorm

$$|u|_{C^\sigma(\bar{\Omega})} = \sup_{x, y \in \Omega} \frac{|u(x) - u(y)|}{|x - y|^\alpha}$$

and for $l \geqslant 0$ an integer and $0 < \sigma < 1$, $C^{l+\sigma}(\bar{\Omega})$ will denote the usual Hölder spaces with the norm

$$\|u\|_{C^{l+\sigma}(\bar{\Omega})} = \|u\|_{C^l(\bar{\Omega})} + \sum_{|\gamma| = l} |D^\gamma u|_{C^\sigma(\bar{\Omega})}.$$

It is well known that if $l + \sigma \leqslant r$ and $u \in C^{l+\sigma}(\bar{\Omega})$ then there exists a $\chi \in S_1^h(\Omega)$ such that $\|u - \chi\|_{C^0(\bar{\Omega})} \leqslant Ch^{l+\sigma}(\bar{\Omega}) \|u\|_{C^{l+\sigma}(\bar{\Omega})}$. This together with (0.6) immediately implies the following

COROLLARY. *Under the conditions of Theorem* 1, *let $u \in C^{l+\sigma}(\bar{\Omega})$ for some integer $l \geqslant 0$ and $0 < \sigma < 1$, then*

(0.8) $$\|u - u_h\|_{L_\infty(\bar\Omega)} \leqslant Ch^{l+\sigma}\left(\ln \frac{1}{h}\right)^{\bar r} \|u\|_{C^{l+\sigma}(\bar\Omega)}.$$

In Nitsche [7] it was shown that if Ω is convex, $u = 0$ on $\partial\Omega$ and $\mathring{S}_r^h(\Omega)$ is taken to be piecewise linear functions ($r = 2$) the one has the estimate

$$\|u - u_h\|_{L_\infty(\Omega)} \leqslant Ch\left(\ln \frac{1}{h}\right)\|u - \chi\|_{W^1_\infty(\Omega)}$$

for any $\chi \in \mathring{S}_r^h(\Omega)$.

In the nonconvex case, again with $u = 0$ on $\partial\Omega$, it was shown in Schatz and Wahlbin [10] (using a much simpler technique) that one may obtain an almost optimal rate of convergence in L_∞ of order $h^{\beta-\epsilon}$ provided $u \in \mathring{W}_2^1 \cap W_2^{1+\beta-\epsilon}(\Omega)$ ($\epsilon > 0$ arbitrary), where $\beta = \pi/\alpha$ and α as before. There it was assumed that near the corner with maximal angle α the solution u behaves like (using polar coordinates (R, θ)) $u \approx c_1 R^\beta \sin \beta\theta$ + smoother terms. The methods used there do not extend to yield the "optimal" rate of convergence if u is smoother. This latter fact is very useful for example in finding pointwise estimates for the error when singular functions are used in the finite element method in conjunction with the usual piecewise polynomial subspaces. This will be the subject of a future publication.

(2) In Schatz and Wahlbin [9] (see Section 2 for more details) it was shown that if $\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega$ then one has an interior estimate (valid for a large class of finite element methods) of the form

(0.9) $$\|u - u_h\|_{L_\infty(\Omega_0)} \leqslant C\left(\ln \frac{1}{h}\right)^{\bar r} \|u - \chi\|_{L_\infty(\Omega_1)} + \|\!|u - u_h|\!\|_{-p,\Omega}.$$

The major point of (0.6) is that it is valid up to boundary for nonsmooth domains. We shall find it convenient to use (0.9) in proving Theorem 2.

An outline of this paper is as follows: In Section 1 we introduce some notation and collect some preliminaries. In Section 2 we shall discuss some properties of the subspaces $S_r^h(\Omega)$ and some preliminary results for the finite element method. Section 3 is devoted to proving Theorem 1 and Section 4 to proving Theorem 2.

In Part II of this study we shall first localize the results presented in Theorems 1 and 2 and use these results to show that inequalities of the type (0.4) and (0.6) "almost" hold when the finite element spaces are defined on a class of meshes which are refined near certain points of the domain. Applications will also be given.

1. Notations and Some Preliminaries. All functions considered in this paper will be real valued. If $1 \leqslant p \leqslant \infty$, then

$$\|u\|_{L_p(\Omega)} \equiv \left(\int_\Omega |u|^p \, dx\right)^{1/p}$$

with the usual modification when $p = \infty$. For each $j = 1, 2$, define the seminorm

$$|u|_{W_p^j(\Omega)} \equiv \left(\sum_{|\alpha|=j} \|D^\alpha u\|^p_{L_p(\Omega)}\right)^{1/p}$$

and the usual norm on the Sobolev space $W_p^k(\Omega)$, k a nonnegative integer,

$$\|u\|_{W_p^k(\Omega)} \equiv \left(\sum_{j=0}^{k} |u|^p_{W_p^j(\Omega)}\right)^{1/p}.$$

$\overset{\circ}{W}{}_p^k(\Omega)$ is the completion of $C_0^\infty(\Omega)$ under the $W_p^k(\Omega)$ norm. For the L_2 inner product we set $(u, v) = \int_\Omega u(x)v(x)\,dx$.

If $\Omega_0 \subseteq \bar{\Omega}$, then for $d > 0$, $N_d(\Omega_0)$ will denote a d neighborhood of Ω_0 relative to $\bar{\Omega}$, i.e.,

$$N_d(\Omega_0) = \{x : x \in \bar{\Omega};\ \mathrm{dist}(x, \Omega_0) < d\}.$$

In the special case that Ω_0 is a point, say $\{x_0\}$, we set $N_d(\Omega_0) = S_d(x_0)$, the intersection of $\bar{\Omega}$ with a sphere of radius d centered at x_0.

We shall need the following version of Poincaré's inequality.

LEMMA 1.1. *Let Ω be a simply connected polygonal domain and $v \in \overset{\circ}{W}{}_2^1(\Omega)$. Then for any $d > 0$ and $\bar{x} \in \partial\Omega$*

$$\|v\|_{L_2(S_d(\bar{x}))} \leqslant 4\pi d\,|v|_{W_2^1(S_d(\bar{x}))}. \tag{1.1}$$

Proof. It is sufficient to consider $v \in C_0^\infty(\Omega)$. Extend v as $v \equiv 0$ outside Ω and introduce polar coordinates (ρ, θ), where $\rho = |x - \bar{x}|$. Then

$$v(x_1, x_2) = v(\bar{x}_1 + \rho\cos\theta,\ \bar{x}_2 + \rho\sin\theta).$$

Since $v_\theta = -v_{x_1}\rho\sin\theta + v_{x_2}\rho\cos\theta$ and v vanishes at some point on each circle $\rho =$ constant, it follows for $0 \leqslant \rho \leqslant d$ that

$$|v(x_1, x_2)|^2 \leqslant \left(\int_0^{2\pi}(|v_{x_1}| + |v_{x_2}|)\rho \cdot d\theta\right)^2 \leqslant 4\pi d^2\int_0^{2\pi}(|v_{x_1}|^2 + |v_{x_2}|^2)\,d\theta;$$

the inequality now follows on integrating this last inequality over $S_d(\bar{x})$.

Remark. If Ω is not simply connected then (1.1) holds for all $d < d_0$ for some $d_0 = d_0(\Omega)$.

We shall frequently use the following inequality which is a consequence of Hölder's inequality. Namely if $\mathrm{diam}(\Omega) \leqslant d$, $1 \leqslant p \leqslant 2$ and $v \in L_2(\Omega)$ then

$$\|v\|_{L_p(\Omega)} \leqslant d^{-(1-2/p)}\|v\|_{L_2(\Omega)}. \tag{1.2}$$

We shall also need some estimates for weak solutions of $-\Delta u = f$ in Ω, $u = 0$ on $\partial\Omega$ when Ω is a polygonal domain. Let $v \in \overset{\circ}{W}{}_2^1(\Omega)$ satisfy

$$D(v, \psi) = (f, \psi) \quad \text{for all } \psi \in \overset{\circ}{W}{}_2^1(\Omega). \tag{1.3}$$

As in the introduction, let $0 < \alpha < 2\pi$ be the maximal interior angle in Ω and set $\beta = \pi/\alpha$.

LEMMA 1.2. *Let v satisfy* (1.3), *then*

(i) (*O. A. Ladyženskaya and N. N. Ural'ceva* [3], *Grisvard* [2]). *For any convex Ω $(1 < \beta)$, $v \in W_2^2(\Omega) \cap \overset{\circ}{W}{}_2^1(\Omega)$ and*

(1.4) $$|v|_{W_2^2(\Omega)} \leqslant \|f\|_{L_2(\Omega)}.$$

(ii) (*Grisvard* [2]). *If* $\frac{1}{2} < \beta < 1$ *and* $f \in L_p(\Omega)$ *for some* $1 < p < 2/(2-\beta)$, *then* $v \in W_p^2(\Omega) \cap \overset{\circ}{W}{}_2^1(\Omega)$ *and there exists a constant* C *(depending on* Ω *and* p*) such that*

(1.5) $$\|v\|_{W_p^2(\Omega)} \leqslant C\|f\|_{L_p(\Omega)}.$$

(iii) *If* $f \in L_2(\Omega)$, $\operatorname{supp}(f) \subset S_d(x_0)$, $x_0 \in \bar{\Omega}$, $\operatorname{dist}(x_0, \partial\Omega) \leqslant d$, *then*

(1.6) $$|u|_{W_2^1(\Omega)} \leqslant 4\pi d\|f\|_{L_2(S_d(x_0))}.$$

Proof of (iii).

$$|u|^2_{W_2^1(\Omega)} = (\nabla u, \nabla u) = (f, u) \leqslant \|f\|_{L_2(S_d(x_0))}\|u\|_{L_2(S_d(x_0))}.$$

There exists a point $\bar{x} \in \partial\Omega$ such that $S_d(x_0) \subset S_{2d}(\bar{x})$, and (1.6) now follows on applying Lemma 1.1.

We shall need local versions of (i) and (ii) of Lemma 1.2.

LEMMA 1.3. *Under the conditions of Lemma* 1.2, *there exists a constant* C *such that if* $\Omega_0 \subseteq \Omega_1 \subseteq \Omega$, $\bar{N}_d(\Omega_0) \subseteq \Omega_1$, $p = 2$ *when* Ω *is convex* ($\beta > 1$), *and* $1 < p < 2/(2-\beta)$ *when* $\frac{1}{2} < \beta < 1$, *then*

(1.7) $$|v|_{W_p^2(\Omega_0)} \leqslant C(\|f\|_{L_p(\Omega_1)} + d^{-1}|v|_{W_p^1(\Omega_1)} + d^{-2}\|v\|_{L_p(\Omega_1)}),$$

where C *is independent of* Ω *if* Ω *is convex.*

Proof. By a straightforward covering argument it is sufficient to consider the case when $\Omega_0 = S_{d/2}(x_0)$, $\Omega_1 = S_d(x_0)$ and $x_0 \in \bar{\Omega}_0$. Let $\omega \in C_0^\infty(S_d(x_0))$, $\omega \equiv 1$ on $B_{2d/3}(x_0)$, $|D^\alpha \omega| \leqslant C/d^{|\alpha|}$, $|\alpha| = 1, 2$, where $B_d(x_0)$ is the open ball of radius d centered at x_0. Then $\omega v \in W_p^2(\Omega) \cap \overset{\circ}{W}{}_2^1(\Omega)$ and (1.7) follows by applying (1.4) and (1.5) to the function ωv instead of v.

2. **Some Properties of the Subspaces and Further Preliminaries.** We shall need some properties of the subspaces $S_r^h(\Omega)$. To begin with, let T be a fixed triangle. We choose the nodes for T to consist of (see [13] and [12]) (i) the vertices of T, (ii) if $r \geqslant 3$, the $r-2$ points on each edge that divides the edge into $r-1$ equal parts, (iii) if $r \geqslant 4$, $(r-3)(r-2)/2$ points in the interior of T. The nodes in any triangle $\tau \in T_h$ are defined by an affine identification of τ and T. The interpolant $u_I \in S_r^h(\Omega)$ of u is defined by $u = u_I$ at the nodal points. We shall collect some well-known properties of the subspaces $S_r^h(\Omega)$. If M_h is any mesh domain, i.e., the union of triangles in T_h and if $u \in W_p^2(M_h)$ for some $1 < p \leqslant 2$, then

(2.1) $$h^{-1}\|u - u_I\|_{L_2(M_h)} + |u - u_I|_{W_2^1(M_h)} \leqslant Ch^{2-2/p}|u|_{W_p^2(M_h)}.$$

If M_h is any mesh domain then for any $\chi \in S^h(M)$

$$\|\chi\|_{W^1_\infty(M_h)} \leqslant Ch^{-1}\|\chi\|_{L_\infty(M_h)} \tag{2.2}$$

and

$$\|\chi\|_{L_\infty(M_h)} \leqslant Ch^{-1}\|\chi\|_{L_2(M_h)}. \tag{2.3}$$

The constant C in (2.1), (2.2) and (2.3) is independent of M_h.

Now let $u_h \in S^h(\Omega)$ be arbitrary, and let $\widetilde{u}_h \in \mathring{S}^h(\Omega)$ be equal to u_h at all interior nodes and $\widetilde{u}_h = 0$ at all boundary nodes. Then obviously,

$$\operatorname{supp}(u_h - \widetilde{u}_h) \subseteq \Lambda_h = \{x : x \in \bar{\Omega}, \operatorname{dist}(x, \partial\Omega) \leqslant h\}; \tag{2.4a}$$

and since the nodal basis is uniform (see [13]), it follows that

$$\|u_h - \widetilde{u}_h\|_{L_\infty(\Omega)} \leqslant C\|u_h\|_{L_\infty(\partial\Omega)}, \tag{2.4b}$$

where C is independent of Ω and depends only on γ and r.

We shall need some local estimates, for the finite element method, up to the boundary in $\mathring{W}^1_2$; see Nitsche and Schatz [8], Schatz and Wahlbin [10] and interior estimates in L_∞, Schatz and Wahlbin [9].

LEMMA 2.1. *Suppose that* (A.1) *is satisfied. There exist positive constants C_1 and k_1 such that*

(i) *If $\Omega_0 \subseteq \Omega_1 \subseteq \bar{\Omega}$, $\bar{N}_d(\Omega_0) \subseteq \Omega_1$, $d \geqslant k_1 h$ and $u \in \mathring{W}^1_2(\Omega)$ and $u_h \in \mathring{S}^h(\Omega)$ satisfy*

$$D(u - u_h, \psi) = 0 \quad \textit{for all } \psi \in \mathring{S}^h(\Omega_1), \tag{2.5}$$

then for h sufficiently small and any $\chi \in \mathring{S}^h(\Omega)$

$$\begin{aligned} &\|u - u_h\|_{W^1_2(\Omega_0)} \\ &\leqslant C(|u - \chi|_{W^1_2(\Omega_1)} + d^{-1}\|u - \chi\|_{L_2(\Omega_1)} + d^{-1}\|u - u_h\|_{L_2(\Omega_1)}). \end{aligned} \tag{2.6}$$

(ii) *If $\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega$, $d = \operatorname{dist}(\Omega_0, \partial\Omega_1) \geqslant k_1 h$, $u \in C(\Omega)$ and $u_h \in S^h(\Omega)$ satisfies* (2.5) (*see Section* 0), *then for h sufficiently small and any $\chi \in S^h(\Omega)$*

$$\|u - u_h\|_{L_\infty(\Omega_0)} \leqslant C\left(\ln \frac{1}{h}\right)^{\bar{r}} (\|u - \chi\|_{L_\infty(\Omega_1)} + d^{-1}\|u - u_h\|_{L_2(\Omega_1)}), \tag{2.7}$$

where

$$\bar{r} = \begin{cases} 1 & \textit{if } r = 2, \\ 0 & \textit{if } r \geqslant 3. \end{cases}$$

In (2.7) *and* (2.6), *C is independent of Ω_0, Ω_1, Ω, u, u_h and h, for h sufficiently small.*

Let us note that if $u_h \in S^h(\Omega)$ satisfies

$$D(u_h, \chi) = 0 \quad \text{for all } \chi \in \mathring{S}^h(\Omega_1)$$

we are at liberty to choose $u \equiv 0$ and $\chi \equiv 0$ in (2.7) and, hence,

$$\|u_h\|_{L_\infty(\Omega_0)} \leq Cd^{-1}\|u_h\|_{L_2(\Omega_1)}, \tag{2.8}$$

where C is as above.

3. **Proof of Theorem 1.** Let $u_h \in S_r^h(\Omega)$ satisfy (0.3), and let $x_0 \in \bar{\Omega}$ be such that $|u_h(x_0)| = \|u_h\|_{L_\infty(\Omega)}$. Set $d = \operatorname{dist}(x_0, \partial\Omega)$. It follows from (2.8) that there exists a constant $k_1 \geq 1$ such that if $d \geq 2k_1 h$

$$|u_h(x_0)| \leq cd^{-1}\|u_h\|_{L_2(S_d(x_0))}.$$

On the other hand if $d \leq 2k_1 h$, then using the inverse property (2.3) we have that since $x_0 \in \tau \in T_n$ for some τ

$$|u_h(x_0)| \leq ch^{-1}\|u_h\|_{L_2(\tau)} \leq ch^{-1}\|u_h\|_{L_2(S_{2h}(x_0))}.$$

Hence

$$\|u_h\|_{L_\infty(\Omega)} \leq c\rho^{-1}\|u_h\|_{L_2(S_\rho(x_0))}, \tag{3.1}$$

where

$$\rho = \max(d, 2h). \tag{3.2}$$

Now

$$\|u_h\|_{L_2(S_\rho(x_0))} = \sup_{\varphi \in C_0^\infty(S_\rho(x_0))} |(u_h, \varphi)|, \ \|\varphi\|_{L_2(S_\rho(x_0))} = 1.$$

Let $v \in \mathring{W}_2^1(\Omega)$ be the unique solution of

$$D(v, \psi) = (\varphi, \psi) \quad \text{for all } \psi \in \mathring{W}_2^1(\Omega), \tag{3.3}$$

and let $v_h \in \mathring{S}_r^h(\Omega)$ be the finite element approximation to v defined by

$$D(v_h, \chi) = (\varphi, \chi) \quad \text{for all } \chi \in \mathring{S}_r^h(\Omega). \tag{3.4}$$

We note that since $\frac{1}{2} < \beta$, by Lemma 1.2, there exists a $4/3 < p_0 \leq 2$, where $p_0 = 2$ if Ω is convex, such that $v \in W_{p_0}^2(\Omega) \cap \mathring{W}_2^1(\Omega)$.

Let $U_h = u_h$ on $\partial\Omega$ satisfy

$$D(U_h, \psi) = 0 \quad \text{for all } \psi \in \mathring{W}_2^1(\Omega). \tag{3.5}$$

Note that U_h satisfies $-\Delta U_h = 0$ and, since $U_h \in W_\infty^1(\partial\Omega)$ and $U_h \in C(\bar{\Omega})$, U_h satisfies a maximum principle. We write $u_h = u_h - U_h + U_h$. Then using (3.2), (3.3), (3.4), and (3.5)

$$\begin{aligned} |(u_h, \varphi)| &\leq |(u_h - U_h, \varphi)| + |(U_h, \varphi)| = |D(u_h - U_h, v)| + |(U_h, \varphi)| \\ &\leq |D(u_h, v)| + \|u_h\|_{L_\infty(\partial\Omega)}\|\varphi\|_{L_1(S_\rho(x_0))} \\ &\leq |D(u_h, v)| + \rho\|u_h\|_{L_\infty(\partial\Omega)}\|\varphi\|_{L_2(\Omega)} \leq |D(u_h, v)| + \rho\|u_h\|_{L_\infty(\partial\Omega)}. \end{aligned} \tag{3.6}$$

In view of (0.3), (3.3) and (3.4) we have for any $\chi \in \overset{\circ}{S}{}_r^h(\Omega)$

$$|D(u_h, v)| = D(u_h, v - v_h) = D(u_h - \chi, v - v_h). \tag{3.7}$$

We now choose $\chi = \widetilde{U}_h$ satisfying (2.4a) and (2.4b). Then, using (2.2) and (2.3),

$$\begin{aligned} |D(u_h - \widetilde{u}_h, v - v_h)| &\leq c|u_h - \widetilde{u}_h|_{W^1_\infty(\Omega)} |v - v_h|_{W^1_1(\Lambda_h)} \\ &\leq ch^{-1}\|u_h - \widetilde{u}_h\|_{L_\infty(\Omega)} |v - v_h|_{W^1_1(\Lambda_n)} \leq ch^{-1}|v - v_h|_{W^1_1(\Lambda_h)} \|u_h\|_{L_\infty(\partial\Omega)}, \end{aligned} \tag{3.8}$$

where as in Section 2, $\Lambda_h = \{x : x \in \bar{\Omega}; \operatorname{dist}(x, \partial\Omega) \leq h\}$. Collecting (3.1)–(3.8), we arrive at

$$\|u_h\|_{L_\infty(\Omega)} \leq c(\rho^{-1}h^{-1}|v - v_h|_{W^1_1(\Lambda_h)} + 1)\|u_h\|_{L_\infty(\partial\Omega)}. \tag{3.9}$$

The proof of Theorem 1 will be complete once we have shown

$$\rho^{-1}h^{-1}|v - v_h|_{W^1_1(\Lambda_h)} \leq C. \tag{3.10}$$

Let $R_0 = \operatorname{diam}\Omega$ and consider the annuli

$$A_j = \{x : x \in \bar{\Omega}, 2^{-(j+1)}R_0 \leq |x - x_0| \leq 2^{-j}R_0\}, \quad j = 0, 1, 2, \text{etc.}$$

and

$$A_j^l = A_{j-l} \cup \cdots \cup A_j \cup A_{j+1} \cup \cdots \cup A_{j+l}, \qquad l = 1, 2, \text{etc.}$$

Set $d_j = R_0 2^{-j}$, and let $1 \leq j \leq J = [\ln_2(R_0/8\rho)] + 1$ where $[\cdot]$ denotes the greatest integer function.

We first note that for any $j = 0, 1, \ldots$, etc.,

$$\operatorname{mes}(A_j \cap \Lambda_h) \leq cd_jh, \tag{3.11}$$

where, in general, c depends on Ω if Ω is nonconvex but is independent of Ω if Ω is convex. This is easy to see in the nonconvex domain where Ω is a fixed polygonal domain; and hence, its boundary is composed of a fixed finite number of straight line segments. To see this for a general convex Ω we first notice that

$$(A \cap \Lambda_h) \subseteq (S_{d_j}(x_0) \cap \Lambda_h) \subseteq \{x : x \in S_{d_j}(x_0), \operatorname{dist}(x, \partial S_{d_j}(x_0)) \leq h\} = G_j.$$

Since Ω is convex, then $S_{d_j}(x_0)$ is convex and the length of $\partial S_{d_j}(x_0) \leq 2\pi d_j$. Hence, $\operatorname{mes}(A_j \cap \Lambda_h) \leq \operatorname{mes}(G_j) \leq 2\pi d_jh$, which proves (3.11).

Using Schwarz's inequality and (3.10),

$$\begin{aligned} &\rho^{-1}h^{-1}\|v - v_h\|_{W^1_1(\Lambda_h)} \\ &\leq \rho^{-1}h^{-1}\left\{\sum_{j=0}^{J} |v - v_h|_{W^1_1(\Lambda_h \cap A_j)} + |v - v_h|_{W^1_1(\Lambda_h \cap S_{8\rho}(x_0))}\right\} \\ &\leq C\left\{\sum_{j=0}^{J} \rho^{-1}h^{-1/2}d_j^{1/2}|v - v_h|_{W^1_2(\Lambda_h \cap A_j)} + \rho^{-1/2}h^{J/2}|v - v_h|_{W^1_2(\Lambda_h \cap S_{8\rho}(x_0))}\right\}. \end{aligned} \tag{3.12}$$

Note first that in view of (2.1), (1.4), (1.5) and (1.2)

$$
\begin{aligned}
|v - v_h|_{W_2^1(\Lambda_h \cap S_{8\rho}(x_0))} &\leqslant |v - v_h|_{W_2^1(\Omega)} \leqslant |v - v_I|_{W_2^1(\Omega)} \\
&\leqslant ch^{2-2/p_0}|v|_{W_{p_0}^2(\Omega)} \leqslant ch^{2-2/p_0}\|\varphi\|_{L_{p_0}(S_\rho(x_0))} \\
&\leqslant ch^{2-2/p_0}\rho^{2/p_0-1}\|\varphi\|_{L_2(S_\rho(x_0))}.
\end{aligned}
$$

Since $\|\phi\|_{L_2(S_\rho(x_0))} = 1$, $p_0 > 4/3$ and $h/\rho < 1$,

$$
h^{-1/2}\rho^{-1/2}|v - v_h|_{W_2^1(\Lambda_h \cap S_{8\rho}(x_0))} \leqslant c\left(\frac{h}{\rho}\right)^{3/2-2/p_0}, \tag{3.13}
$$

where c is independent of Ω if Ω is convex.

Let $Z = \{j : 1 \leqslant j \leqslant J;\ \Lambda_h \cap A_j \neq \emptyset\}$. For $j \in Z$ we apply (2.6) to the domains A_j and A_j^1, respectively, and obtain using (1.4), (1.5) and (2.1)

$$
\begin{aligned}
|v - v_h|_{W_2^1(\Lambda_h \cap A_j)} &\leqslant c\{|v - v_I|_{W_2^1(A_j^1)} + d_j^{-1}\|v - v_I\|_{L_2(A_j^1)} \\
&\qquad + d_j^{-1}\|v - v_h\|_{L_2(A_j^1)}\} \\
&\leqslant c\{h^{2-2/p_0}|v|_{W_{p_0}^2(A_j^2)} + d_j^{-1}\|v - v_h\|_{L_2(A_j^1)}\}.
\end{aligned} \tag{3.14}
$$

Since v is harmonic in A_j^3, we have using (1.7) and (1.2) that

$$
h^{2-2/p_0}|v|_{W_{p_0}^2(A_j^2)} \leqslant ch^{2-2/p_0}d_j^{2/p_0-1}(d_j^{-1}|v|_{W_2^1(A_j^3)} + d_j^{-2}\|v\|_{L_2(A_j^3)}).
$$

Since $\operatorname{diam}(A_j^3) \leqslant 4d_j$, $A_j^3 \cap \Lambda_h \neq 0$ and, therefore, $\operatorname{dist}(A_j^3, \partial\Omega) \leqslant h$, there exists a point $\bar{x}_j \in \partial\Omega$ such that $A_j^3 \subseteq S_{8d_j}(\bar{x}_j)$. Hence, by (1.1) and (1.6)

$$
h^{2-2/p_0}|v|_{W_{p_0}^2(A_j^2)} \leqslant ch^{2-2/p_0}d_j^{2/p_0-2}|v|_{W_2^1(A_j^3)} \leqslant c\left(\frac{h}{d_j}\right)^{2-2/p_0}\rho. \tag{3.15}
$$

We now estimate the second term on the right of (3.11). For $j \in Z$ and $\bar{x}_j$ as above

$$
\begin{aligned}
\|v - v_h\|_{L_2(A_j^3)} &\leqslant \|v - v_h\|_{L_2(S_{8d_j}(\bar{x}_j))} \\
&= \sup_{\eta \in C_0^\infty(S_{8d_j}(\bar{x}_j))} |(v - v_h, \eta)|,\ \|\eta\|_{L_2(S_{8d_j}(\bar{x}_j))} = 1.
\end{aligned} \tag{3.16}
$$

Let $\omega \in \mathring{W}_2^1$ satisfy $D(\omega, \psi) = (\eta, \psi)$ for all $\psi \in \mathring{W}_2^1(\Omega)$ (note that $\omega \in W_{p_0}^2(\Omega)$), and let $\psi_h \in \mathring{S}^h(\Omega)$ satisfy $D(\omega_h, \chi) = (\eta, \chi)$ for all $\chi \in \mathring{S}^h(\Omega)$. Then in view of (1.4), (1.5), (1.2) and (2.1)

$$|(v - v_h, \eta)| = |D(v - v_h, \omega)| = |D(v - v_h, \omega - \omega_h)|$$
$$\leqslant |v - v_h|_{W_2^1(\Omega)} |\omega - \omega_h|_{W_2^1(\Omega)} \leqslant |v - v_I|_{W_2^1(\Omega)} |w - w_I|_{W_2^1(\Omega)}$$
$$\leqslant ch^{4-4/p_0} \|\varphi\|_{L_{p_0}(\Omega)} \|\eta\|_{L_{p_0}(\Omega)} \leqslant ch^{4-4/p_0} \rho^{2/p_0 - 1} d_j^{2/p_0 - 1}.$$

From this (3.13), (3.12) and (3.11) we obtain after summing over $j \in Z$

$$\sum_{j\in Z} \rho^{-1} h^{-1/2} d_j^{1/2} |v - v_h|_{W_2^1(\Lambda_h \cap A_j)}$$

(3.17)
$$\leqslant c \left[h^{3/2 - 2/p_0} \sum_{j=1}^{J} \frac{1}{d_j^{3/2 - 2/p_0}} + \frac{h^{7/2 - 2/p_0}}{\rho^{2 - 2/p_0}} \sum_{j=1}^{J} \frac{1}{d_j^{3/2 - 2/p_0}} \right].$$

Since $p_0 > 4/3$, $3/2 - 2/p_0 > 0$, $7/2 - 4/p_0 > 0$, $d_j \geqslant \rho$ we obtain from (3.14), (3.10) and (3.8) that

$$\rho^{-1} h^{-1} |v - v_h|_{W_1^1(\Lambda_h)} \leqslant c \left\{ \left(\frac{h}{\rho}\right)^{3/2 - 2/p_0} + \left(\frac{h}{\rho}\right)^{7/2 - 2/p_0} \right\},$$

where in view of (3.2), the inequality (3.9) follows which completes the proof.

4. Proof of Theorem 2. Extend u to $\bar{\tilde{\Omega}}$ as a continuous function on $\bar{\tilde{\Omega}}$ (again calling it u) such that $u = 0$ on $\partial\tilde{\Omega}$ and

(4.1)
$$\|u\|_{L_\infty(\tilde{\Omega})} \leqslant C \|u\|_{L_\infty(\Omega)}.$$

Let $\tilde{u}_h \in \mathring{S}_r^h(\tilde{\Omega})$ be the unique solution of

(4.2)
$$D(u - \tilde{u}_h, \eta) = 0 \quad \text{for all } \eta \in \mathring{S}^h(\tilde{\Omega}).$$

Now

(4.3)
$$\|u - u_h\|_{L_\infty(\Omega)} \leqslant \|u - \tilde{u}_h\|_{L_\infty(\Omega)} + \|\tilde{u}_h - u_h\|_{L_\infty(\Omega)}.$$

Apply the interior estimate (2.7) to the domains Ω and $\tilde{\Omega}$, respectively, then for the first term on the right of (4.3)

(4.4)
$$\|u - \tilde{u}_h\|_{L_\infty(\Omega)} \leqslant C \left(\ln \frac{1}{h}\right)^{\bar{r}} \|u\|_{L_\infty(\tilde{\Omega})} + \|u - \tilde{u}_h\|_{L_2(\tilde{\Omega})}.$$

Assume for the moment that we have shown

(4.5)
$$\|u - \tilde{u}_h\|_{L_2(\tilde{\Omega})} \leqslant C \|u\|_{L_\infty(\tilde{\Omega})},$$

then from (4.4), (4.5) and (4.1) we have

(4.6)
$$\|u - \tilde{u}_h\|_{L_\infty(\Omega)} \leqslant C \left(\ln \frac{1}{h}\right)^{\bar{r}} \|u\|_{L_\infty(\Omega)}.$$

Consider the second term on the right of (4.3). Since $D(\tilde{u}_h - u_h, \eta) = 0$ for all

$\eta \in \overset{\circ}{S}{}^h_r(\Omega)$, i.e., $\widetilde{u}_h - u_h$ is discrete harmonic in Ω, we have from Theorem 1 that

$$\|\widetilde{u}_h - u_h\|_{L_\infty(\Omega)} \leqslant C\|\widetilde{u}_h - u_h\|_{L_\infty(\partial\Omega)}$$

$$\leqslant C[\|u - \widetilde{u}_h\|_{L_\infty(\partial\Omega)} + \|u - u_h\|_{L_\infty(\partial\Omega)}].$$

The first term on the right may be bounded as in (4.6). Since $u_h = u_I$ on $\partial\Omega$ and $\|u_I\|_{L_\infty(\partial\Omega)} \leqslant \|u\|_{L_\infty(\partial\Omega)}$, we arrive at

$$\|\widetilde{u}_h - u_h\|_{L_\infty(\Omega)} \leqslant C\left(\ln \frac{1}{h}\right)^{\bar{r}} \|u\|_{L_\infty(\Omega)}.$$

Combining this with (4.6) and using (4.3), we have

$$\|u - u_h\|_{L_\infty(\Omega)} \leqslant C\left(\ln \frac{1}{h}\right)^{\bar{r}} \|u\|_{L_\infty(\Omega)};$$

and the result would follow on applying this result to $u - \chi$ with $u - u_h = u - \chi - (u_h - \chi)$.

Let us now prove (4.5). We write

$$\|u - u_h\|^2_{L_2(\widetilde{\Omega})} = (u - u_h, u - u_h) = D(u - u_h, \psi),$$

where $D(\psi, \eta) = (u - u_h, \eta)$ for all $\eta \in \overset{\circ}{W}{}^1_2(\widetilde{\Omega})$.

Let $\psi_h \in \overset{\circ}{S}{}^h_r(\widetilde{\Omega})$ satisfy $D(\psi - \psi_n, \chi) = 0$ for all $\chi \in \overset{\circ}{S}{}^h(\widetilde{\Omega})$, then using this and integrating by parts

$$\|u - u_h\|^2_{L_2(\widetilde{\Omega})} = D(u - u_h, \psi - \psi_h) = D(u, \psi - \psi_h)$$

$$(4.7)\qquad = \sum_{\tau \in \widetilde{T}^h} \left(\int_{\partial\tau} u \frac{\partial(\psi - \psi_h)}{\partial n}\, ds - \int_\tau u\Delta(\psi - \psi_h)\, dx\right)$$

$$\leqslant \sum_{\tau \in \widetilde{T}^h} \left(\left\|\frac{\partial(\psi - \psi_h)}{\partial n}\right\|_{L_1(\partial\tau)} + \|\psi - \psi_h\|_{W^2_1(\tau)}\right) \|u\|_{L_\infty(\Omega_c)}.$$

Since the triangulation is quasi-uniform, it follows (see [9]) that for each $\tau \in \widetilde{T}^h$

$$\left\|\frac{\partial(\psi - \psi_h)}{\partial n}\right\|_{L_1(\partial\tau)} \leqslant C(\|\psi - \psi_h\|_{W^2_1(\tau)} + h^{-1}\|\psi - \psi_h\|_{W^1_1(\tau)});$$

and we obtain from (4.7)

$$(4.8)\quad \|u - u_h\|^2_{L_2(\Omega_c)} \leqslant C\left\{h^{-1}\|\psi - \psi_h\|_{W^1_1(\widetilde{\Omega})} + \sum_{\tau \in \widetilde{T}_h} \|\psi - \psi_h\|_{W^2_1(\tau)}\right\} \|u_h\|_{L_\infty(\Omega_c)}.$$

For the first term on the right-hand side of (4.8) we have from the definition of ψ_h, (2.1) and (1.4) that

$$\|\psi - \psi_h\|_{W^1_1(\widetilde{\Omega})} \leqslant \|\psi - \psi_h\|_{W^1_2(\widetilde{\Omega})} \leqslant C\|\psi - \psi_I\|_{W^1_2(\widetilde{\Omega})} \leqslant Ch|\psi|_{W^2_2(\widetilde{\Omega})}$$

$$(4.9)\qquad \leqslant Ch\|u - u_h\|_{L_2(\widetilde{\Omega})}.$$

In order to estimate the second term on the right of (4.8) we first note that the interpolant ψ_I of ψ has the property that

(4.10) $$\|\psi - \psi_I\|_{W_1^2(\tau)} \leq C\|\psi\|_{W_1^2(\tau)},$$

and for any $\chi \in S_r^h(\Omega)$

(4.11) $$\|\chi\|_{W_1^2(\tau)} \leq Ch^{-1}\|\chi\|_{W_1^1(\tau)}.$$

Hence, using (4.10) and (4.11),

$$\begin{aligned}\|\psi - \psi_h\|_{W_1^2(\tau)} &\leq \|\psi - \psi_I\|_{W_1^2(\tau)} + \|\psi_I - \psi_h\|_{W_1^2(\tau)}\\ &\leq C(\|\psi\|_{W_1^2(\tau)} + h^{-1}\|\psi - \psi_I\|_{W_1^1(\tau)} + h^{-1}\|\psi - \psi_h\|_{W_1^1(\tau)})\\ &\leq C(\|\psi\|_{W_1^2(\tau)} + h^{-1}\|\psi - \psi_h\|_{W_1^1(\tau)}).\end{aligned}$$

Now using (4.9),

$$\sum_{\tau\in\widetilde{T}_h} \|\psi - \psi_h\|_{W_1^2(\tau)} \leq C\|u - u_h\|_{L_2(\widetilde{\Omega})}.$$

This together with (4.9) and (4.8) proves (4.5) which completes the proof of (0.6).

In order to prove (0.7) we use (0.6) and (2.2) and write

$$\begin{aligned}\|u - u_h\|_{W_\infty^1(\Omega)} &\leq \|u - u_I\|_{W_\infty^1(\Omega)} + \|u_I - u_h\|_{W_\infty^1(\Omega)}\\ &\leq C(\|u - u_I\|_{W_\infty^1(\Omega)} h^{-1}\|u_I - u_h\|_{L_\infty(\Omega)})\\ &\leq C(\|u - u_I\|_{W_\infty^1(\Omega)} + h^{-1}\|u - u_I\|_{L_\infty(\Omega)} + h^{-1}\|u - u_h\|_{L_\infty(\Omega)})\\ &\leq C\left(\|u - u_I\|_{W_\infty^1(\Omega)} + h^{-1}\left(\ln \frac{1}{h}\right)^{\bar{r}} \|u - u_I\|_{L_\infty(\Omega)}\right)\\ &\leq C\left(\ln \frac{1}{h}\right)^{\bar{r}} \|u\|_{W_\infty^1(\Omega)}.\end{aligned}$$

The desired result (0.7) now follows on applying this last inequality to $u - \chi$ for any $\chi \in S_r^h(\Omega)$ and writing $u - u_h = u - \chi - (u_h - \chi)$.

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