MATHEMATICS OF COMPUTATION, VOLUME 34, NUMBER 149
JANUARY 1980, PAGES 277–301

On Maximal Finite Irreducible Subgroups of $GL(n, \mathbf{Z})$ V. The Eight Dimensional Case and a Complete Description of Dimensions Less Than Ten

By Wilhelm Plesken and Michael Pohst

Abstract. All maximal finite (absolutely) irreducible subgroups of $GL(8, \mathbf{Z})$ are determined up to $\mathbf{Z}$-equivalence. Moreover, we present a full set of representatives of the $\mathbf{Z}$-classes of the maximal finite irreducible subgroups of $GL(n, \mathbf{Z})$ for $n \leqslant 9$ by listing generators of the groups, the corresponding quadratic forms fixed by these groups, and the shortest vectors of these forms.

1. Introduction. The present paper completes our discussion of maximal finite ($\mathbf{C}$-)irreducible subgroups of $GL(8, \mathbf{Z})$ which we began in Part IV [15]. There are 26 $\mathbf{Z}$-classes of such groups described in Theorem (4.1) as $\mathbf{Z}$-automorphism groups of certain quadratic forms.

The major part of this note is concerned with finding the quadratic forms F of degree 8, the automorphism groups A of which are irreducible and satisfy the condition: each irreducible subgroup of A has no $\mathbf{Q}$-reducible subgroup of index two. (All other forms of interest were already obtained in Part IV [15].) The procedure is nearly the same as in Part I [15]. First, we determine essentially all minimal irreducible finite subgroups of $GL(8, \mathbf{Z})$ up to $\mathbf{Q}$-equivalence satisfying the condition from above (Section 2). Then we compute the $\mathbf{Z}$-classes of the natural representation modules of these groups, respectively, the $\prec$-maximal centerings of the corresponding lattices, by the centering algorithm [15, Part I]. The centerings are listed on the microfiche at the end of this issue. A detailed description of the output and the associated quadratic forms in which we are mainly interested are given in Section 3. Finally, the automorphism groups of these forms and of the ones obtained in Part IV [15] are derived in Section 4. They are the representatives of the $\mathbf{Z}$-classes of the maximal finite ($\mathbf{C}$-)irreducible subgroups of $GL(8, \mathbf{Z})$.

In an appendix and on the second part of the attached microfiche we present a complete list of the results for all degrees $2 \leqslant n \leqslant 9$.

The extensive electronic computations were carried out on the CDC Cyber 76 of the Rechenzentrum of the Universität zu Köln and on the CDC Cyber 175 of the Rechenzentrum of the RWTH Aachen. For various parts of this paper, especially for

Received November 27, 1978.
AMS (MOS) subject classifications (1970). Primary 20C10, 20H15.
Key words and phrases. Integral matrix groups.

calculations in matrix and permutation groups, we made use of the implementation of group theoretical algorithms in the Aachen-Sydney GROUP System [4].

2. Minimal Irreducible Finite Subgroups of $GL(8, \mathbf{Z})$. Because of the results of Part IV [15] we are no longer interested in all $\mathbf{Q}$-classes of minimal irreducible finite subgroups of $GL(8, \mathbf{Z})$ but only in those containing groups G with the property: (β) G has no subgroup of index two which is $\mathbf{Q}$-reducible. In this paragraph we shall derive a set of representatives of the $\mathbf{Q}$-classes of these groups. Often it is more convenient to compute a $\mathbf{Q}$-irreducible subgroup H of G which fixes—up to scalar multiples—exactly one quadratic form. They are obtained more easily and their number is smaller. The character χ of the natural representation of H can be of two types:

$\chi = 2\psi$, where ψ is irreducible, rational and of Schur-index 2, or

$\chi = \psi_1 + \psi_2$, where ψ_i is irreducible ($i = 1, 2$) and ψ_2 is the complex conjugate of ψ_1. We do not check in each case, if H is really contained in an irreducible group G. This can, however, be decided by the results of Section 4, where we compute the automorphism groups of the obtained quadratic forms.

Let Δ be the natural representation of a minimal irreducible finite subgroup G of $GL(8, \mathbf{Z})$ with property (β), and let N be a maximal abelian normal subgroup of G. Applying Theorem (3.1) of [15, Part I] (an integral version of Clifford's Theorem) we have $\Delta|_N = \Gamma_1 \dotplus \cdots \dotplus \Gamma_r$, where $\Gamma_1, \ldots, \Gamma_r$ are integral representations of N satisfying $\Gamma_i \sim_{\mathbf{Q}} k\Delta_i$ ($i = 1, \ldots, r$; $k \in \mathbf{N}$) with Δ_i being $\mathbf{Q}$-irreducible, integral, and inequivalent of the same degree m. Furthermore, $\Gamma_1(N) = \cdots = \Gamma_r(N)$ holds. As a consequence, we must discuss all possible solutions of the equation $8 = k\, m\, r$:

	(i)	(ii)	(iii)	(iv)	(v)	(vi)	(vii)	(viii)	(ix)	(x)
k	1	1	1	1	2	2	2	4	4	8
m	1	2	4	8	1	2	4	1	2	1
r	8	4	2	1	4	2	1	2	1	1

In the cases (i)–(iii) and (vi)–(viii) there are no groups G. We prove this except for case (vii) by the following lemmas some of which will also be used in the other cases.

(2.1) Lemma. *If $G \leqslant GL(2^n, \mathbf{Q})$ is an irreducible 2-group, then G has a subgroup of index 2 which is $\mathbf{Q}$-reducible.*

Proof. By a result of Vol'vacev [1] G is conjugate to a subgroup $\widetilde{G}$ of the iterated wreath product

$$\langle -1 \rangle \sim \underbrace{C_2 \sim \cdots \sim C_2}_{n}$$

which has a subgroup of index 2 consisting of block diagonal matrices. The intersection with $\widetilde{G}$ is certainly a $\mathbf{Q}$-reducible subgroup of index 2 in $\widetilde{G}$. Q.E.D.

(2.2) Lemma. *G/N is isomorphic to a minimal transitive permutation group of degree 8 in case $k = 1$.*

Proof. Since the Schur-indices of representations of abelian groups are equal to 1, the restriction of Δ to N is the sum of eight 1-dimensional inequivalent complex representations which are permuted faithfully by G/N. G is irreducible, if and only if G/N acts transitively. This fact implies the result as in the proof of Theorem (3.2) in [15, Part I]. Q.E.D.

(2.3) LEMMA. *For prime numbers p the minimal transitive permutation groups of degree p^α ($\alpha \in \mathbf{N}$) are p-groups.*

For a proof see [15, Part III, Lemma (2.1)].

(2.4) LEMMA. *In the cases* (i)–(iii) *G does not satisfy* (β), *i.e. G has a subgroup of index* 2 *which is* **Q**-*reducible.*

Proof. By (2.2) and (2.3) G/N is a 2-group in each case. In case (i) N is also a 2-group; hence, G is a 2-group and violates (β) because of (2.1). In cases (ii) and (iii) G/N acts imprimitively on the absolutely irreducible constituents of $\Delta|_N$. One can easily find block stabilizers the inverse images of which yield a **Q**-reducible subgroup of index 2. Q.E.D.

(2.5) LEMMA. *In the cases* (vi) *and* (viii) *no group G exists which fulfills* (β).

Proof. In case (viii) N is obviously isomorphic to the Klein-four-group. The centralizer of N in G is **Q**-reducible and of index 2. In case (vi) G permutes the **C**-irreducible constituents of $\Delta|_N$ imprimitively, since the constituents of Γ_1 form a block. Clearly, the block stabilizer S is a **Q**-reducible subgroup of index 2 in G. (S is also the centralizer of N in G.) Q.E.D.

The remaining cases are discussed one after the other.

Case (iv). (Compare Case (ii) in [15, Part II, p. 555f].) N must be cyclic with $\varphi(|N|) = 8$ (φ denotes Euler's φ-function) and $|G| = 8|N|$. Therefore, $|N| \in \{16, 24, 20, 15, 30\}$. For $|N| = 16$, G is a 2-group and violates (β) by (2.1). We discuss $|N| = 24$ in greater detail, since it is the most complicated case.

(2.6) LEMMA. *For $|N| = 24$ we obtain exactly one group G_1 with five generators:*

$$g_1 = \begin{pmatrix} 0 & 1 & 1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & -3 & 0 & 2 & 0 & 0 \\ 0 & 2 & 2 & -4 & 0 & 2 & 0 & 1 \\ 0 & 3 & 3 & -6 & 0 & 3 & 1 & 0 \\ 0 & 2 & 2 & -4 & -1 & 3 & 1 & 0 \\ 0 & 1 & 2 & -3 & -1 & 3 & 0 & 0 \\ -1 & 1 & 2 & -2 & -1 & 2 & 0 & 0 \\ 0 & 1 & 1 & -1 & -1 & 1 & 0 & 0 \end{pmatrix},$$

$$g_2 = \begin{pmatrix} 0 & 2 & 0 & -1 & 0 & 1 & -1 & 0 \\ 1 & 2 & 0 & -2 & 1 & 1 & -1 & 0 \\ 0 & 3 & 1 & -3 & 1 & 2 & -2 & 0 \\ 1 & 4 & 0 & -4 & 2 & 3 & -3 & 0 \\ 0 & 3 & 0 & -3 & 2 & 2 & -2 & 0 \\ 0 & 2 & 0 & -2 & 2 & 1 & -2 & 0 \\ 0 & 2 & 0 & -2 & 2 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 & 0 & -1 \end{pmatrix},$$

$$g_3 = \begin{pmatrix} 1 & 0 & -2 & 0 & 1 & 0 & 0 & 1 \\ 1 & -1 & -3 & 1 & 1 & 0 & 1 & 0 \\ 2 & -1 & -4 & 1 & 1 & 1 & 0 & 1 \\ 3 & -2 & -6 & 2 & 1 & 1 & 1 & 1 \\ 2 & -2 & -5 & 2 & 1 & 1 & 0 & 1 \\ 2 & -1 & -4 & 1 & 1 & 1 & 0 & 0 \\ 1 & -1 & -2 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

$$g_4 = \begin{pmatrix} 0 & 0 & 0 & 1 & -2 & 0 & 2 & -1 \\ 0 & -1 & -1 & 2 & -2 & 0 & 3 & -2 \\ 0 & -1 & -1 & 3 & -3 & -1 & 4 & -2 \\ 1 & -2 & -2 & 4 & -4 & -1 & 6 & -3 \\ 1 & -2 & -1 & 3 & -3 & -1 & 5 & -3 \\ 0 & -2 & 0 & 2 & -2 & -1 & 4 & -2 \\ 0 & -1 & 0 & 1 & -1 & -1 & 3 & -2 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \end{pmatrix},$$

$$g_5 = \begin{pmatrix} 0 & 1 & 1 & -2 & 1 & -1 & 1 & 0 \\ 0 & 0 & 2 & -2 & 1 & -2 & 2 & 0 \\ 1 & 1 & 2 & -3 & 1 & -2 & 2 & 0 \\ 1 & 1 & 4 & -5 & 2 & -3 & 3 & 0 \\ 0 & 1 & 4 & -4 & 1 & -2 & 2 & 0 \\ 0 & 1 & 3 & -3 & 1 & -2 & 1 & 1 \\ 0 & 0 & 2 & -2 & 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 & -1 & 0 & 0 \end{pmatrix}, \quad \text{where } N = \langle g_1, g_5\rangle.$$

Proof. G must be an extension of C_{24} by its automorphism group (acting in the natural way). We have $C_{24} = \langle a, b \mid a^3 = b^8 = [a, b] = 1\rangle$ and $\mathrm{Aut}(C_{24}) = \langle \alpha, \beta, \gamma\rangle$ with $a^\alpha = a$, $b^\alpha = b^3$, $a^\beta = a$, $b^\beta = b^{-1}$, $a^\gamma = a^{-1}$, $b^\gamma = b$.

The cohomology group $H^2(\langle\alpha\rangle, C_{24}) \cong \{x \in C_{24} \mid x^\alpha = x\}/\{xx^\alpha \mid x \in C_{24}\}$ is trivial. Hence, there is only one extension E_1 of C_{24} by $\langle\alpha\rangle$. The center $Z(E_1)$ of E_1 is $\langle a\rangle \times \langle b^4\rangle$. Similarly, $H^2(\langle\beta\rangle, Z(E_1)) \cong C_2$; and there are at most two extensions E_2, $\widetilde{E}_2$ of E_1 by $\langle\beta\rangle$. We have $Z(E_2) = Z(\widetilde{E}_2) = Z(E_1)$. Finally, $H^2(\langle\gamma\rangle, Z(E_1)) \cong C_2$;

hence, there are at most two extensions in each case. Thus, we end up with at most four extensions altogether. We give a construction for each of them: Let $T = D_{12}$ and $\widetilde{T} = \langle a, b \mid a^6 = b^4 = 1, a^3 = b^2, a^b = a^{-1}\rangle$ be the two extensions of C_6 by $\mathrm{Aut}(C_6)$ and

$$U = \langle a, b, c \mid a^8 = b^2 = c^2 = 1, a^b = a^3, a^c = a^{-1}, [b, c] = 1\rangle,$$

$$\widetilde{U} = \langle a, b, c \mid a^8 = b^2 = c^4 = 1, a^b = a^3, a^c = a^{-1}, a^4 = c^2, [b, c] = 1\rangle,$$

be the two extensions of C_8 by $\mathrm{Aut}(C_8)$. Then the central products $T \curlyvee U$, $T \curlyvee \widetilde{U}$, $\widetilde{T} \curlyvee U$, $\widetilde{T} \curlyvee \widetilde{U}$ yield the four extensions of C_{24} by $\mathrm{Aut}(C_{24})$.*

However, $T \curlyvee \widetilde{U}$, $\widetilde{T} \curlyvee U$ have no faithful $\mathbf{Z}$-representation of degree 8. T, $\widetilde{T}$ have only one faithful character of degree 2, the one of T being of Schur-index 1 over $\mathbf{R}$ and the one of $\widetilde{T}$ being of Schur-index 2. Similarly, U, $\widetilde{U}$ have only one faithful character of degree 4, the one of U having Schur-index 1 and the other one 2. The faithful characters of $T \curlyvee U$, $T \curlyvee \widetilde{U}$, $\widetilde{T} \curlyvee U$, $\widetilde{T} \curlyvee \widetilde{U}$ are outer tensor products of these two and four dimensional characters. Hence, their Schur-indices over $\mathbf{R}$ are 1 in case of $T \curlyvee U$ and $\widetilde{T} \curlyvee \widetilde{U}$. For $G \cong T \curlyvee U$ one easily sees that G has a reducible subgroup of order 2 (isomorphic to $T \curlyvee H$, where H is a subgroup of index 2 of U). By some lengthy computations one obtains an embedding of $\widetilde{T} \curlyvee \widetilde{U}$ into the Weyl-group of the root system E_8. Q.E.D.

(2.7) LEMMA. *For $|N| = 20$ there is no group G.*

Proof. One obtains four extensions of C_{20} by $\mathrm{Aut}(C_{20})$ in exactly the same way as in (2.6). The corresponding characters of two of them have Schur-index 2 over $\mathbf{R}$. One of the other two groups

$$\left(\cong D_8 \times \mathrm{Aff}(1, 5) \text{ with } \mathrm{Aff}(1, 5) = \left\{\begin{pmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix} \middle| \alpha, \beta \in \mathbf{Z}_5, \alpha \neq 0\right\}\right)$$

has a $\mathbf{Q}$-reducible subgroup of index 2.

The last group is isomorphic to the central product $T_1 \curlyvee T_2$ of the quaternion group $T_1 \cong Q_8$ and a nonsplit extension of $\mathrm{Aff}(1, 5)$ by C_2. $T_1 \curlyvee T_2$ has a nonfaithful integral representation of degree 8. Namely, the centralizer of the subgroup of $GL(8, \mathbf{Z})$ isomorphic to T_2 in $\mathbf{Q}^{8\times 8}$ is isomorphic to the quaternion algebra $\left(\frac{-3,\ -5}{\mathbf{Q}}\right)$ in which $Q_8 \cong T_1$ cannot be embedded, since $\left(\frac{-3,\ -5}{\mathbf{Q}}\right) \ncong \left(\frac{-1,\ -1}{\mathbf{Q}}\right)$. Q.E.D.

(2.8) LEMMA. *For $|N| = 15$ we obtain exactly one group*

$$G_2 = \left\{ a \otimes b \middle| b \in \left\langle \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle, a \in \left\langle \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \right\rangle \right\}.$$

*For the notation see Huppert [11, p. 49].

Proof. By applying the Schur-Zassenhaus Theorem one gets $G \cong S_3 \times \mathrm{Aff}(1, 5)$. Q.E.D.

(2.9) LEMMA. *For $|N| = 30$ there is no group G.*

The proof is analogous to the one of (2.7).

Case (v). The restriction $\Delta|_N$ of Δ to the maximal normal abelian subgroup N is given by $\Delta|_N = \Sigma_{i=1}^4 2\Gamma_i$, where the Γ_i are 1-dimensional and inequivalent. Hence, N is clearly a subgroup of $\{\mathrm{diag}(a_1, a_2, a_3, a_4) | a_i \in \langle -I_2 \rangle\}$ of index 1 or 2.

(2.10) LEMMA. *In Case* (v) *there are two* **Q**-*classes of minimal irreducible subgroups of $GL(8, \mathbf{Z})$ both containing the* **Q**-*irreducible group*

$$H_1 = \left\langle \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \otimes I_2, \mathrm{diag}(-1, -1, 1, 1) \otimes I_2 \right\rangle.$$

Proof. As a consequence of Schur's Lemma G fixes a quadratic form $\mathrm{diag}(X_1, X_2, X_3, X_4)$. Hence, we may assume that G is a subgroup of the automorphism group of $I_4 \otimes X$ with $X \in \{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 2 & -1 \\ -1 & 2 \end{smallmatrix}\right)\}$, since $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 2 & -1 \\ -1 & 2 \end{smallmatrix}\right)$ are—up to multiples and **Z**-equivalence—the only forms of degree two with an irreducible automorphism group. Also, by Schur's Lemma the elements of the centralizer $C_G(N)$ are of the form $\mathrm{diag}(k_1, k_2, k_3, k_4)$ with $k_i \in \mathrm{Aut}_{\mathbf{Z}}(X)$ $(i = 1, \ldots, 4)$. Even $C_G(N) = N$ holds. For a proof we first assume that $C_G(N)$ is not abelian. Then $C_G(N)$ would be a subdirect product of four copies of dihedral groups of order 8 for $X = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ or of order 12 for $X = \left(\begin{smallmatrix} 2 & -1 \\ -1 & 2 \end{smallmatrix}\right)$. In each case we get a normal abelian subgroup containing N properly, namely of exponent 4 in the first case, of exponent 6 in the second. Hence $C_G(N)$ is abelian, but since it is normal and contains N it must be N itself.

The factor group G/N is isomorphic to a permutation group of degree 4, since it acts on $\{\Gamma_1, \ldots, \Gamma_4\}$ faithfully. By Lemma (2.1) G cannot be a 2-group, therefore, $3 | (G: N)$. On the other hand, 8 divides $(G: N)$ by Ito's Theorem [11, p. 570]. Hence $G/N \cong S_4$.

Next we show $X = \left(\begin{smallmatrix} 2 & -1 \\ -1 & 2 \end{smallmatrix}\right)$. Namely, for $X = I_2$ the inertia group of Γ_1 had epimorphic images isomorphic to D_8 and an extension of C_2^3 by D_6. Hence

$$\left(\frac{8 \cdot 48}{2} 4 \right) \Bigg| |G|.$$

On the other hand, $|G| = 4!|N| \big| 4!16$, obviously a contradiction.

Now we claim that G is isomorphic to a subgroup of index 2 of the wreath product $C_2 \sim S_4$. We consider the faithful representation $\overline{\Delta}$: $G \longrightarrow GL(8, \mathbf{Z}_3)$: $(g_{ij}) \mapsto (g_{ij} + 3\mathbf{Z})$. This representation is reducible, namely $I_4 \otimes \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ multiplied by a suitable permutation matrix transforms $\overline{\Delta}$ into $\left(\begin{smallmatrix} \overline{\Delta}_1 & * \\ 0 & \overline{\Delta}_2 \end{smallmatrix}\right)$, where the $\overline{\Delta}_i$ are faithful and monomial of degree 4. But the group of all monomial matrices of degree 4 over $\mathbf{Z}_3$ is isomorphic to $C_2 \sim S_4$. Hence, G is isomorphic to a subgroup of $C_2 \sim S_4$. The rest of the proof follows from the character tables of $C_2 \wr S_4$ and its subgroups in [3]. Q.E.D.

Case (vii). We have $\Delta|_N = 2\Gamma$ where Γ is a $\mathbf{Q}$-irreducible representation of degree 4, hence N is a cyclic group with $\varphi(|N|) = 4$, i.e. $|N| \in \{8, 12, 5, 10\}$. The centralizer $C_G(N)$ of N in G is contained in the commuting algebra of N in $\mathbf{Q}^{8\times 8}$ which is isomorphic to the ring of 2×2-matrices over the $|N|$th cyclotomic field. Since N is a maximal abelian normal subgroup, we obtain $\Delta|_{C_G(N)} \sim_{\mathbf{C}} \Delta_1 \dotplus \cdots \dotplus \Delta_4$, where the Δ_i are inequivalent irreducible, algebraic conjugate representations of degree 2. Moreover, $G/C_G(N) \cong \mathrm{Aut}(N)$ is of order 4.

(2.11) LEMMA. *There is no minimal irreducible subgroup of $GL(8, \mathbf{Z})$ in Case* (vii).

Proof. If $\Delta_1(C_G(N))$ is an imprimitive subgroup of $GL(2, \mathbf{C})$, it has either a characteristic abelian subgroup properly containing $\Delta_1(N)$ or it is a product of $\Delta_1(N)$ and a quaternion group of eight elements. In both cases N cannot be a maximal abelian normal subgroup of G. Hence, $\Delta_1(C_G(N))$ is a primitive subgroup of $GL(2, \mathbf{C})$.

In Blichfeldt's book [2] we find a list of all finite primitive subgroups of $PSL(2, \mathbf{C})$. From these we obtain the finite primitive subgroups of $GL(2, \mathbf{C})$ in the following way. For each group K in the list take those subgroups $\widetilde{K}$ of the group generated by the matrices representing K and some scalar matrices of finite order which satisfy $\widetilde{K}/Z(\widetilde{K}) \cong K$. Then $\Delta_1(C_G(N))$ is isomorphic to a subgroup of a central product of a cyclic group with one of the groups $SL(2, 3)$, $GL(2, 3)$, $SL(2, 5)$. In each of these cases the proper subgroup of G generated by N and a 2-Sylow subgroup is still irreducible, since all of the primitive finite subgroups of $GL(2, \mathbf{C})$ contain the quaternion group of eight elements Q_8. Q.E.D.

Case (ix). The restriction $\Delta|_N$ of Δ to the maximal abelian normal subgroup N is rationally equivalent to 4Γ, where Γ is an integral representation of degree 2. Hence, N is a cyclic group of order 4, 3, or 6. Analogous to Case (vii) we find $(G: C_G(N)) = 2$ and $\Delta|_{C_G(N)} \sim_{\mathbf{C}} \Delta_1 \dotplus \Delta_2$, where the Δ_i are inequivalent irreducible, algebraic conjugate representations of degree 4. There are exactly three possibilities for Δ_1: It is monomial, imprimitive of minimal block-size 2, or primitive as a complex representation. We consider them separately.

(2.12) LEMMA. *If Δ_1 is monomial, G contains a group which is rationally equivalent to one of the following four groups*:

$$\begin{aligned} H_2 = \langle &\mathrm{diag}(1, -1, -1, 1, -1, 1, 1, -1)D((18)(253764)),\\ &\mathrm{diag}(-1, 1, 1, -1, 1, -1, 1, -1)D((12)(34)(56)(78))\rangle \\ \cong\ & \widetilde{S}_4 \curlyvee C_4, \end{aligned}$$

where $\curlyvee$ stands for central product and D denotes the natural permutation representation of S_8, and $\widetilde{S}_4$ is the binary octahedral group;

$$H_3 = \left\langle h_1 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 & -1 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & -1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}, \right.$$

$$\left. h_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & -1 & 2 & -1 & -1 & 1 & 0 & 1 \\ -1 & -2 & 1 & -1 & 0 & 0 & -1 & 0 \\ -1 & -1 & 2 & -1 & -1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & -1 & 0 & 2 & 0 \\ -1 & -1 & 1 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 1 & -1 & -1 & -1 \end{pmatrix} \right\rangle \cong \widetilde{S}_4 \times C_3;$$

$$H_4 = \left\langle I_4 \otimes \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \otimes I_2, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \otimes I_2 \right\rangle$$

$$\cong GL(2,3) \curlyvee C_4;$$

$$H_5 = \left\langle I_4 \otimes \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \otimes I_2, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \otimes I_2 \right\rangle$$

$$\cong GL(2,3) \times C_3.$$

Proof. From Clifford's Theorem we conclude that G can be transformed into a (complex) monomial matrix group. All matrices corresponding to diagonal matrices are contained in N, since N is a maximal abelian normal subgroup. Hence, $C_G(N)/N$ is isomorphic to a transitive permutation group of degree 4. Δ_1 can be considered as an irreducible projective representation of a transitive subgroup P of S_4. Because of $|P| > 4^2$ we have $S_4 = P \cong C_G(N)/N$. Therefore, $C_G(N)$ is a central product of N with the binary octahedral group $\widetilde{S}_4$ or with $GL(2, 3)$. In both cases $C_G(N)$ is already $\mathbf{Q}$-irreducible and fixes only one quadratic form up to scalar multiples. Therefore, we obtain four groups $H_2 \cong \widetilde{S}_4 \curlyvee C_4$, $H_3 \cong \widetilde{S}_4 \times C_3$, $H_4 \cong GL(2, 3) \curlyvee C_4$, and $H_5 \cong GL(2, 3) \times C_3$. Q.E.D.

(2.13) Lemma. *If the minimal block-size of Δ_1 is 2, there is no group G.*

*

Blichfeldt's notation	Order	Isomorphism type		K/K'
(A)	$60 \cdot 2$	$SL(2, 5) \cong \widetilde{A}_5$		–
(B)	60	A_5	*	–
(C)	$360 \cdot 2$	$SL(2, 9) \cong \widetilde{A}_6$		–
(D)	$2520 \cdot 2$	$\widetilde{A}_7$		–
(E)	$168 \cdot 2$	$SL(2, 7)$		–
(F)	$25920 \cdot 2$	$Sp(4, 3)$		–
(G)	$120 \cdot 2$	$\widetilde{S}_5$		C_2
(H)	120	S_5	*	C_2
(K)	$720 \cdot 2$	$\widetilde{S}_6$		C_2
1°	$144 \cdot 2$	$\widetilde{A}_4 \mathrm{Y} \widetilde{A}_4$	*	$C_3 \times C_3$
2°	$288 \cdot 2$	$\widetilde{S}_4 \diamond \widetilde{S}_4$	*	C_2
3°	$288 \cdot 2$	$\widetilde{A}_4 \mathrm{Y} \widetilde{S}_4$		$C_2 \times C_3$
4°	$720 \cdot 2$	$\widetilde{A}_4 \mathrm{Y} \widetilde{A}_5$		C_3
5°	$576 \cdot 2$	$\widetilde{S}_4 \mathrm{Y} \widetilde{S}_4$		$C_2 \times C_2$
6°	$1440 \cdot 2$	$\widetilde{S}_4 \mathrm{Y} \widetilde{A}_5$		C_2
7°	$3600 \cdot 2$	$\widetilde{A}_5 \mathrm{Y} \widetilde{A}_5$		–
8°	$576 \cdot 2$	$(2^\circ \vert C_2)_s$	*	$C_2 \times C_2$
9°	$576 \cdot 2$	$(2^\circ \vert C_2)_{ns}$		C_4
10°	$288 \cdot 2$	$(1^\circ \vert C_2)$	*	$C_2 \times C_3$
11°	$7200 \cdot 2$	$(7^\circ \vert C_2)$		C_2
12°	$1152 \cdot 2$	$(5^\circ \vert C_2)$		$C_2 \times C_2$
13°	$80 \cdot 2$	$(E \vert C_5)$		C_5
14°	$160 \cdot 2$	$(E \vert D_{10})$		C_2
15°	$360 \cdot 2$	$(E \vert \mathrm{Aff}(1, 5))$		C_4
16°	$960 \cdot 2$	$(E \vert A_5)$		–
17°	$1920 \cdot 2$	$(E \vert S_5)$		C_2
18°	$960 \cdot 4$	$(C_4 \mathrm{Y} E \vert A_5)$		–
19°	$1920 \cdot 4$	$(C_4 \mathrm{Y} E \vert S_5)$		C_2
20°	$5760 \cdot 4$	$(C_4 \mathrm{Y} E \vert A_6)$		–
21°	$11520 \cdot 4$	$(C_4 \mathrm{Y} E \vert S_6)$		C_2

Proof. By Clifford's theory G can be transformed into a subgroup of the wreath product $H \wr S_4$, where H is a primitive subgroup of $GL(2, \mathbf{C})$. The matrices of G corresponding to the block diagonal matrices in $H \wr S_4$ form a subgroup S of index 2 in

*For an explanation of this table see next page.

$C_G(N)$. From the lattice of normal subgroups of the finite primitive subgroups of $GL(2, \mathbf{C})$ (see proof of Lemma (2.11)) and the maximality of N as a normal abelian subgroup of G we conclude that S is a central product of N with one of the groups $SL(2, 3)$, $\widetilde{S}_4$, $GL(2, 3)$ or $SL(2, 5)$. In case of $SL(2, 3)$ the group G becomes monomial. In cases of $\widetilde{S}_4$ and $GL(2, 3)$ the product of N and a 2-Sylow subgroup of G is still irreducible. In case of $SL(2, 5)$ the normalizer of a 5-Sylow subgroup is still irreducible; compare also the proof of (2.11). Q.E.D.

Next, we must discuss the case of Δ_1 being primitive. It is useful to present part of Blichfeldt's results [2] on finite primitive subgroups of $PSL(4, \mathbf{C})$. We give a list of all finite subgroups K of $GL(4, \mathbf{C})$ of minimal order such that all primitive subgroups of $PSL(4, \mathbf{C})$ can be derived by factoring out the center $Z(K)$ of K. (Compare also the proof of (2.11).) The orders of the groups K in column 2 are written as products $(K: Z(K)) \cdot |Z(K)|$. The groups which can be obtained as subgroups of $GL(4, \mathbf{Z})$ are marked by an asterisk in column 3. Column 5 gives the structure of the commutator factor groups which is important for the construction of all primitive finite subgroups of $GL(4, \mathbf{C})$ from the groups of this list. The symbol $\curlyvee$ denotes the central product, $\Diamond$ denotes the central product with common factor group C_2, $(2^\circ | C_2)$ denotes an extension of 2° by C_2. The subscripts s, ns refer to the extension to be split or nonsplit, respectively. E denotes the central product of the quaternion group of order eight with the dihedral group of order eight: $E = Q_8 \curlyvee D_8$.

We are now able to prove

(2.14) LEMMA. *If Δ_1 is primitive, G contains one of the following three groups*:

$$H_6 = \left\langle h_3 = \begin{pmatrix} -1 & 0 & 0 & 1 & -2 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -3 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -4 & 4 & -2 & 0 \\ 0 & 1 & -2 & 3 & -6 & 5 & -2 & 0 \\ 0 & 0 & -2 & 3 & -5 & 4 & -1 & 0 \\ 0 & 0 & -2 & 3 & -4 & 3 & -1 & 0 \\ 0 & 0 & -1 & 2 & -3 & 2 & -1 & 1 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \end{pmatrix}, \right.$$

$$\left. h_4 = \begin{pmatrix} 0 & 1 & 1 & -1 & 0 & -1 & 0 & 2 \\ -1 & 0 & 2 & -1 & 0 & -1 & 0 & 3 \\ 0 & 0 & 2 & -1 & 0 & -2 & 0 & 4 \\ 0 & 0 & 3 & -2 & 0 & -2 & 0 & 6 \\ 0 & 0 & 3 & -2 & 0 & -1 & -1 & 5 \\ 0 & 0 & 3 & -2 & 0 & -1 & 0 & 3 \\ 0 & 0 & 2 & -2 & 1 & -1 & 0 & 2 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 1 \end{pmatrix} \right\rangle \cong SL(2, 5),$$

where $h_3^3 = h_4^2 = (h_3 h_4)^5 = -I_8$;

$$H_7 = \left\langle \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}, I_2 \otimes \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \right\rangle \cong C_4 \times A_5,$$

$$H_8 = \left\langle \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}, I_2 \otimes \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \right\rangle \cong C_3 \times A_5.$$

Proof. The centralizer of N is a central product of N and one of the groups derived from the list above. In all cases $C_G(N)$ is already $\mathbf{Q}$-irreducible as a subgroup of $GL(8, \mathbf{Z})$. The character of group (A) is rational of Schur index 2. Hence, (A) yields a $\mathbf{Q}$-irreducible subgroup of $GL(8, \mathbf{Z})$ fixing a quadratic form which is unique up to scalar multiples. This subgroup is $\mathbf{Q}$-equivalent to H_6. The groups (C), (D), (F), (G) and (K) need not be considered since they contain (A) [6, p. 307]. Group (B) yields the groups H_7 and H_8. (H) contains (B) and, therefore, need not be considered. (E) does not occur, since the field generated by the character values is not quadratic.

All of the groups $1°$–$21°$ contain one of the extra-special groups of order 32 as a subgroup. Therefore the proper subgroup of G generated by N and a 2-Sylow subgroup of G would still be irreducible. Q.E.D.

Case (x). In this final case we have $\Delta|_N = 8\Gamma$, where Γ is an integral representation of degree 1. Hence, N is a subgroup of $\langle -I_8 \rangle$. We subdivide the case into four sections according to the minimal degree of the blocks of the complex natural representation of G.

(2.15) LEMMA. *If G can be transformed into a monomial subgroup of $GL(8, \mathbf{C})$, G is $\mathbf{Q}$-equivalent to*

$$G_3 = \left\langle g_6 = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}, \right.$$

$$\left. g_7 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \right\rangle \cong PSL(2, 7),$$

where $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right) \mapsto g_6$, $\left(\begin{smallmatrix}1&0\\2&1\end{smallmatrix}\right) \mapsto g_7$ *defines an epimorphism of* $SL(2, 7)$ *onto* G_3; *or* G *contains a subgroup which is* Q-*equivalent to*

$$H_9 = \left\langle h_5 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \right.$$

$$\left. h_6 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix} \right\rangle \cong SL(2, 7),$$

where $h_5 \mapsto \left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$, $h_6 \mapsto \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ *defines the isomorphism.*

Proof. The natural representation Δ of G can be transformed into a complex monomial representation. Let $\varphi\colon G \longrightarrow S_8$ be the associated permutation representation. Since $\ker \varphi$ is a normal abelian subgroup of G we get $\ker \varphi = N$. We first assume that N is trivial. Then G is isomorphic to a transitive permutation group of degree 8. Since the imprimitive permutation groups of degree 8 are solvable, $\varphi(G)$ must be primitive. Hence G is isomorphic to one of the groups $PSL(2, 7)$, $PGL(2, 7)$, A_8, or S_8 [18]. But A_8 and S_8 have no irreducible characters of degree 8 and $PGL(2, 7)$ contains $PSL(2, 7)$, yielding only G_3.

Now we assume $N = \langle -I_8 \rangle$. Some lengthy but elementary arguments show that $\varphi(G)$ is a primitive permutation group, because $N = \ker \varphi$ is the only abelian normal subgroup of G. Again, if $\varphi(G)$ is one of the two primitive solvable permutation groups of degree 8 (which are isomorphic to extensions of elementary abelian groups of order 8 by C_7 or Aff(1, 7) [18]) then G must have an abelian normal subgroup properly containing N. Hence, G must be isomorphic to a nonsplit central extension of $PSL(2, 7)$, $PGL(2, 7)$, A_8, or S_8. However, A_8 and S_8 have no irreducible projective characters of degree 8 [17]. In the other two cases the commutator subgroup G' of G is isomorphic to $SL(2, 7)$ ([16]; [11, p. 641ff.]). By inspection of the character table of $SL(2, 7)$ [16] we see that G must contain H_9 as a subgroup of index 2. Q.E.D.

Next we assume that G is C-equivalent to a subgroup $\widetilde{G}$ of $K \sim S_4$, where K is one of the finite primitive subgroups of $GL(2, \mathbf{C})$ which were already described in the proof of (2.11). Let $\varphi\colon \widetilde{G} \longrightarrow S_4$ denote the permutation representation derived from the embedding of $\widetilde{G}$ into $K \sim S_4$.

(2.16) LEMMA. *If G can be transformed into an imprimitive subgroup $\widetilde{G}$ of $GL(8, \mathbf{C})$ of minimal block-size 2, G has a group H of index 2 which is rationally equivalent to*

$$H_{10} = \left\langle h_7 = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ -1 & 1 & -1 & 1 & 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & -1 \\ 1 & -1 & 0 & 0 & -1 & 1 & 1 & -1 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \right.$$

$$\left. h_8 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 & -1 & 0 & -1 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix} \right\rangle$$

$$\cong SL(2, 3) \curlyvee SL(2, 3),$$

where the epimorphism of $SL(2, 3) \times SL(2, 3)$ onto H_{10} is given by

$$\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}\right) \mapsto h_7, \quad \left(\begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right) \mapsto h_8.$$

Proof. First we want to determine $\ker \varphi$ which is certainly a subgroup of $\{\mathrm{diag}(a_1, \ldots, a_4) | a_i \in K\}$. Let π_i: $\mathrm{diag}(a_1, \ldots, a_4) \mapsto a_i$ denote the projection of $\ker \varphi$ onto the ith component ($i = 1, \ldots, 4$). From the character relations we get $|G| = |\ker \varphi||\varphi(\widetilde{G})| \geqslant 8^2 + |\varphi(G)|$, hence,

$$|\ker \varphi| \geqslant \frac{8^2}{|\varphi(\widetilde{G})|} + 1 \geqslant \frac{8^2}{24} + 1 \geqslant 3.$$

Therefore, $\ker \varphi$ contains N properly and is not abelian. Furthermore, the π_i are irreducible representations of $\ker \varphi$. $\pi_1(\ker \varphi)$ must be isomorphic either to Q_8, $SL(2, 3)$, $\widetilde{S}_4$, $GL(2, 3)$, or $SL(2, 5)$, since all other irreducible finite subgroups of $GL(2, \mathbf{C})$ have a characteristic abelian subgroup of order bigger than 2 which would yield a normal abelian subgroup of G containing N properly. Since the nontrivial normal subgroups of $\pi_1(\ker \varphi)$ all contain the nontrivial center of $\pi_1(\ker \varphi)$ we conclude that the π_i are isomorphisms because of the general structure theory of subdirect products. An application of Clifford's theory provides that all of the π_i are equivalent representations (note: none of the five groups above have more than two faithful inequivalent representations of degree 2). Since the sum of the characters of the π_i has to be rational, we end up with ker φ isomorphic to Q_8 or $SL(2, 3)$.

By Clifford's theory the natural representation of $\widetilde{G}$ is equivalent to the tensor product of two irreducible projective representations Δ_1 and Δ_2 of $\widetilde{G}$ of degrees 4 and 2, where Δ_1 can even be considered as a projective representation of $\varphi(\widetilde{G})$. Since S_4 is the only permutation group of degree 4 which has an irreducible projective representation of degree 4, we obtain $\varphi(\widetilde{G}) = S_4$; more precisely, $\widetilde{G}$ is a nonsplit extension of Q_8 or $SL(2, 3)$ by S_4. In the first case some computations show that G has an elementary abelian normal subgroup of order 8, contradicting the maximality of N. In the second case the centralizer $C_G(\ker \varphi)$ of ker φ is easily seen to be isomorphic to $SL(2, 3)$, and the character of the restriction of the natural representation of $\widetilde{G}$ contains both complex faithful characters of $SL(2, 3)$ with multiplicity 2. Moreover, the product $\ker \varphi \cdot C_G(\ker \varphi)$ is a subgroup of index 2 in $\widetilde{G}$ which is equivalent to the **Q**-irreducible group H_{10}. Q.E.D.

Next we assume that G is **C**-equivalent to a subgroup $\widetilde{G}$ of $K \sim S_2$, where K is a finite primitive subgroup of $GL(4, \mathbf{C})$ for which we refer to the list preceding Lemma (2.14). Again let φ: $\widetilde{G} \longrightarrow S_2$ denote the permutation representation of $\widetilde{G}$ coming from the embedding of $\widetilde{G}$ into the wreath product $G \sim S_2$. Clearly [12, p. 86], the natural representation of $\widetilde{G}$ restricted to ker φ is the sum of two inequivalent irreducible representations Δ_1, Δ_2 of ker φ. Unlike Lemma (2.16) it is now immediate that $\Delta_i(\ker \varphi)$ is a primitive subgroup of $GL(4, \mathbf{C})$.

(2.17) Lemma. *If G can be transformed into an imprimitive subgroup $\widetilde{G}$ of $GL(8, \mathbf{C})$ of minimal block-size 4, G contains a group H which is rationally equivalent to H_6 (obtained in (2.14)) or*

$$H_{11} = \left\langle h_9 = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ -1 & 1 & -1 & 1 & 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & -1 \\ 1 & -1 & 0 & 0 & -1 & 1 & 1 & -1 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \right.$$

$$\left. h_{10} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & -1 & 1 & -1 & 1 & -1 \end{pmatrix} \right\rangle$$

$\cong SL(2, 3) \curlyvee SL(2, 3)$,

where the epimorphism of $SL(2, 3) \times SL(2, 3)$ *onto* H_{10} *is given by*

$$\left(\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, \begin{pmatrix}-1 & -1\\ 0 & -1\end{pmatrix}\right) \mapsto h_9, \quad \left(\begin{pmatrix}-1 & -1\\ 0 & -1\end{pmatrix}, \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\right) \mapsto h_{10};$$

$$H_{12} = \left\langle h_9, h_{11} = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & -1 & 1\\ 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0\\ -1 & 1 & 1 & -1 & 1 & -1 & 1 & -1\\ -1 & 0 & 1 & 0 & 1 & 0 & 1 & 0\\ 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0\\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0\\ 0 & 0 & -1 & 1 & -1 & 1 & -1 & 1\\ 0 & 0 & -1 & 0 & -1 & 0 & -1 & 0 \end{pmatrix}\right\rangle$$

$$\cong SL(2, 3) \curlyvee SL(2, 3),$$

where $h_9 \mapsto h_9$, $h_{10} \mapsto h_{11}$ *defines an isomorphism of* H_{12} *onto* H_{11};

$$H_{13} = \left\langle h_{12} = \begin{pmatrix} -1 & 0 & 1 & 1 & 0 & 0 & 0 & 0\\ 0 & -1 & -1 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1\\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 1\\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1\\ 0 & 0 & 0 & 0 & -1 & 1 & 1 & 0 \end{pmatrix}, \right.$$

$$\left. h_{13} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1\\ 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0\\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1\\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1\\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \end{pmatrix}\right\rangle \cong 13^{\circ},$$

where the conjugates of h_{12} *generate an extraspecial group* $Q_8 \curlyvee D_8$ *of order* 32, *on which* h_{13} *induces an automorphism of order* 5 *by conjugation.*

Proof. If the Δ_i are not faithful, $\Delta_i(\ker \varphi)$ contains the group (B) $\cong A_5$. Namely, all the other finite primitive subgroups of $GL(4, \mathbf{C})$ have a nontrivial center which is contained in each of the other nontrivial normal subgroups. This, however, yields a normal subgroup of G containing N properly (compare also the proof of (2.16)). Now it is easy to see that ker φ contains a subgroup isomorphic to $A_5 \times A_5$. Moreover, $\Delta_i(A_5)$ can be chosen integral from which one can conclude that either G was not minimal irreducible or contains a **Q**-reducible subgroup of index 2. Therefore, the Δ_i are faithful algebraic conjugate representations of ker φ which can be constructed over a quadratic number field.

We must now check the list preceding Lemma (2.14) for possible candidates of $\Delta_1(\ker \varphi)$. Groups (A) and (C), (D), (F), (G), (K) which contain (A) can only provide a group G containing H_5 (compare Lemma (2.14)). The argument at the beginning of this proof can be applied to the groups (B) and (H). Finally, (E) yields H_9. The remaining groups $1°$–$21°$ are treated separately.

Ad $1°$. Since the center of $1°$ and the commutator factor group are relatively prime, ker φ is isomorphic to $1°$. This group has nine faithful irreducible representations of degree 4 one of which is rational, whereas the others can be constructed over $\mathbf{Q}(\sqrt{-3})$ and fall into four pairs of complex conjugate representations. One of the subgroups of $GL(8, \mathbf{Z})$ derived from these has already been listed in (2.16). The other two are H_{11} and H_{12}.

Ad $2°$. This group can already be conjugated into $GL(4, \mathbf{Z})$. Therefore, we only have to deal with a subgroup of index 2 of $\langle iI_4, 2°\rangle$ ($i^2 = -1$) the center of which has order 2 and which is not isomorphic to $2°$. If $\Delta_1(\ker \varphi)$ is equal to this group, the 2-Sylow subgroup of G would already be irreducible by an argument outlined in $3°$.

Ad $3°$. In this case the 2-Sylow subgroup of G would still be irreducible as one concludes by observing that $3°$ contains an extra special group of order 32 and that the character of $3°$ is irrational on certain elements of order 2.

Ad $4°$. The argument of $3°$ can be applied to the normalizer of the 5-Sylow subgroup.

Ad $5°$, $9°$, $12°$. The same argument as in $3°$ applies.

Ad $6°$. The character values of Δ_1 are not contained in a quadratic number field.

Ad $7°$, $11°$. The same argument as in $4°$ applies.

Ad $8°$. The same argument as in $2°$ applies.

Ad $10°$. Here we either obtain a group containing one of the groups from $1°$ or the same argument as in $2°$ holds.

Ad $13°$–$21°$. All these groups contain $13°$ which already yields an irreducible subgroup H_{13} of $GL(8, \mathbf{Z})$ with a unique primitive quadratic form (note: the character of $13°$ is rational of Schur-index 2). Q.E.D.

Finally, we assume that G is primitive as a subgroup of $GL(8, \mathbf{C})$. We rely on results of Huffman and Wales [9] in case $7 \nmid |G|$ and on results of Feit [7], [8] in case $7 \mid |G|$.

(2.18) LEMMA. *If G is* **C**-*primitive, each group in the* **Q**-*class of G fixes a quadratic form which is integrally equivalent to one of the forms F_i ($i = 1, \ldots, 26$) obtained in* [15, *Part* IV] *respectively in the next paragraph.*

Proof. First we assume $7 \nmid |G|$. By [9] $G/Z(G) \cong PSL(2, 9) \cong A_6$ or G is a tensor product of two primitive groups of degree 2 and 4. One checks easily that neither $PSL(2, 9)$ nor $SL(2, 9)$ has an irreducible integral representation of degree 8. Since the finite primitive subgroups of $GL(2, \mathbf{C})$ and $GL(4, \mathbf{C})$ contain proper irreducible subgroups, G cannot be minimal irreducible.

Hence $7 \,\|\, |G|$. We must check whether G is one of the groups listed in Theorem A in [8]. The groups listed under A(i), A(iii), and A(iv) cannot be transformed into subgroups of $GL(8, \mathbf{Z})$; compare also [7]. The groups in A(ii) have an irreducible 2-subgroup. Some of the groups in A(v) yield minimal irreducible subgroups of $GL(8, \mathbf{Z})$ (having $PSL(2, 7)$ as a nonabelian composition factor); all of them were already treated (H_9, G_3). A(vi) yields $SL(2, 8)$ as a minimal irreducible subgroup of $GL(8, \mathbf{Z})$. The representation comes from the 2-transitive permutation representation of $SL(2, 8)$ on nine elements. However, the centerings which one gets are the same as those of S_9 by [13, Theorem 5.1]. The forms derived are multiples of F_5, F_{10}, F_{13}. This also settles A(viii), since in that case G is isomorphic to A_9 or S_9. Also, the groups in A(ix) contain A_9 and, therefore, are not minimal irreducible. Those groups of A(vii) which are rational certainly contain H_6. Finally, A(x) yields a group H isomorphic to $Sp(6, 2)$, which is a subgroup of $W(E_8) = \mathrm{Aut}_{\mathbf{Z}}(F_5)$ [7, Theorem 4.4], since $Sp(6, 2)$ contains a subgroup isomorphic to A_8 ($\cong GL(4, 2) \cong SO^+(6, 2)$). Therefore, H contains a subgroup isomorphic to A_8 or to the covering group $\tilde{A}_8$ of A_8. In the second case H contains H_6 of Lemma (2.14). The first case easily leads to a contradiction if one computes the lattices invariant under A_8 by the methods in [14]. Q.E.D.

3. Computation of the Z-Classes. From the centerings of the groups G_i, H_j ($i = 1, \ldots, 3$; $j = 1, \ldots, 13$) we obtain the quadratic forms $F_{18}, \ldots, F_{26}$ which together with $F_1, \ldots, F_{17}$ from Part IV [15] form a full set of representatives of the Z-classes of the positive definite primitive integral quadratic forms of degree 8 with irreducible automorphism groups. The forms are given by their matrices: ($J_k \in \mathbf{Z}^{k \times k}$ has all entries one.)

$$F_{18} = (3I_2 - J_2) \otimes (5I_4 - J_4); \qquad \det F_{18} = 1^2 5^2 15^4;$$

$$F_{19} = (3I_2 - J_2) \otimes (I_4 + J_4); \qquad \det F_{19} = 1^4 3^2 15^2;$$

$$F_{20} = \begin{pmatrix} 4 & 0 & 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 4 & 0 & 1 & 0 & -1 & -1 & 1 \\ 0 & 0 & 4 & -1 & 0 & 1 & -1 & 1 \\ 1 & 1 & -1 & 4 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 4 & -1 & 1 & 1 \\ 1 & -1 & 1 & 1 & -1 & 4 & 1 & 0 \\ -1 & -1 & -1 & 0 & 1 & 1 & 4 & 1 \\ 1 & 1 & 1 & -1 & 1 & 0 & 1 & 4 \end{pmatrix}, \qquad \det F_{20} = 1^2 2^2 6^4;$$

$$F_{21} = \begin{pmatrix} 3 & 1 & 0 & 0 & 0 & -1 & -1 & 1 \\ 1 & 3 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & -1 & -1 & 1 & 0 & 1 \\ 0 & 0 & -1 & 3 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 3 & -1 & 1 & 1 \\ -1 & 0 & 1 & 0 & -1 & 3 & 1 & 0 \\ -1 & 0 & 0 & -1 & 1 & 1 & 3 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 3 \end{pmatrix}, \qquad \det F_{21} = 1^4 3^2 6^2;$$

$$F_{22} = \begin{pmatrix} 6 & -2 & 2 & 3 & 3 & -1 & 1 & 0 \\ -2 & 6 & -2 & -1 & 1 & 3 & 3 & 0 \\ 2 & -2 & 6 & 3 & 1 & 1 & -1 & 2 \\ 3 & -1 & 3 & 6 & 1 & 1 & 2 & 3 \\ 3 & 1 & 1 & 1 & 6 & 0 & 3 & 1 \\ -1 & 3 & 1 & 1 & 0 & 6 & 3 & 3 \\ 1 & 3 & -1 & 2 & 3 & 3 & 6 & 3 \\ 0 & 0 & 2 & 3 & 1 & 3 & 3 & 6 \end{pmatrix}, \qquad \det F_{22} = 1^2 2^2 6^2 12^2;$$

$$F_{23} = \begin{pmatrix} 8 & -4 & -1 & 2 & 2 & -4 & -1 & 2 \\ -4 & 8 & 2 & -4 & -1 & 2 & -1 & -1 \\ -1 & 2 & 8 & -4 & -4 & 2 & 2 & -1 \\ 2 & -4 & -4 & 8 & 2 & -1 & -1 & -1 \\ 2 & -1 & -4 & 2 & 8 & -4 & 2 & -1 \\ -4 & 2 & 2 & -1 & -4 & 8 & -1 & 2 \\ -1 & -1 & 2 & -1 & 2 & -1 & 8 & -4 \\ 2 & -1 & -1 & -1 & -1 & 2 & -4 & 8 \end{pmatrix}, \qquad \det F_{23} = 1 \cdot 3^4 21^3;$$

$$F_{24} = \begin{pmatrix} 6 & -1 & 0 & 1 & -1 & 1 & -1 & 0 \\ -1 & 6 & -1 & 0 & -2 & 1 & 2 & 2 \\ 0 & -1 & 6 & 3 & 3 & 2 & -2 & 3 \\ 1 & 0 & 3 & 6 & 1 & 1 & 1 & 2 \\ -1 & -2 & 3 & 1 & 6 & 3 & -1 & 2 \\ 1 & 1 & 2 & 1 & 3 & 6 & 2 & 3 \\ -1 & 2 & -2 & 1 & -1 & 2 & 6 & 1 \\ 0 & 2 & 3 & 2 & 2 & 3 & 1 & 6 \end{pmatrix}, \qquad \det F_{24} = 1^3 7^4 21;$$

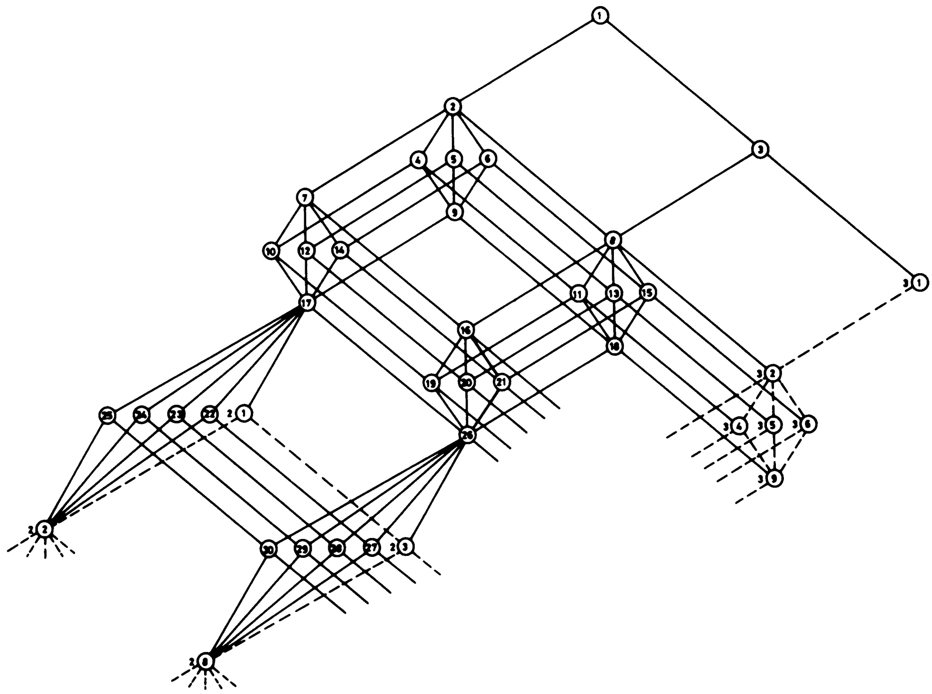
$$F_{25} = \begin{pmatrix} 4 & 2 & 2 & -1 & 0 & 1 & 0 & -1 \\ 2 & 4 & 1 & 0 & -1 & 1 & 1 & -1 \\ 2 & 1 & 4 & -1 & 2 & 0 & 2 & 1 \\ -1 & 0 & -1 & 4 & 0 & 2 & 1 & 0 \\ 0 & -1 & 2 & 0 & 4 & 1 & 1 & 1 \\ 1 & 1 & 0 & 2 & 1 & 4 & 1 & 1 \\ 0 & 1 & 2 & 1 & 1 & 1 & 4 & 2 \\ -1 & -1 & 1 & 0 & 1 & 1 & 2 & 4 \end{pmatrix}, \qquad \det F_{25} = 1^5 7^2 21;$$

$$F_{26} = \begin{pmatrix} 14 & 1 & 7 & -4 & 4 & 4 & -5 & 1 \\ 1 & 14 & -4 & 7 & -4 & -4 & -1 & 5 \\ 7 & -4 & 14 & -5 & 5 & 5 & -1 & -4 \\ -4 & 7 & -5 & 14 & -5 & -5 & 4 & 1 \\ 4 & -4 & 5 & -5 & 14 & -1 & 5 & -1 \\ 4 & -4 & 5 & -5 & -1 & 14 & -1 & 5 \\ -5 & -1 & -1 & 4 & 5 & -1 & 14 & 5 \\ 1 & 5 & -4 & 1 & -1 & 5 & 5 & 14 \end{pmatrix}, \qquad \det F_{26} = 1 \cdot 3^2 21^5.$$

The centerings of the groups G_i, H_j ($i = 1, \ldots, 3; j = 1, \ldots, 13$) are listed on the attached microfiche. We chose the example of H_1 to explain the output. Let a_1, a_2, a_3 be the generators of H_1, Δ the natural representation of H_1, Δ_{11} and Δ_{12} the constituents of Δ modulo $p_1 = 2$, Δ_{21} the only constituent of Δ modulo $p_2 = 3$. These input data are printed in the output as follows:

	Generators		
	a_1	a_2	a_3
	Constituents mod 2		
No. 1	$\Delta_{11}(a_1)$	$\Delta_{11}(a_2)$	$\Delta_{11}(a_3)$
No. 2	$\Delta_{12}(a_1)$	$\Delta_{12}(a_2)$	$\Delta_{12}(a_3)$
	Constituents mod 3		
No. 1	$\Delta_{21}(a_1)$	$\Delta_{21}(a_2)$	$\Delta_{21}(a_3)$.

Then the bases of all $\prec$-maximal centerings $C(i)$ of H_1 are printed (expressed as coordinate columns with respect to the standard basis of the natural representation module $C(1) = \mathbf{Z}^{8 \times 1}$), together with the quadratic form $\overline{C(i)}^t \cdot F \cdot \overline{C(i)}$, where F is the matrix of the quadratic form fixed by H_1 and $\overline{C(i)}$ denotes the matrix of the basis of $C(i)$. The elementary divisors of both matrices follow. Then the names of the maximal centerings $C(j)$ of $C(i)$ are printed which are of 2- respectively 3-power index in $C(i)$. Also, the isomorphism types of $C(i)/C(j)$ are given (e.g., prime 2 constituent no. 2 leads to $C(j)$). In case $C(i)/C(j)$ is not absolutely irreducible $C(j)$ is listed several times. Thus, we obtain the following lattice of centerings for H_1:



The quadratic forms in the output are not necessarily multiples of the forms $F_1, \ldots, F_{26}$ but only $\mathbf{Z}$-equivalent to them. These equivalences have been checked partly by hand and partly by machine. The reader can identify a form in the output with one of the F_i via the elementary divisors of the corresponding matrices. $F_1 = I_8$ and the Weyl form F_5 are the only two forms with the same elementary divisors. However, F_1 is odd and F_5 is even. We should mention that we have added a reduction subroutine to the original program of the centering algorithm described in Part I [15]. This subroutine reduces the bases of the centerings with respect to the associated form. It became necessary since the entries of the $\overline{C(i)}$ obtained by the earlier version of the program became too large, especially for high class numbers.

In four cases, namely in case of H_6, H_7, H_9, and H_{13}, the centering algorithm does not terminate by the usual test, i.e. there are infinitely many $\prec$-maximal centerings. (Note: the H_i $(i = 1, \ldots, 13)$ are not $\mathbf{C}$- but only $\mathbf{Q}$-irreducible.) In these cases we have to make sure that we already found sufficiently many centerings.

(3.1) LEMMA. *The quadratic forms obtained from the centerings of H_6, H_{13} are multiples of F_5 which is the Weyl-form of the root system E_8.*

Proof. Let L_1, L_2 be two centerings of $\mathbf{Z}^{8\times 1}$ with respect to H_6 (H_{13}). If L_1 and L_2 belong to the same genus, then clearly the determinants of the primitive positive forms associated with L_1, L_2 are equal. For both groups H_6, H_{13} the $\mathbf{Q}$-class of the natural representation can split into at most two genera. We prove this for H_6; for H_{13} the proof is similar.

The output shows that the lattice of 3-centerings of H_6 is linearly ordered.

Therefore, we have at most two isomorphism classes of $\mathbf{Z}_3^* H_6$-modules lying in the $\mathbf{Q}_3$-class of $(\mathbf{Z}_3^*)^{8\times 1}$. (For prime numbers p we denote the field of p-adic numbers by $\mathbf{Q}_p$, the ring of p-adic integers by $\mathbf{Z}_p^*$.) For $p = 2, 5$ the output shows that $(\mathbf{Z}_p^*)^{8\times 1}$ becomes reducible ([19, Proposition 6.2] and the well-known results of Maranda). $(\mathbf{Z}_p^*)^{8\times 1}$ is also reducible for all other prime numbers p by standard arguments because of $p \nmid |H_6| = 2^4 \cdot 3 \cdot 5$. Therefore, by Theorem 1.6 in [14] there is only one $\mathbf{Z}_p^*$-class in the $\mathbf{Q}_p^*$-class of $(\mathbf{Z}_p^*)^{8\times 1}$ for $p \neq 3$. Hence, we have at most two genera.

The only quadratic forms in the output concerning H_6 (H_{13}) are multiples of F_5. Every quadratic form belonging to a centering of H_6 (H_{13}) is even, since the natural representation of H_6 (H_{13}) has no 1-dimensional constituent modulo 2. Hence, all forms must be multiples of F_5, F_5 being the only positive definite even form of degree 8 with determinant 1. Q.E.D.

(3.2) LEMMA. *The class number of the natural representation of H_7 is four. The isomorphism types of lattices are represented by $C(i)$ for $i \in \{1, 3, 4, 7\}$ (compare microfiche).*

Proof. The matrix $A := \begin{pmatrix} 0 & -I_4 \\ I_4 & 0 \end{pmatrix}$ centralizes H_6. Because of $C(2) = (I_8 + A)\mathbf{Z}^{8\times 1}$, $C(8) = (I_8 - 2A)\mathbf{Z}^{8\times 1}$, and $C(9) = (I_8 + 2A)\mathbf{Z}^{8\times 1}$ these centerings are isomorphic to $C(1)$. Therefore, all centerings of H_7 which are not contained in $C(2)$, $C(8)$ or $C(9)$ contain representatives of the centerings of $C(1)$. But these are given by $C(i)$, $i \in \{1, 3, 4, 7\}$. No two of them can be isomorphic as can be seen from the lattice of centerings. Q.E.D.

A similar but slightly more complicated argument shows that a set of representatives for the isomorphism classes of centerings of H_9 is given by $C(1)$, $C(2)$, $C(4)$, $C(5)$, and $C(8)$. (Note: The centralizer of $\overline{C(2)}^{-1} H_9 \overline{C(2)}$ in $\mathbf{Z}^{8\times 8}$ is isomorphic to $\mathbf{Z}[(1 + \sqrt{-7})/2]$.)

4. The Irreducible Maximal Finite Subgroups of $GL(8, \mathbf{Z})$. There are—up to $\mathbf{Z}$-equivalence—26 such groups. They fall into 16 $\mathbf{Q}$-classes. For the derivation we use freely the results in four dimensions [3].

(4.1) THEOREM. *The irreducible maximal finite subgroups of $GL(8, \mathbf{Z})$ are $\mathbf{Z}$-equivalent to the automorphism groups of the quadratic forms $F_1, \ldots, F_{26}$.*

(i) $\mathrm{Aut}_{\mathbf{Z}}(F_1)$, $\mathrm{Aut}_{\mathbf{Z}}(F_2)$, *and* $\mathrm{Aut}_{\mathbf{Z}}(F_4)$ *are* $\mathbf{Q}$*-equivalent. They are isomorphic to the wreath product $C_2 \sim S_8$ of order $2^8 8!$.* $\mathrm{Aut}_{\mathbf{Z}}(F_1)$ *is the full monomial group of degree* 8.

(ii) $\mathrm{Aut}_{\mathbf{Z}}(F_3)$ *is isomorphic to the wreath product $W(F_4) \sim C_2$ of the Weyl group of the root system F_4 with C_2 of order $1152^2 2$.*

(iii) $\mathrm{Aut}_{\mathbf{Z}}(F_5)$ *is isomorphic to the Weyl group of the root system E_8 of order* $2^{14}3^5 5^2 7$.

(iv) $\mathrm{Aut}_{\mathbf{Z}}(F_6)$ *is isomorphic to the direct product $S_3 \times W(F_4)$ of the symmetric group on three elements and the Weyl group of the root system F_4. It is of order* $3!1152$.

(v) $\mathrm{Aut}_{\mathbf{Z}}(F_7)$ *is isomorphic to the wreath product $D_{12} \sim S_4$ of the dihedral group of order* 12 *with the symmetric group on four elements. It is of order* $12^4 4!$.

(vi) $\mathrm{Aut}_{\mathbf{Z}}(F_8) \sim_{\mathbf{Q}} \mathrm{Aut}_{\mathbf{Z}}(F_{11})$. *They are isomorphic to* $C_2 \times (S_3 \sim S_4)$ *of order* $2 \cdot 6^4 4!$.

(vii) $\mathrm{Aut}_{\mathbf{Z}}(F_9)$ *is isomorphic to the wreath product* $(C_2 \times (S_3 \sim C_2)) \sim C_2$ *of order* $144^2 2$.

(viii) $\mathrm{Aut}_{\mathbf{Z}}(F_{10}) \sim_{\mathbf{Q}} \mathrm{Aut}_{\mathbf{Z}}(F_{13})$. *Both are isomorphic to* $C_2 \times S_9$ *of order* $2 \cdot 9!$.

(ix) $\mathrm{Aut}_{\mathbf{Z}}(F_{12})$ *is isomorphic to the direct product* $C_2 \times (S_3 \sim S_3)$ *of order* $2 \cdot 6^3 3!$.

(x) $\mathrm{Aut}_{\mathbf{Z}}(F_{14}) \sim_{\mathbf{Q}} \mathrm{Aut}_{\mathbf{Z}}(F_{16})$. *Both are isomorphic to the wreath product* $(C_2 \times S_5) \sim C_2$ *of order* $240^2 2$.

(xi) $\mathrm{Aut}_{\mathbf{Z}}(F_{15})$ *is isomorphic to an extension of the central product of* $SL(2, 5)$ *with itself by* C_2. *It is of order* $2 \cdot 60^2 2$.

(xii) $\mathrm{Aut}_{\mathbf{Z}}(F_{17})$ *is isomorphic to the direct product* $C_2 \times (S_5 \sim C_2)$ *of order* $2 \cdot (5!)^2 2$.

(xiii) $\mathrm{Aut}_{\mathbf{Z}}(F_{18}) \sim_{\mathbf{Q}} \mathrm{Aut}_{\mathbf{Z}}(F_{19})$. *Both groups are isomorphic to the direct product* $C_2 \times S_5 \times S_3$ *of order* $2 \cdot 5! \cdot 3!$.

(xiv) $\mathrm{Aut}_{\mathbf{Z}}(F_{20}) \sim_{\mathbf{Q}} \mathrm{Aut}_{\mathbf{Z}}(F_{21})$. *They are isomorphic to the Weyl group* $W(\mathsf{F}_4)$ *of the root system* F_4 *of order* 1152.

(xv) $\mathrm{Aut}_{\mathbf{Z}}(F_{22})$ *is isomorphic to a subdirect product of* S_3 *with the Weyl group* $W(\mathsf{F}_4)$. *It is of order* $1152 \cdot 6 \cdot 2^{-1}$.

(xvi) $\mathrm{Aut}_{\mathbf{Z}}(F_{23})$, $\mathrm{Aut}_{\mathbf{Z}}(F_{24})$, $\mathrm{Aut}_{\mathbf{Z}}(F_{25})$, *and* $\mathrm{Aut}_{\mathbf{Z}}(F_{26})$ *are rationally equivalent. They are isomorphic to the direct product* $C_2 \times PGL(2, 7)$ *of order* $2 \cdot 336$.

Proof. Ad(i). Compare Theorem (6.1) in Part I [15].

Ad(ii). This follows immediately from [5].

Ad(iii). Compare [10, p. 66].

Ad(iv). F_6 is the Kronecker product of $\left(\begin{smallmatrix} 2 & -1 \\ -1 & 2 \end{smallmatrix}\right)$ with Weyl form of the root system F_4. Therefore, $\mathrm{Aut}_{\mathbf{Z}}(F_6)$ certainly contains a subgroup $H \cong S_3 \times W(\mathsf{F}_4)$. We must show that H is already the full automorphism group. F_6 contains 36 vectors of shortest length–up to sign–in its lattice. These are permuted transitively by H. The stabilizer H_1 in $\mathrm{Aut}_{\mathbf{Z}}(F_6)$ of one of these vectors operates on a sublattice of index 4 on which F_6 induces the quadratic form $\left(\begin{smallmatrix} 2 & -1 \\ -1 & 2 \end{smallmatrix}\right) \otimes 2I_4$. Hence, H_1 is rationally equivalent to a subgroup of the stabilizer in $\mathrm{Aut}_{\mathbf{Z}}(\left(\begin{smallmatrix} 2 & -1 \\ -1 & 2 \end{smallmatrix}\right)) \sim S_4$ of the lattice belonging to F_6. The order of this stabilizer is $2^5 3$ which proves $H = \mathrm{Aut}_{\mathbf{Z}}(F_6)$.

Ad(v). This follows immediately from [5].

Ad(vi). The forms $9F_8^{-1}$ and F_{11} are integrally equivalent. Hence, their automorphism groups are rationally equivalent. Similar arguments as in the proof of Theorem (4.1) in Part III [15] show that $\mathrm{Aut}_{\mathbf{Z}}(F_8)$ acts on a lattice with the induced form $I_4 \otimes \left(\begin{smallmatrix} 2 & -1 \\ -1 & 2 \end{smallmatrix}\right)$. Therefore, $\mathrm{Aut}_{\mathbf{Z}}(F_8)$ is rationally equivalent to a subgroup of $\mathrm{Aut}_{\mathbf{Z}}(\left(\begin{smallmatrix} 2 & -1 \\ -1 & 2 \end{smallmatrix}\right)) \sim S_4$, more precisely to

$$\left\{ g \in \mathrm{Aut}_{\mathbf{Z}}\left(I_4 \otimes \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\right) \middle| (1, \ldots, 1)g \equiv \pm(1, \ldots, 1) \bmod 3 \right\}.$$

Ad(vii). This follows also from [5].

Ad(viii). The forms $9F_{10}^{-1}$ and F_{13} are integrally equivalent. Therefore, their automorphism groups are rationally equivalent. F_{13} contains–up to sign–nine vectors of shortest length. The result now follows as in the proof of Theorem (4.1)(vii) in Part III [15].

Ad (ix). Clearly, the automorphism group of

$$F_{12} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \otimes \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \otimes \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

contains all matrices $g_1 \otimes g_2 \otimes g_3$ with $g_i \in \mathrm{Aut}_{\mathbf{Z}}((\begin{smallmatrix} 2 & -1 \\ -1 & 2 \end{smallmatrix}))$, $i = 1, 2, 3$, as well as those permutation matrices which conjugate $g_1 \otimes g_2 \otimes g_3$ into $g_{\pi(1)} \otimes g_{\pi(2)} \otimes g_{\pi(3)}$ $(\pi \in S_3)$. These generate a subgroup H of $\mathrm{Aut}_{\mathbf{Z}}(F_{12})$ with $H \cong C_2 \times (S_3 \sim S_3)$. To verify $H = \mathrm{Aut}_{\mathbf{Z}}(F_{12})$ we note that F_{12} has 27 vectors–up to sign–of shortest length. Similar arguments as used in the proof of Theorem (4.1) in Part III [15] then show that $\mathrm{Aut}_{\mathbf{Z}}(F_{12})$ has a centering with induced quadratic form $I_4 \otimes (\begin{smallmatrix} 2 & -1 \\ -1 & 2 \end{smallmatrix})$ and, hence, is rationally equivalent to a subgroup of $\mathrm{Aut}_{\mathbf{Z}}((\begin{smallmatrix} 2 & -1 \\ -1 & 2 \end{smallmatrix})) \sim S_4$.

Ad(x). The forms $5F_{14}^{-1}$ and F_{16} are integrally equivalent; hence, their automorphism groups rationally equivalent. The rest follows from [5].

Ad(xi). First we describe the construction of a group H which will turn out to be $\mathbf{Z}$-equivalent to the full automorphism group of F_{15}. The group $SL(2, 5)$ has a faithful representation of degree two which can be realized over a quadratic extension E of $\mathbf{Q}(\sqrt{5})$. From this representation and the regular representation of $E/\mathbf{Q}$ we obtain a $\mathbf{Q}$-irreducible representation Δ of $SL(2, 5)$. The enveloping algebra of $\Delta(SL(2, 5))$ is isomorphic to a quaternion algebra over $\mathbf{Q}(\sqrt{5})$. Its centralizer in $\mathbf{Q}^{8\times 8}$ is isomorphic to the same quaternion algebra. Therefore, we get a subgroup H' of $GL(8, \mathbf{Q})$ and, hence, of $GL(8, \mathbf{Z})$ which is isomorphic to a central product of $SL(2, 5)$ by itself. The centralizer of H' in $\mathbf{Q}^{8\times 8}$ is isomorphic to $\mathbf{Q}(\sqrt{5})$ and H' fixes a two dimensional space of quadratic forms in which one finds a form F being $\mathbf{Z}$-equivalent to F_{15}. After this it is not difficult to obtain an automorphism g of F, $g \notin H'$, $g^2 \in H'$. Set $H = \langle H', g\rangle$.

We now want to verify that H is the full automorphism group of F. The lattice of F contains–up to sign–60 vectors of minimum length. Each of the subgroups of H' isomorphic to $SL(2, 5)$ operates transitively and regularly on these vectors. (Note: the enveloping algebra of $\Delta(SL(2, 5))$ is a division algebra.) Some lengthy computations now yield that the stabilizer in $\mathrm{Aut}_{\mathbf{Z}}(F)$ of one of the vectors is isomorphic to the symmetric group on five elements.

Ad(xii). The same argument as in the proof of (vi) yields that $\mathrm{Aut}_{\mathbf{Z}}(F_{17})$ is rationally equivalent to

$$\{g \in \mathrm{Aut}_{\mathbf{Z}}(5I_4 - J_4) \sim C_2 \,|\, (1, \ldots, 1)g \equiv \pm(1, \ldots, 1) \bmod 5\}.$$

Ad(xiii). The forms $15F_{18}^{-1}$ and F_{19} are integrally equivalent; hence, their automorphism groups rationally equivalent. Using the fact that F_{18} has–up to sign–15 vectors of minimum length, one easily derives

$$\mathrm{Aut}_{\mathbf{Z}}(F_{18}) = \{g \otimes h | g \in \mathrm{Aut}_{\mathbf{Z}}(5I_4 - J_4),\ h \in \mathrm{Aut}_{\mathbf{Z}}(3I_2 - J_2)\}.$$

Ad(xiv). The forms $6F_{20}^{-1}$ and F_{21} are integrally equivalent; hence, their automorphism groups rationally equivalent. Similar arguments as in case (vi) show that $\mathrm{Aut}_{\mathbf{Z}}(F_{20})$ is rationally equivalent to

$$\{g \in \mathrm{Aut}_{\mathbf{Z}}(F_6) | (1\ 0\ 0\ 1\ 1\ 0\ 1\ 0)g \equiv (1\ 0\ 0\ 1\ 1\ 0\ 1\ 0) \bmod 2\}$$

which turns out to be a subgroup H of index 6 of $\mathrm{Aut}_{\mathbf{Z}}(F_6) \cong S_3 \times W(\mathsf{F}_4)$ with $H \cong W(\mathsf{F}_4)$.

Ad(xv). As in case (xiv) $\mathrm{Aut}_{\mathbf{Z}}(F_{22})$ is rationally equivalent to a subgroup H of $\mathrm{Aut}_{\mathbf{Z}}(F_6) \cong S_3 \times W(\mathsf{F}_4)$. More precisely, H is the biggest subgroup of $\mathrm{Aut}_{\mathbf{Z}}(F_6)$ leaving the sublattice L' of index 2^2 of $\mathbf{Z}^{8 \times 1}$ invariant given by the kernel of the $\mathbf{Z}$-epimorphism φ: $\mathbf{Z}^{8\times 1} \longrightarrow \mathbf{Z}_2^{2 \times 1}$ described by the matrix $\left(\begin{smallmatrix} 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \end{smallmatrix}\right)$. Now the result follows easily.

Ad(xvi). The forms $21F_{23}^{-1}$ and F_{24} as well as $21F_{25}^{-1}$ and F_{26} are integrally equivalent. As in Theorem (4.1) case (vii) from Part III [15] one sees that all four automorphism groups are rationally equivalent. From the way we obtained these forms we know that the group G_3 in Lemma (2.15) is a subgroup of the automorphism group of F_{23}. This and the fact that F_{23} has—up to sign—21 vectors of minimum length lead to the desired result by some elementary calculations. (Note: $\langle -I_8 \rangle \times G_3$ is already transitive on the vectors of minimum length.) Q.E.D.

5. Appendix: List of Maximal Finite Irreducible Subgroups of $GL(n, \mathbf{Z})$ for $n \leqslant 9$. The second part of the attached microfiche** contains a complete list of a set of representatives $\{G(i) | i = 1, \ldots, k(n)\}$ of the $\mathbf{Z}$-classes of maximal finite irreducible subgroups of $GL(n, \mathbf{Z})$ for $2 \leqslant n \leqslant 9$. For each group $G(i)$ the output gives the following information:

(1) The matrix $F(i) \in \mathbf{Z}^{n \times n}$ of the primitive quadratic form fixed by $G(i)$, i.e. $g^T F(i) g = F(i)$ for all $g \in G(i)$, the greatest common divisor of the entries of $F(i)$ is one, $G(i) = \mathrm{Aut}_{\mathbf{Z}}(F(i))$.

(2) Generating matrices for $G(i)$.

(3) Elementary divisors of $F(i)$.

(4) The order of $G(i)$.

(5) The vectors of minimum length (up to sign) of $F(i)$ as coordinate columns with respect to the natural basis.

The number $k(n)$ of groups for each dimension n is

n	2	3	4	5	6	7	8	9
$k(n)$	2	3	6	7	17	7	26	20

In almost all cases we chose the same quadratic forms $F(i)$ which already occur in this paper or in Parts I–IV [15], except for

F_6, F_7 of degree $n = 5$, since the forms given in Part I [15] were not reduced;

F_{10} of degree $n = 6$, where some signs in Part II [15] are missing. It should read correctly $F_{10} = (3I_2 - J_2) \otimes (I_3 + J_3)$.

**Headline: Irred(ucible) Maximal F(inite) U(nimodular) Groups of Degree 2–9.

We also correct three further misprints in this paper. On page 564 the determinants of F_{10}, F_{11} are $4^2 3^3$, $4^4 3^3$ respectively, and on page 570 the (5, 5)-entry of $B(T_2)$ has to be 1 instead of 0.

RWTH Aachen
Lehrstuhl D für Mathematik
Templergraben 64
5100 Aachen, West Germany

Mathematisches Institut der Universität zu Köln
Weyertal 86–90
5000 Köln 41, West Germany

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