MATHEMATICS OF COMPUTATION, VOLUME 35, NUMBER 152
OCTOBER 1980, PAGES 1063–1079

On the Coupling of Boundary Integral and Finite Element Methods

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Abstract. Let Ω^c be the complementary of a bounded regular domain in $\mathbf{R}^2$ of boundary Γ. We consider the problem

$$\text{(1)} \qquad \begin{cases} \Delta u = f; & \text{in } \Omega^c, \\ u|_\Gamma = u_0, \end{cases}$$

where f has its support in a bounded subdomain Ω_1 of Ω^c. Let Γ_2 be the common boundary of Ω_1 and $\Omega_2 = \Omega^c - \Omega_1$. We solve the problem (1) by using an equivalent system of equations involving an integral equation on Γ_2 coupled with the equation:

$$\text{(2)} \qquad \begin{cases} \Delta u = f & \text{in } \Omega_1, \\ u|_\Gamma = u_0, \\ u|_{\Gamma_2} = \lambda. \end{cases}$$

We introduce a finite element approximation of Eq. (2) and of the integral equation and we prove optimal error estimates.

Introduction. The purpose of this note is to analyze a procedure obtained by coupling the boundary integral method (cf. [4], [5], [7], [8], [12]) and the usual finite element method. Such coupled procedures have been proposed by e.g. Silvester-Hsieh [10] and Zienkiewicz et al. [11] for the numerical solution of problems in unbounded domains. As a typical example let us consider a problem of the form

$$\begin{cases} Au = f & \text{in } \Omega^c, \\ u = u_0 & \text{on } \Gamma, \end{cases}$$

where Ω is a bounded domain in the plane with boundary Γ, Ω^c is the unbounded complement of Ω, and A is an elliptic differential operator. Let us further assume that Ω^c can be divided into a bounded part Ω_1 and an unbounded part Ω_2, with common boundary Γ_2 (see Figure 1), so that $f = 0$ in Ω_2 and A is linear and has constant coefficients in Ω_2 while A may be nonlinear or have variable coefficients in the bounded part Ω_1. Then the unbounded part Ω_2 can be taken into account using an integral equation on the boundary Γ_2, and an approximate solution can be found using a conventional finite element discretization of Ω_1 together with a discretization along Γ_2. Below we shall analyze a model problem of this type.

For numerical experiments and references into the engineering literature on this subject, we refer to [11].

Received February 22, 1979; revised December 13, 1979.
1980 *Mathematics Subject Classification*. Primary 65N99.

1. A Model Problem. Let us consider the following exterior Dirichlet problem:

$$(1.1)\qquad \begin{cases} -\Delta u = f & \text{in } \Omega^c, \\ u = 0 & \text{on } \Gamma, \end{cases}$$

where Ω is a bounded domain in $\mathbf{R}^2$ with smooth boundary Γ and Ω^c is the complement of $\Omega \cup \Gamma$. Let us assume that the support of f is bounded and that $f \in L^2(\Omega^c)$. It is known (see e.g. [3], [6]) that the problem (1.1) admits a unique solution $u \in W^1(\Omega^c)$, where

$$W^1(\Omega^c) = \{v\colon (1 + |x|^2)^{-1/2}(1 + \log\sqrt{1 + |x|^2})^{-1} v \in L^2(\Omega^c),\ \nabla v \in [L^2(\Omega^c)]^2\},$$

and that this solution has the following asymptotic behavior:

$$(1.2)\qquad \begin{cases} u(x) = \alpha + \mathcal{O}\left(\dfrac{1}{|x|}\right), & |x| \longrightarrow \infty, \\ \nabla u(x) = \mathcal{O}\left(\dfrac{1}{|x|^2}\right), & |x| \longrightarrow \infty, \end{cases}$$

where α is a constant.

Let now Γ_2 be a smooth curve dividing Ω^c into an unbounded part Ω_2 and a bounded part Ω_1 containing the support of f (see Figure 1). Then (1.1) can alternatively be formulated as follows:

$$\begin{cases} -\Delta u_1 = f & \text{in } \Omega_1, & (1.3a) \\ -\Delta u_2 = f = 0 & \text{in } \Omega_2, & (1.3b) \\ u_1 = u_2 & \text{on } \Gamma_2, & (1.3c) \\ \dfrac{\partial u_1}{\partial n} = \dfrac{\partial u_2}{\partial n} = \lambda & \text{on } \Gamma_2, & (1.3d) \\ u_1 = 0 & \text{on } \Gamma, & (1.3e) \end{cases}$$

where $u_i = u|\Omega_i$, $i = 1, 2$, and $\partial/\partial n$ denotes the outward normal derivative to $\Gamma_2 = \partial\Omega_2$ (see Figure 1). The equations (1.3a) and (1.3b) signify a decomposition into two problems in the separate domains Ω_1 and Ω_2, while (1.3c) and (1.3d) reflect the appropriate *coupling* of these two problems.

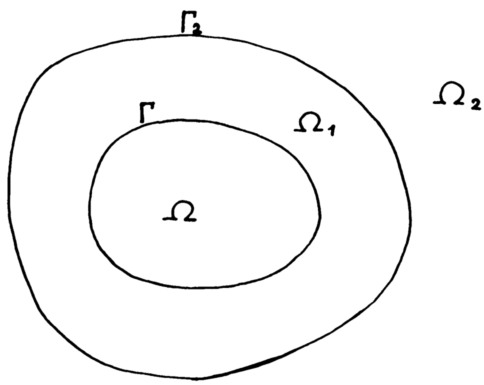

FIGURE 1

2. A Variational Formulation of the Model Problem. Let us now give a variational formulation of (1.2). Since $-\Delta u = f$ in Ω_1, we find, using Green's formula, that

$$a(u, v) + \langle v, \lambda\rangle = (f, v) \quad \forall v \in W, \tag{2.1}$$

where

$$\lambda = \frac{\partial u}{\partial n} |\Gamma_2, \qquad a(u, v) = \int_{\Omega_1} \nabla u \cdot \nabla v \, dx,$$

$$\langle v, \lambda\rangle = \int_{\Gamma_2} v\lambda \, ds, \qquad (f, v) = \int_{\Omega_1} fv \, dx,$$

$$W = \{v \in H^1(\Omega_1): v = 0 \text{ on } \Gamma\}.$$

Moreover, since $-\Delta u = 0$ in Ω_2, we find, using Green's formula and (1.2), that (cf. [6]),

$$\tfrac{1}{2}u(x) = \int_{\Gamma_2} u(y) G_n(x, y)\, ds_y - \int_{\Gamma_2} \lambda(y) G(x, y)\, ds_y + \alpha, \qquad x \in \Gamma_2, \tag{2.2}$$

$$u(x) = \int_{\Gamma_2} u(y) G_n(x, y)\, ds_y - \int_{\Gamma_2} \lambda(y) G(x, y)\, ds_y + \alpha, \qquad x \in \Omega_2, \tag{2.3}$$

where

$$G(x, y) = \frac{1}{2\pi} \log|x - y|, \qquad x \neq y,$$

is the Green's function associated with the two-dimensional Laplacian and

$$G_n(x, y) = \frac{\partial}{\partial n_y} G(x, y), \qquad x \neq y,\ y \in \Gamma_2,$$

with n_y being the outward unit normal to Γ_2 at $y \in \Gamma_2$. Let us observe that (1.2) together with (2.3) imply that $\int_{\Gamma_2} \lambda \, ds = 0$, since otherwise $u(x)$ would behave like $c \log|x|$, $c \neq 0$, as $|x| \longrightarrow \infty$, thus contradicting (1.2).

Now, formally multiplying (2.2) by the function $\mu(x)$ satisfying $\int_{\Gamma_2} \mu \, ds = 0$, and integrating over Γ_2, we find that

$$b(\lambda, \mu) - \frac{1}{2}\langle u, \mu\rangle + \langle G_n u, \mu\rangle = 0, \tag{2.4}$$

where

$$\begin{aligned} b(\lambda, \mu) &= -\int_{\Gamma_2} \int_{\Gamma_2} \lambda(y) \mu(x) G(x, y)\, ds_x ds_y, \\ G_n u(x) &= \int_{\Gamma_2} u(x) G_n(x, y)\, ds_y. \end{aligned} \tag{2.5}$$

We recall (see [6]) that b is a continuous bilinear form on $H^{-1/2}(\Gamma_2) \times H^{-1/2}(\Gamma_2)$. Moreover, b is H-elliptic with

$$H = \{\mu \in H^{-1/2}(\Gamma_2): \langle 1, \mu\rangle = 0\},$$

i.e., there exists a positive constant β such that

$$b(\mu, \mu) \geqslant \beta|\mu|^2_{-1/2}, \qquad \mu \in H. \tag{2.6}$$

Here $\langle\cdot,\ \cdot\rangle$ denotes the duality between $H^{1/2}(\Gamma_2)$ and $H^{-1/2}(\Gamma_2)$ and we use the notation

$$|\cdot|_s = \|\cdot\|_{H^s(\Gamma_2)}.$$

Recalling (2.1) and (2.4), we are thus led to the following variational formulation of (1.2): *Find* $(u, \lambda) \in W \times H$ *such that*

$$\begin{cases} a(u, v) + \langle v, \lambda\rangle = (f, v) & \forall v \in W, \quad (2.7\text{a}) \\ 2b(\lambda, \mu) - \langle u, \mu\rangle + 2\langle G_n u, \mu\rangle = 0 & \forall \mu \in H. \quad (2.7\text{b}) \end{cases}$$

Let us now analyze this problem. First, recalling the trace theorem:

$$|\gamma v|_{s-1/2} \leqslant C_s\|v\|_s \quad \forall v \in H^s(\Omega_1), \tag{2.8}$$

where $s > 1/2$, $\gamma v = v|_{\Gamma_2}$ and $\|\cdot\|_s = \|\cdot\|_{H^s(\Omega_1)}$, it follows that $\langle\cdot,\ \cdot\rangle$ is a continuous bilinear form on $W \times H$. Further, since $v = 0$ on Γ if $v \in W$, it follows that $a(\cdot, \cdot)$ is W-elliptic, i.e., there is a positive constant β' such that

$$a(v, v) \geqslant \beta'\|v\|_1^2 \quad \forall v \in W. \tag{2.9}$$

Moreover, since

$$G_n(x, y) = -\frac{n_y\cdot(x-y)}{|x-y|^2}, \qquad x, y \in \Gamma_2, \tag{2.10}$$

and Γ is smooth, it follows that G_n is pseudo-homogeneous of degree zero and thus (see [9]) the integral operator G_n defined by (2.5) is smoothing. More precisely, one has

$$|G_n v|_{s+1} \leqslant C_s|v|_s \quad \forall v \in H^s(\Gamma_2). \tag{2.11}$$

In order to analyze (2.7), it is convenient to introduce the following simplified problem obtained by omitting the term $\langle G_n u, \mu\rangle$: *Given* $\hat{g} = (g_1, g_2, g_3)$ *find* $(w, \theta) \in W \times H$ *such that*

$$\begin{cases} a(w, v) + \langle v, \theta\rangle = (g_1, v) + \langle v, g_2\rangle & \forall v \in W, \quad (2.12\text{a}) \\ 2b(\theta, \mu) - \langle w, \mu\rangle = \langle g_3, \mu\rangle & \forall \mu \in H. \quad (2.12\text{b}) \end{cases}$$

We shall see that due to (2.11) the full problem (2.7) is a compact perturbation of the simplified problem (2.12).

Let us now formulate (2.7) and (2.12) as operator equations. To this end we introduce the continuous bilinear forms

$$A, B, K\colon V \times V \longrightarrow \mathbf{R},$$

where $V = W \times H$, and the corresponding continuous linear mappings

$$A, B, K\colon V \longrightarrow V',$$

defined by

$$B(\hat{u}, \hat{v}) \equiv [B\hat{u}, \hat{v}] \equiv a(u, v) + \langle v, \lambda\rangle - \langle u, \mu\rangle + 2b(\lambda, \mu),$$

$$K(\hat{u}, \hat{v}) \equiv [K\hat{u}, \hat{v}] = \langle G_n u, \mu\rangle \quad \forall \hat{u} = (u, \lambda), \hat{v} = (v, \mu) \in V,$$

$$A = B + K,$$

where $[\cdot, \cdot]$ denotes the duality between V and V', the dual of V. Then (2.12) can be formulated

$$B\hat{u} = \hat{g}, \qquad \hat{u} = (u, \lambda), \tag{2.13}$$

i.e.,

$$B(\hat{u}, \hat{v}) = [\hat{g}, \hat{v}] \quad \forall \hat{v} = (v, \lambda) \in V, \tag{2.14}$$

and (2.7) is equivalent to

$$A\hat{u} = \hat{f}, \tag{2.15}$$

i.e.,

$$A(\hat{u}, \hat{v}) = [\hat{f}, \hat{v}] \quad \forall \hat{v} \in V, \tag{2.16}$$

with $\hat{f} = (f, 0, 0)$.

Let us note that the bilinear form $B(\cdot, \cdot)$ is V-elliptic; by (2.6) and (2.9) we have that

$$\begin{aligned} B(\hat{v}, \hat{v}) &= a(v, v) + 2b(\mu, \mu) \\ &\geq \beta' \|v\|_1^2 + 2\beta|\mu|_{-1/2}^2 \geq \beta'' \|\hat{v}\|_V^2 \quad \forall \hat{v} \in V, \end{aligned} \tag{2.17}$$

where $\beta'' = \min(\beta', 2\beta)$ and $\|\cdot\|_V$ denotes the norm in V, i.e.,

$$\|\hat{v}\|_V^2 = (\|v\|_1^2 + |\mu|_{-1/2}^2)^{1/2}.$$

LEMMA 1. *The mapping B: $V \longrightarrow V'$ is an isomorphism. Moreover, for $k \geq 0$ the mapping*

$$B^{-1}: H^{k-1}(\Omega_1) \times H^{k-1/2}(\Gamma_2) \times H^{k+1/2}(\Gamma_2) \longrightarrow H^{k+1}(\Omega_1) \times H^{k-1/2}(\Gamma_2),$$

defined by $B\hat{u} = \hat{g}$ is continuous.

Proof. The first statement of the lemma follows directly from the V-ellipticity (2.17). The regularity result is proved in the Appendix below. □

Let us now return to the original problem $A\hat{u} = \hat{f}$. Since $A = B + K$, this problem can be written after applying B^{-1}:

$$(I + B^{-1}K)\hat{u} = B^{-1}\hat{f}, \tag{2.18}$$

where I: $V \longrightarrow V$ is the identity mapping. Now, recalling (2.8) and (2.11), it follows that K: $V \longrightarrow \{0\} \times \{0\} \times H^{3/2}(\Gamma_2)$, is continuous. Therefore, using Lemma 1 with $k = 1$, we see that $B^{-1}K$: $V \longrightarrow H^2(\Omega_1) \times H^{1/2}(\Gamma_2)$ is continuous. Since $H^2(\Omega_1) \times H^{1/2}(\Gamma_2)$ is compactly embedded in V, it follows that $B^{-1}K$: $V \longrightarrow V$ is compact and hence (2.18) is an equation of the Fredholm second kind. Thus, to prove existence of a solution to (2.18) or the equivalent original problem (2.15), it is sufficient to prove uniqueness. With this observation it is easy to prove

LEMMA 2. *The mapping* $A: V \longrightarrow V'$ *is an isomorphism. Moreover, for* $k \geqslant 0$ *the mapping*

$$A^{-1}: H^{k-1}(\Omega_1) \times H^{k-1/2}(\Gamma_2) \times H^{k+1/2}(\Gamma_2) \longrightarrow H^{k+1}(\Omega_1) \times H^{k-1/2}(\Gamma_2)$$

is continuous.

Proof. To prove uniqueness of the solution of the equation $A\hat{u} = \hat{f}$, let us assume that $\hat{w} = (w, \theta) \in V$ and $A\hat{w} = 0$, i.e.,

$$\left\{\begin{array}{ll} a(w, v) + \langle v, \theta\rangle = 0 \quad \forall v \in W, & (2.19\text{a}) \\ 2b(\theta, \mu) - \langle w, \mu\rangle + 2\langle G_n w, \mu\rangle = 0 \quad \forall \mu \in H. & (2.19\text{b}) \end{array}\right.$$

From (2.19a) it follows that

$$\left\{\begin{array}{ll} -\Delta w = 0 & \text{in } \Omega_1, \\ \dfrac{\partial w}{\partial n} = 0 & \text{on } \Gamma_2. \end{array}\right.$$

Let now $\widetilde{w} \in W^1(\Omega_2)$ be the harmonic extension of w to Ω_2, i.e.,

$$\left\{\begin{array}{ll} -\Delta \widetilde{w} = 0 & \text{in } \Omega_2, \\ \widetilde{w} = \gamma w & \text{on } \Gamma. \end{array}\right. \tag{2.20}$$

Then, by an argument similar to that leading to (2.4), it follows that

$$2b(\widetilde{\theta}, \mu) - \langle w, \mu\rangle + 2\langle G_n w, \mu\rangle = 0, \qquad \mu \in H, \tag{2.21}$$

where $\widetilde{\theta} = \partial\widetilde{w}/\partial n|_{\Gamma_2} \in H$. Combining (2.19b) and (2.21) we find that $b(\widetilde{\theta} - \theta, \mu) = 0$ $\forall \mu \in H$, and thus (2.6) shows that $\theta = \widetilde{\theta}$. But this means that if w is extended to Ω^c by putting $w = \widetilde{w}$ in Ω_2, then $\Delta w = 0$ in Ω^c, $w \in W^1(\Omega^c)$, and $w = 0$ on Γ so that $w \equiv 0$ and the uniqueness follows. Thus, for any $\hat{g} \in V'$, the equation $A\hat{w} = \hat{g}$ has a unique solution and the continuity of $A^{-1}: V' \longrightarrow V$ follows from the closed graph theorem. This proves the first statement of the lemma.

To prove the regularity result, we use induction on k. Thus assume that the statement holds for $k = m - 1$. Let us consider the equation $A\hat{w} = \hat{g}$, where $\hat{g} \in H^{m-1}(\Omega_1) \times H^{m-1/2}(\Gamma_2) \times H^{m+1/2}(\Gamma_2)$. By the induction hypothesis, we then have $\hat{w} \in H^m(\Omega_1) \times H^{m-3/2}(\Gamma_2)$ so that by (2.11) $K\hat{w} \in \{0\} \times \{0\} \times H^{m+1/2}(\Gamma_2)$. But the equation $A\hat{w} = \hat{g}$ can be written

$$B\hat{w} = \hat{g} - K\hat{w},$$

and thus by Lemma 1 we conclude that $\hat{w} \in H^{m+1}(\Omega_1) \times H^{m-1/2}(\Gamma_2) \times H^{m+1/2}(\Gamma_2)$. Therefore A^{-1} maps $H^{m-1}(\Omega_1) \times H^{m-1/2}(\Gamma_2) \times H^{m+1/2}(\Gamma_2)$ into $H^{m+1}(\Omega_1) \times H^{m-1/2}(\Gamma_2)$ and the continuity of the mapping follows from the closed graph theorem. This completes the induction step and thus the proof of the lemma. □

We shall also need the corresponding result for the adjoints B^*, A^*: $V \longrightarrow V'$ defined by

$$[A^*\hat{v}, \hat{w}] = [A\hat{w}, \hat{v}],$$

$$[B^*\hat{v}, \hat{w}] = [B\hat{w}, \hat{v}] \quad \forall \hat{v}, \hat{w} \in V.$$

LEMMA 3. *The mappings B^*, A^*: $V \longrightarrow V'$ are isomorphisms and, for $k \geqslant 0$,*

$$B^{*-1}, A^{*-1}: H^{k-1}(\Omega_1) \times H^{k-1/2}(\Gamma_2) \times H^{k+1/2}(\Gamma_2) \longrightarrow H^{k+1}(\Omega_1) \times H^{k-1/2}(\Gamma_2)$$

are continuous.

The proof is parallel to the proofs of Lemmas 1 and 2.

Remark. As pointed out by the referee, if the outer boundary Γ_2 is a circle, then one can solve the equation (2.3) in λ explicitly. More precisely, in this case (2.3) takes the form

$$-\frac{1}{\pi}\int_0^{2\pi} \widetilde{\lambda}(\theta)\log\left(2\left|\sin\frac{\theta-\eta}{2}\right|\right) R\,d\theta = \gamma(\eta),$$

where $\widetilde{\lambda}(\theta) = \lambda(R\cos\theta, R\sin\theta)$ and γ is determined by u. This integral equation has the explicit solution

$$\widetilde{\lambda}(\theta) = \frac{1}{2\pi R}\int_0^{2\pi}\frac{d\gamma}{d\eta}\operatorname{cotan}\left(\frac{\eta-\theta}{2}\right)d\eta - \frac{1}{2\pi R(\log R^2)}\int_0^{2\pi}\gamma\,d\eta,$$

which makes it possible to eliminate λ from (2.7) and thus obtain an equation involving only u. To see if such a procedure is advantageous from a numerical point of view requires further investigation. □

3. The Coupled Procedure. Error Estimates. Let us now consider a finite element method based on the variational formulation (2.7). Let $W_h \subset W$ and $H_h \subset H$ be finite-dimensional spaces depending on the positive parameter h and set $V_h = W_h \times H_h$. Let $A_h(\cdot\,,\ \cdot)$ be a bilinear form approximating $A(\cdot\,,\ \cdot)$ and consider the following discrete problem: *Find $\hat{u}_h = (u_h, \lambda_h) \in V_h$ such that*

$$A_h(\hat{u}_h, \hat{v}) = (f, v) \quad \forall \hat{v} \in V_h. \tag{3.1}$$

We shall assume that the spaces W_h and H_h satisfy the following approximation hypothesis: For any positive ϵ, there exists a constant C such that

$$\operatorname*{Inf}_{v \in W_h} \|w - v\|_1 \leqslant Ch^s\|w\|_{s+1+\epsilon}, \qquad 0 \leqslant s \leqslant k, \tag{3.2a}$$

$$\operatorname*{Inf}_{\mu \in H_h} |\theta - \mu|_{-1/2} \leqslant Ch^s|\theta|_{s-1/2}, \qquad 0 \leqslant s \leqslant k, \tag{3.2b}$$

where k is a positive integer. This will correspond to using piecewise polynomials of degree k for W_h and degree $k - 1$ for H_h (cf. Example 1 below). Note that the functions in H_h may be chosen to be discontinuous while the functions in W_h will have to be continuous. Furthermore, we shall assume that there is a constant C such that

$$|A(\hat{v}, \hat{w}) - A_h(\hat{v}, \hat{w})| \leqslant Ch^k\|\hat{v}\|_V\|\hat{w}\|_V \quad \forall \hat{v}, \hat{w} \in V_h. \tag{3.3}$$

In Example 1 below we shall in detail consider a finite element method satisfying (3.2) and (3.3) with $k = 1$.

We shall now prove that for h small enough the problem (3.1) admits a unique solution $\hat{u}_h$ and then estimate the error $\hat{u} - \hat{u}_h$. The crucial result is then the following:

LEMMA 4. *There is a positive constant c such that for h small enough*

$$\sup_{\hat{v} \in V_h; \hat{v} \neq 0} \frac{A_h(\hat{w}, \hat{v})}{\|\hat{v}\|_V} \geqslant c\|\hat{w}\|_V \quad \forall \hat{w} \in V_h. \tag{3.4}$$

Proof. Given $\hat{w} \in V_h$ there exists by Lemma 3 $\hat{\psi} = (\psi, H) \in V$ such that

$$A(\hat{v}, \hat{\psi}) = (\hat{w}, \hat{v})_V, \qquad \hat{v} \in V, \tag{3.5}$$

where $(\cdot\,, \cdot)_V$ denotes the scalar product in V, and

$$\|\hat{\psi}\|_V \leqslant C\|\hat{w}\|_V. \tag{3.6}$$

In fact, $\hat{\psi} = A^{*-1}J\hat{w}$, where $J\colon V \longrightarrow V'$, is the canonical mapping defined by $[J\hat{w}, \hat{v}] = (\hat{w}, \hat{v})_V$, $\forall \hat{v}, \hat{w} \in V$. Furthermore, again by Lemma 3, there exists $\hat{\psi}_h = (\psi_h, H_h) \in V_h$ such that

$$B(\hat{v}, \hat{\psi} - \hat{\psi}_h) = 0 \quad \forall \hat{v} \in V_h, \tag{3.7}$$

and

$$\|\hat{\psi}_h\|_V \leqslant C\|\hat{\psi}\|_V. \tag{3.8}$$

Now, using (3.7) and (3.5) with $v = w$, we find that

$$\begin{aligned} A(\hat{w}, \hat{\psi}_h) &= B(\hat{w}, \hat{\psi}_h) + K(\hat{w}, \hat{\psi}_h) \\ &= B(\hat{w}, \hat{\psi}) + K(\hat{w}, \hat{\psi}_h) = A(\hat{w}, \hat{\psi}) + K(\hat{w}, \hat{\psi}_h - \hat{\psi}) \\ &= \|\hat{w}\|_V^2 + \langle G_n\hat{w}, H_h - H\rangle \\ &\geqslant \|\hat{w}\|_V^2 - |G_n w|_{3/2}|H - H_h|_{-3/2} \\ &\geqslant \|\hat{w}\|_V^2 - \|\hat{w}\|_V|H - H_h|_{-3/2}, \end{aligned} \tag{3.9}$$

since by (2.11) and (2.8),

$$|G_n w|_{3/2} \leqslant C|w|_{1/2} \leqslant C\|w\|_1 \leqslant C\|\hat{w}\|_V.$$

In order to estimate $|H - H_h|_{-3/2}$, we shall use the usual duality argument: Given $\nu \in H^{3/2}(\Gamma_2)$ let $\hat{\varphi} = B^{-1}\hat{\nu}$ where $\hat{\nu} = (0, 0, \nu)$, i.e.,

$$B(\hat{\varphi}, \hat{v}) = \langle \nu, \mu\rangle \quad \forall \hat{v} = (v, \mu) \in V. \tag{3.10}$$

By Lemma 2 we then have

$$\|\hat{\varphi}\|_{H^2(\Omega_1)\times H^{1/2}(\Gamma_2)} \leqslant C|\nu|_{3/2}. \tag{3.11}$$

Hence, taking $\hat{v} = \hat{\psi} - \hat{\psi}_h$ in (3.10) and using (3.7), we find that for any $\hat{\varphi}_h \in V_h$,

$$\begin{aligned}\langle \nu, H - H_h \rangle &= B(\hat{\varphi}, \hat{\psi} - \hat{\psi}_h) = B(\hat{\varphi} - \hat{\varphi}_h, \hat{\psi} - \hat{\psi}_h)\\ &\leqslant C\|\hat{\varphi} - \hat{\varphi}_h\|_V \|\hat{\psi} - \hat{\psi}_h\|_V \leqslant C\|\hat{\varphi} - \hat{\varphi}_h\|_V \|\hat{w}\|_V,\end{aligned}$$

where the last inequality follows from (3.6) and (3.8). Thus, using (3.2) with $s = 1 - \epsilon$ together with (3.11), it follows that

$$\langle \nu, H - H_h \rangle \leqslant Ch^{1-\epsilon}|\nu|_{3/2}\|\hat{w}\|_V, \qquad \nu \in H^{3/2}(\Gamma_2),$$

which proves that

$$|H - H_h|_{-3/2} \leqslant Ch^{1-\epsilon}\|\hat{w}\|_V,$$

where $0 < \epsilon < 1$. Returning to (3.9) we thus have

$$A(\hat{w}, \hat{\psi}_h) \geqslant \|\hat{w}\|_V^2 - Ch^{1-\epsilon}\|\hat{w}\|_V^2 = (1 - Ch^{1-\epsilon})\|\hat{w}\|_V^2.$$

Finally, recalling (3.3), we conclude that

$$A_h(\hat{w}, \hat{\psi}_h) = A(\hat{w}, \hat{\psi}_h) + A_h(\hat{w}, \hat{\psi}_h) - A(\hat{w}, \hat{\psi}_h) \geqslant (1 - Ch^{1-\epsilon})\|\hat{w}\|_V^2.$$

Since $\|\hat{\psi}_h\|_V \leqslant C\|\hat{w}\|_V$, this proves that (3.4) holds for h sufficiently small and the proof is complete. □

We can now prove

THEOREM 1. *For h sufficiently small the discrete problem* (3.1) *admits a unique solution $\hat{u}_h \in V_h$ and we have the following error estimate*:

$$\|\hat{u} - \hat{u}_h\|_V \leqslant Ch^k\|u\|_{k+1+\epsilon}.$$

Proof. Uniqueness, and hence existence, of a solution of (3.1) for h sufficiently small follows directly from Lemma 4. Furthermore, using (2.16), (3.1), (3.3) and Lemma 4, we see that for any $\hat{v}_h \in V_h$,

$$\begin{aligned}\|\hat{u}_h - \hat{v}_h\|_V &\leqslant C \sup_{\hat{v} \in V_h} \frac{A_h(\hat{u}_h - \hat{v}_h, \hat{v})}{\|\hat{v}\|_V}\\ &= C \sup_{\hat{v} \in V_h} \frac{A(\hat{u} - \hat{v}_h, \hat{v}) + A(\hat{v}_h, \hat{v}) - A_h(\hat{v}_h, \hat{v})}{\|\hat{v}\|_V}\\ &\leqslant C\|\hat{u} - \hat{v}_h\|_V + Ch^k\|\hat{v}_h\|_V.\end{aligned}$$

Thus, choosing $\hat{v}_h$ according to (3.2), we find that

$$\begin{aligned}\|\hat{u} - \hat{u}_h\|_V &\leqslant \|\hat{u} - \hat{v}_h\|_V + \|\hat{u}_h - \hat{v}_h\|_V\\ &\leqslant Ch^k(\|u\|_{k+1+\epsilon} + |\lambda|_{k-1/2}).\end{aligned}$$

Finally, by the trace theorem (2.8), we have

$$|\lambda|_{k-1/2} \leqslant C\|u\|_{k+1},$$

and the proof is complete. □

Let us now exhibit a natural finite element method satisfying (3.2) and (3.3) with $k = 1$.

Example 1. Let Γ_2 be chosen so that $\bar{\Omega} \cup \Omega_1$ is convex and let $\Omega_1^h \subset \Omega_1$ be a polygonal domain approximating Ω_1 according to Figure 2. Let Γ^h and Γ_2^h be the corresponding polygonal approximations of Γ and Γ_2 so that $\partial\Omega_1^h = \Gamma^h \cup \Gamma_2^h$. Let $S_h = \{S\}$ be the sides of Γ_2^h, let h be the maximal length of the sides $S \in S_h$, and define

$$\widetilde{H}_h = \{\mu \in L^2(\Gamma_h)\colon \mu|_S \text{ is constant}, S \in S_h, \langle 1, \mu\rangle_h = 0\},$$

where $\langle v, \mu\rangle_h = \int_{\Gamma^h} v\mu\, ds$. Further, let $T_h = \{T\}$ be a regular* triangulation of Ω_1^h with maximal sidelength at most h and define

$$\widetilde{W}_h = \{v \in H^1(\Omega_1^h)\colon v|_T \text{ is linear}, T \in T_h, v = 0 \text{ on } \Gamma^h\}.$$

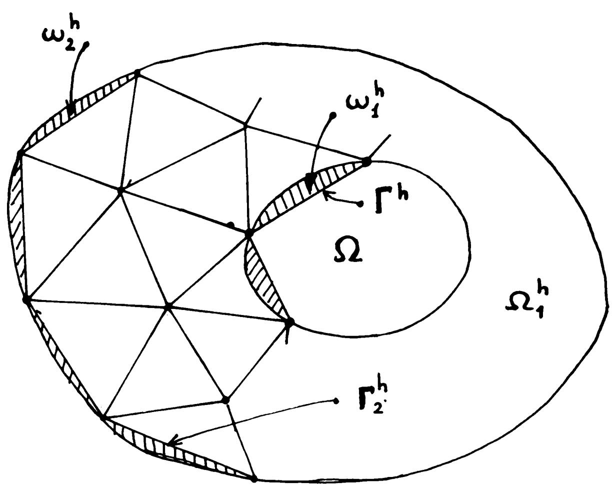

FIGURE 2

In order to formulate a discrete analogue of (2.7), using the spaces $\widetilde{W}_h$ and $\widetilde{H}_h$ and replacing boundary integrals along Γ by integrals along the polygonal boundary Γ_2^h, we have to rewrite the term $\langle G_n u, \mu\rangle$. To this end we note that taking $u \equiv 1$ in (2.2) shows that

$$\int_{\Gamma_2} G_n(x, y)\, ds_y = -\tfrac{1}{2}, \qquad x \in \Gamma_2.$$

Hence recalling (2.10), we have

$$\begin{aligned} \langle G_n u, \mu\rangle &= \int_{\Gamma_2}\int_{\Gamma_2} G_n(x, y)\,[u(y) - u(x)]\,\mu(x)\, ds_y\, ds_x - \tfrac{1}{2}\langle u, \mu\rangle \\ &= -\frac{1}{2\pi}\int_{\Gamma_2}\int_{\Gamma_2} \frac{n_y \cdot (x - y)}{|x - y|^2}(u(y) - u(x))\mu(x)\, ds_y\, ds_x - \tfrac{1}{2}\langle u, \mu\rangle \\ &\equiv d(u, \mu) - \tfrac{1}{2}\langle u, \mu\rangle, \end{aligned} \tag{3.12}$$

* All angles of the triangles $T \in T_h$ are bounded below uniformly in h.

with the obvious definition of $d(u, \mu)$. As a discrete analogue of the form $d(\cdot, \cdot)$, we now introduce the form $d_h(\cdot, \cdot)$ defined by

$$\tilde{d}_h(w, \mu) = -\frac{1}{2\pi}\int_{\Gamma_2^h}\int_{\Gamma_2^h}\frac{n_{yh}\cdot(x-y)}{|x-y|^2}\,(w(y)-w(x))\mu(x)\,ds_y\,ds_x,$$

where n_{hy} is linear on each $S \in S_h$ and n_{hy} interpolates the normal n_y at the vertices of Γ_2^h. We note that, since functions in W_h are Lipschitz continuous, the form $\tilde{d}_h(\cdot, \cdot)$ is well defined on $W_h \times H_h$.

We can now formulate the following discrete analogue of (2.7): *Find* $(\tilde{u}_h, \tilde{\lambda}_h) \in \tilde{W}_h \times \tilde{H}_h$ *such that*

$$\begin{cases} a_h(\tilde{u}_h, v) + \langle v, \tilde{\lambda}_h\rangle_{\tilde{h}} = (f, v)_h \quad \forall v \in \tilde{W}_h, & (3.13\text{a}) \\ 2\tilde{b}_h(\tilde{\lambda}_h, \mu) - 2\langle\tilde{u}_h, \mu\rangle_{\tilde{h}} + 2\tilde{d}_h(\tilde{u}_h, \mu) = 0 \quad \forall \mu \in \tilde{H}_h, & (3.13\text{b}) \end{cases}$$

where

$$a_h(u, v) = \int_{\Omega_1^h} \nabla u \cdot \nabla v\,dx,$$

$$\langle v, \lambda\rangle_{\tilde{h}} = \int_{\Gamma_2^h} v\lambda\,ds,$$

$$\tilde{b}_h(\lambda, \mu) = -\int_{\Gamma_2^h}\int_{\Gamma_2^h}\lambda(y)\mu(x)G(x, y)\,ds_y\,ds_x.$$

The problem (3.13) will lead to a nonsymmetric linear system of equations where we have one unknown per node in the triangulation T_h of Ω_1^h and one unknown per side of the polygonal boundary Γ_2^h. The coefficients corresponding to the forms $a_h(\cdot, \cdot)$ and $\langle\cdot, \cdot\rangle_{\tilde{h}}$ are easy to compute. Algorithms for computing the coefficients corresponding to the forms $\tilde{b}_h(\cdot, \cdot)$ and $\tilde{d}_h$ can be found in [2].

Let us now show that the problem (3.13) can be put into the form (3.1) with assumptions (3.2) and (3.3) fulfilled. First, in order to convert the spaces $\tilde{W}_h$ and $\tilde{H}_h$ into subspaces $W_h \subset W$ and $H_h \subset H$, we introduce the mapping $\psi\colon \Gamma_2^h \longrightarrow \Gamma_2$, where $\psi(x)$ is the point on Γ_2 closest to the point $x \in \Gamma_2^h$. For h small enough ψ is clearly a bijection. Now, using the mapping ψ^{-1} to transform integrals along Γ_2^h to integrals along Γ_2, we have

$$\int_{\Gamma_2^h}\mu\,ds = \int_{\Gamma_2}\mu\circ\psi^{-1}J(\psi^{-1})\,ds,$$

where $J(\psi^{-1}) = |\partial\psi^{-1}/\partial s|$, and $\partial/\partial s$ denotes differentiation in the tangential direction to Γ_2. We now define

$$H_h = \{\mu\colon \mu = J(\psi^{-1})\tilde{\mu}\circ\psi^{-1},\ \tilde{\mu} \in \tilde{H}_h\}.$$

Note that if $\mu = J(\psi^{-1})\tilde{\mu}\circ\psi^{-1} \in H_h$ with $\tilde{\mu} \in \tilde{H}_h$, then

$$0 = \int_{\Gamma_2^h}\tilde{\mu}\,ds = \int_{\Gamma_2}\tilde{\mu}\circ\psi^{-1}J(\psi^{-1})\,ds = \int_{\Gamma_2}\mu\,ds,$$

so that $H_h \subset H$.

Furthermore, we extend each function $\widetilde{w} \in \widetilde{W}_h$ defined in Ω_1^h to a function w defined in Ω_1 by setting $w = 0$ in the "skin" ω_1^h with boundary $\Gamma \cup \Gamma^h$ and finally by setting $w(y) = w(x)$ for y on the line segment between $x \in \Gamma_2^h$ and $\psi(x)$ thus defining w in the "skin" ω_2^h with boundary $\Gamma_2 \cup \Gamma_2^h$ (see Figure 2). We denote by W_h the set of functions obtained in this way.

By changing integrations from Γ_2^h to Γ_2 and using the definitions of H_h and W_h, the problem (3.13) can now be formulated as follows: *Find* $(u_h, \lambda_h) \in W_h \times H_h$ *such that*

$$\begin{cases} a_h(u_h, v) + \langle v, \lambda_h \rangle = (f, v) \quad \forall v \in W_h, & (3.14a) \\ 2b_h(\lambda_h, \mu) - 2\langle u_h, \mu \rangle + 2d_h(u_h, \mu) = 0 \quad \forall \mu \in H_h, & (3.14b) \end{cases}$$

where

$$b_h(\lambda, \mu) = -\int_{\Gamma_2} \int_{\Gamma_2} \lambda(y)\mu(x) \log|(\psi^{-1}(y) - \psi^{-1}(x))| \, ds_y \, ds_x,$$

$$d_h(u, \mu) = -\frac{1}{2\pi} \int_{\Gamma_2} \int_{\Gamma_2} \frac{n_{yh} \cdot (\psi^{-1}(x) - \psi^{-1}(y))}{|\psi^{-1}(x) - \psi^{-1}(y)|^2} (u(y) - u(x))\mu(x) \times J(\psi^{-1}(y)) \, ds_y \, ds_x.$$

Let us now check that the assumptions (3.2) and (3.3) are satisfied with $k = 1$ and

$$A_h(v, w) = a_h(v, w) + \langle w, \lambda \rangle - 2\langle v, \mu \rangle + 2b_h(\lambda, \mu) + 2d_h(v, \mu).$$

To prove (3.2a) let $w \in H^2(\Omega_1)$ be given and let $w_h \in \widetilde{W}$ interpolate w at the nodes of T_h. Then, by well-known interpolation theory (see [1]),

$$\|w - w_h\|_{H^1(\Omega_1^h)} \leqslant Ch\|w\|_2.$$

By Sobolev's embedding theorem we have, for any $\epsilon > 0$,

$$\|\nabla w\|_{L^\infty(\Omega_1)} \leqslant C\|w\|_{2+\epsilon},$$

and hence also

$$\|\nabla w_h\|_{L^\infty(\Omega_1)} \leqslant C\|w\|_{2+\epsilon}.$$

Since the area of $\Omega_1 - \Omega_1^h$ is of the order $O(h^2)$, this proves that

$$\|w - w_h\|_{H^1(\Omega_1 \backslash \Omega_1^h)} \leqslant Ch\|w\|_{2+\epsilon},$$

and thus (3.2a) follows. For a proof of (3.2b) we refer to [6].

It remains to prove (3.3). First, since by the construction of W_h,

$$\|w_h\|_{H^1(\omega_2^h)} \leqslant Ch^{1/2}\|w_h\|_1, \qquad w_h \in W_h,$$

we find that

$$|a(w_h, v_h) - a_h(w_h, v_h)| = \int_{\omega_2^h} \nabla w_h \cdot \nabla v_h \, dx \leqslant Ch\|w_h\|_1 \|v_h\|_1. \tag{3.15}$$

Next, since (see [5])

$$\left| \frac{|\psi^{-1}(x) - \psi^{-1}(y)|}{|x - y|} - 1 \right| \leqslant Ch^2, \tag{3.16}$$

it follows easily that

$$|b(\nu, \mu) - b_h(\nu, \mu)| \leqslant Ch^2 |\nu|_0 \, |\mu|_0. \tag{3.17}$$

By the "inverse estimate" (see [6])

$$|\mu|_0 \leqslant Ch^{-1/2} |\mu|_{-1/2}, \qquad \mu \in H_h, \tag{3.18}$$

we thus have

$$|b(\nu, \mu) - b_h(\nu, \mu)| \leqslant Ch |\nu|_{-1/2} \, |\mu|_{-1/2}. \tag{3.19}$$

Finally, using (3.16) we find that, for $\epsilon > 0$,

$$\begin{aligned} |d(w, \mu) - d_h(w, \mu)| &\leqslant Ch^2 \int_{\Gamma_h^2} \int_{\Gamma_h^2} \frac{|w(y) - w(x)|}{|x - y|} \mu(x) \, ds_x \, ds_y \\ &\leqslant Ch^2 \left(\int_{\Gamma_2^h} \int_{\Gamma_2^h} \frac{|\mu(x)|^2}{|x - y|^{1-\epsilon}} \, ds_x \, ds_y \right)^{1/2} \left(\int_{\Gamma_2^h} \int_{\Gamma_2^h} \frac{|w(y) - w(x)|^2}{|x - y|^{1+\epsilon}} \, ds_x \, ds_y \right)^{1/2} \\ &\equiv Ch^2 F_1 F_2, \end{aligned}$$

where we have used Cauchy's inequality and F_1 and F_2 are defined in the obvious way. Integrating with respect to y in the factor F_1, we get $F_1 \leqslant |\mu|_0$. Further (see [5]), for $0 < \epsilon < 2$, $F_2 \leqslant C|w|_{\epsilon/2}$, and therefore, taking $\epsilon = 1$, we have again, using (3.18),

$$|d(w, \mu) - d_h(w, \mu)| \leqslant Ch^2 |w|_{1/2} \, |\mu|_0 \leqslant Ch^{3/2} \|w\|_1 \, |\mu|_{-1/2}. \tag{3.20}$$

Combining (3.15), (3.19), and (3.20), it follows that (3.3) is valid with $k = 1$ and thus the verification is complete.

We can also construct analogous methods satisfying (3.2) and (3.3) for $k > 1$ using polynomials of degree k for $\widetilde{W}_h$ and degree $k - 1$ for $\widetilde{H}_h$. In such a case the domain Ω_1 will be approximated by a domain Ω_1^h with piecewise polynomial boundary $\Gamma^h \cup \Gamma_2^h$ of degree k approximating $\Gamma \cup \Gamma_2$. In the triangulation of Ω_1^h, it is then natural to use isoparametric elements of degree k with one curved edge along $\Gamma^h \cup \Gamma_2^h$ (cf. [1]). □

4. Error Estimates in Weaker Norms. Let us now, using a duality argument, prove an error estimate in a norm weaker than the norm in V. We shall then make the following assumption: For any $\epsilon > 0$ there exists a constant C such that if $\hat{w}_h \in V_h$ interpolates $\hat{w} \in X$, then

$$|A(\hat{u}_h, \hat{w}_h) - A_h(\hat{u}_h, \hat{w}_h)| \leqslant Ch^{k+1-\epsilon} \|\hat{u}\|_X \|\hat{w}\|_X, \tag{4.1}$$

where $X = H^2(\Omega_1) \times H^{1/2}(\Gamma_2)$. We have

THEOREM 2. *For any* $\epsilon > 0$ *there exists a constant* C *such that if* $\hat{u} \in H^{k+1}(\Omega_1) \times H^{k-1/2}(\Gamma_2)$, $k \geq 1$, *then*

$$\|\hat{u} - \hat{u}_h\|_{L^2(\Omega_1)\times H^{-3/2}(\Gamma_2)} \leq Ch^{k+1-\epsilon}\|u\|_{k+1+\epsilon}.$$

Proof. Given $\hat{\varphi} \in L^2(\Omega_1) \times H^{3/2}(\Gamma_2)$, let $\hat{\psi} \in V$ satisfy

$$A(\hat{v}, \hat{\psi}) = [\hat{v}, \hat{\varphi}] \quad \forall \hat{v} \in V. \tag{4.2}$$

By Lemma 3 we then have

$$\|\hat{\psi}\|_X \leq C\|\hat{\varphi}\|_{L^2(\Omega_1)\times H^{3/2}(\Gamma_2)}.$$

Taking $\hat{v} = \hat{u} - \hat{u}_h$ in (4.2), recalling (2.16), (3.1), and (3.2), and using (4.1), letting $\hat{\psi}_h \in V_h$ interpolate $\hat{\psi}$, we find that

$$\begin{aligned}
[\hat{u} - \hat{u}_h, \hat{\varphi}] &= A(\hat{u}, \hat{u}_h, \hat{\psi}) \\
&= A(\hat{u} - \hat{u}_h, \hat{\psi} - \hat{\psi}_h) + A_h(\hat{u}_h, \hat{\psi}_h) - A(\hat{u}_h, \hat{\psi}_h) \\
&\leq \|\hat{u} - \hat{u}_h\|_V \|\hat{\psi} - \hat{\psi}_h\|_V + Ch^{k+1-\epsilon}\|\hat{u}\|_X \|\hat{\psi}\|_X \\
&\leq C(h^{1-\epsilon}\|\hat{u} - \hat{u}_h\|_V + h^{k+1-\epsilon}\|\hat{u}\|_X)\|\hat{\varphi}\|_{L^2(\Omega_1)\times H^{3/2}(\Gamma_2)}.
\end{aligned}$$

Together with Theorem 1, this proves that

$$[\hat{u} - \hat{u}_h, \hat{\varphi}] \leq Ch^{k+1-\epsilon}\|\hat{\varphi}\|_{L^2(\Omega_1)\times H^{3/2}(\Gamma_2)} \quad \forall \hat{\varphi} \in L^2(\Omega_1) \times H^{3/2}(\Gamma_2),$$

and the lemma follows. □

Remark. It is easy to see that the method of Example 1 satisfies (4.1) with $k = 1$. Furthermore, if we define $u_h(x)$ for $x \in \Omega_2$ by

$$u_h(x) = \frac{1}{2}\int_{\Gamma_2^h} (\tilde{u}_h(y) - \tilde{u}_h(x)) \frac{n_{yh} \cdot (x - y)}{|x - y|^2}\, ds_y - \frac{1}{2}\int_{\Gamma_2^h} \tilde{\lambda}_h(y) G(x, y)\, ds_y,$$

then, for all $x \in \Omega_2$ with $\operatorname{dist}(x, \Gamma_2) \geq \delta > 0$ and h sufficiently small, we have $|u(x) - u_h(x)| \leq C_x h^{2-\epsilon}$, where the constant C_x depends on $\operatorname{dist}(x, \Gamma)$ (cf. [7]). □

5. A Symmetrized Procedure. The solution of the original problem (1.1) can be characterized as the solution of the minimization problem

$$\min_{w\in W^1(\Omega^c)} \left\{ \frac{1}{2}\int_{\Omega^c} |\nabla w|^2\, dx - \int_{\Omega^c} fw\, dx \right\}.$$

Since $f = 0$ in Ω_2, this problem can be formulated in the following way:

$$\min_{w\in V} \left\{ \frac{1}{2}\int_{\Omega_1} |\nabla w|^2\, dx + \frac{1}{2}\int_{\Omega_2} |\nabla \widetilde{w}|^2\, dx - \int_{\Omega_1} fw\, dx \right\}, \tag{5.1}$$

where $\widetilde{w} \in W^1(\Omega_2)$ is the harmonic extension of w according to (2.20). Since $\widetilde{w}$ is harmonic in Ω_2, we have by Green's formula $\int_{\Omega_2} |\nabla\widetilde{w}|^2 = \langle w, Dw\rangle$, where D: $H^{1/2}(\Gamma_2) \longrightarrow H$ is the continuous operator defined by $Dw = \partial\widetilde{w}/\partial n|_{\Gamma_2}$. Thus (5.1) can be formulated as follows:

$$\min_{w\in W}\left\{\frac{1}{2}\int_{\Omega_1}|\nabla w|^2\,dx+\frac{1}{2}\langle w, Dw\rangle-\int_{\Omega_1}fw\right\}.$$

The solution $u\in W$ of this problem is characterized by the relation

$$a(u, v)+\frac{1}{2}\{\langle v, Du\rangle+\langle u, Dv\rangle\}=(f, v)\quad\forall v\in W. \tag{5.2}$$

Recalling the formulation (2.7), we have that (2.7b) can equivalently be written $\lambda = Du$ and thus (2.7c) becomes

$$a(u, v)+\langle v, Du\rangle=(f, v)\quad\forall v\in W. \tag{5.3}$$

By Green's formula we have

$$\langle v, Du\rangle=\langle u, Dv\rangle,\qquad v, u\in W, \tag{5.4}$$

and hence (5.2) and (5.3) are equivalent. Thus the problem (5.3) obtained from (2.7), eliminating the variable λ, is in fact *symmetric.* Let us check if the discretized problem (3.13) of Example 1 has the same feature. Introducing the mapping $D_h\colon W_h\longrightarrow H_h$ defined by

$$b_h(D_hw_h, \mu)-\langle w_h, \mu\rangle+d_h(w_h, \mu)=0\quad\forall\mu\in H_h,$$

the problem (3.13) can be written: *Find* $u_h\in W_h$ *such that*

$$a_h(u_h, v)+\langle v, D_hu_h\rangle=(f, v)\quad\forall v\in W_h. \tag{5.5}$$

Now, in contrast to (5.4), we have in general $\langle v, D_hw\rangle\neq\langle w, D_hv\rangle$, and thus (5.5) will in general lead to a nonsymmetric system of equations.

In order to obtain a symmetric problem, which will facilitate the incorporation of the coupled procedure into existing finite element codes, it is natural to consider the following variant of (5.5): *Find* $u_h\in W_h$ *such that*

$$a_h(u_h, v)+\tfrac{1}{2}\{\langle v, D_hu_h\rangle+\langle u_h, D_hv\rangle\}=(f, v)\quad\forall v\in W_h, \tag{5.6}$$

or, equivalently, the minimization problem:

$$\min_{w\in W_h}\{\tfrac{1}{2}a_h(w, w)+\tfrac{1}{2}\langle w, D_hw\rangle-(f, w)\}.$$

The problem (5.6) can also be formulated: *Find* $\hat{u}_h\in V_h$ *such that*

$$\widetilde{A}_h(\hat{u}_h, \hat{v})=(f, v)\quad\forall\hat{v}\in V_h, \tag{5.7}$$

where

$$\widetilde{A}_h(\hat{w}, \hat{v})=A_h(\hat{w}, \hat{v})+\tfrac{1}{2}\{\langle w, D_hv\rangle-\langle v, D_hw\rangle\}.$$

We shall prove the following lemma which extends the result of Section 3 to the symmetrized problems (5.6), (5.7).

LEMMA 3. *There exists a constant C such that for $\hat{v}, \hat{w} \in \hat{V}_h$,*

$$|A_h(\hat{w}, \hat{v}) - \widetilde{A}_h(\hat{w}, \hat{v})| \leqslant Ch\|\hat{v}\|_V\|\hat{w}\|_V.$$

Proof. By the definition of $D_h v$, it follows easily that

$$\delta_h \equiv 2|A_h(\hat{w}, \hat{v}) - \widetilde{A}_h(\hat{w}, \hat{v})| = |\langle w, D_h v\rangle - \langle v, D_h w\rangle|$$

$$= |d_h(w, D_h v) - d_h(v, D_h w)|.$$

On the other hand, by (5.4) and the definition of D, we have

$$d(w, Dv) = d(v, Dw), \qquad v, w \in W,$$

and thus

$$\delta_h \leqslant |d_h(w, D_h v) - d(w, D_h v)| + |d_h(v, D_h w) - d(v, D_h w)|$$

$$+ |d(w, D_h v - Dv) - d(v, D_h w - Dw)|$$

$$\equiv \delta_{h1} + \delta_{h2} + \delta_{h3},$$

with obvious notation. The first two terms can be estimated using (3.20). Rewriting the remaining term using (3.12), we get

$$\delta_{h3} = |\langle G_n w, D_h v - Dv\rangle - \langle G_n v, D_h w - Dw\rangle|$$

$$\leqslant C[|G_n w|_{3/2}|(D - D_h)v|_{-3/2} + |G_n v|_{3/2}|(D - D_h)v|_{-3/2}].$$

Now, by a standard duality argument (see e.g. [6]), we have that

$$|(D - D_h)v|_{-3/2} \leqslant Ch|v|_{1/2} \leqslant Ch\|v\|_1.$$

Moreover,

$$|G_n v|_{3/2} \leqslant C|v|_{1/2} \leqslant C\|v\|_1,$$

and thus $\delta_{h3} \leqslant Ch\|v\|_1\|w\|_1$, which completes the proof. □

Remark. The results of Section 4 can also easily be extended to the problem (5.7). □

6. Appendix. We shall here briefly indicate a proof of the regularity result of Lemma 1. We want to prove that, for $k \geqslant 0$,

$$\|w\|_{k+1} + |\theta|_{k-1/2} \leqslant C(\|g_1\|_{k-1} + |g_2|_{k-1/2} + |g_3|_{k+1/2}) \tag{6.1}$$

if $(w, \theta) \in W \times H$ satisfies (2.12). To this end let us first reformulate (2.12b): We have (cf. [4]) that $\theta \in H$ satisfies (2.12b) if and only if

$$\theta = \left[\frac{\partial\varphi}{\partial n}\right] = \frac{\partial\varphi}{\partial n}\bigg|_{\mathrm{int}\,\Gamma_2} - \frac{\partial\varphi}{\partial n}\bigg|_{\mathrm{ext}\,\Gamma_2}, \tag{6.2}$$

where $\varphi \in W^1(\mathbf{R}^2)$ satisfies

$$-\Delta\varphi = 0 \qquad \text{in } \mathbf{R}^2\backslash\Gamma_2, \tag{6.3a}$$

$$\varphi = \tfrac{1}{2}(w + g_3) + c \qquad \text{on } \Gamma_2; \tag{6.3b}$$

here $[\partial\varphi/\partial n]$ denotes the jump in the normal derivative across Γ_2 and c is a suitable constant. By Green's formula we get from (6.2) and (6.3a) that

$$D(\varphi, \psi) - \langle\psi, \theta\rangle = 0 \quad \forall\psi \in W^1(\mathbf{R}^2), \tag{6.4}$$

where

$$D(\varphi, \psi) = \int_{\mathbf{R}^2} \nabla\varphi \cdot \nabla\psi \, dx.$$

Recalling (2.12a), we also have

$$a(w, v) + \langle v, \theta\rangle = (g_1, v) + \langle v, g_2\rangle \quad \forall v \in W. \tag{6.5}$$

Now, to prove (6.1) for $k = 0$, we take $v = w$ in (6.5), $\psi = \varphi$ in (6.4) multiply by two and add. Using (6.3b) we then obtain

$$a(w, w) + 2D(\varphi, \varphi) = (g_1, w) + \langle w, g_2\rangle + \langle g_3, \theta\rangle,$$

and thus (cf. [6])

$$\|w\|_1^2 + \|\varphi\|_{W^1(\mathbf{R}^2)}^2 \leqslant C[(\|g_1\|_{-1} + |g_2|_{-1/2})\|w\|_1 + |\theta|_{-1/2}|g_3|_{1/2}].$$

This proves (6.1) in the case $k = 0$. For $k \geqslant 1$ we parallel the argument in [8] using the $W \times W^1(\mathbf{R}^2)$-ellipticity of the form

$$\breve{D}(\breve{w}, \breve{v}) = a(w, v) + 2D(\varphi, \Psi), \qquad \breve{w} = (w, \varphi), \qquad v = (\breve{v}, \Psi) \in W \times W_0^1(\mathbf{R}^2). \ \square$$

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