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Variational Crimes and L^∞ Error Estimates in the Finite Element Method*

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Abstract. In order to numerically solve a second-order linear elliptic boundary value problem in a bounded domain, using the finite element method, it is often necessary in practice to violate certain assumptions of the standard variational formulation. Two of these "variational crimes" will be emphasized here and it will be shown that optimal L^∞ error estimates still hold. The first "crime" occurs when a nonconforming finite element method is employed, so that smoothness requirements are violated at interelement boundaries. The second "crime" occurs when numerical integration is employed, so that the bilinear form is perturbed. In both cases, the "patch test" is crucial to the proof of L^∞ estimates, just as it was in the case of mean-square estimates.

1. Introduction. The proof of optimal pointwise error estimates for the finite element method has been the subject of intensive research in recent years. For example, see [1]–[9], where optimal (or nearly optimal) L^∞ error estimates were established for second-order linear elliptic boundary value problems. For a more comprehensive survey of the literature concerning this subject, see [10] or [11].

For practical implementation of the finite element method it is often useful, or even necessary, to violate some of the basic assumptions of the underlying Galerkin method. These violations were referred to as variational crimes in [12]. It is the purpose of this paper to establish optimal L^∞ error estimates for finite element methods when variational crimes are committed.

The first crime discussed in [12] occurs when a nonconforming finite element method is employed. (This means that the trial functions fail to satisfy required smoothness conditions on interelement boundaries.) The second occurs when numerical quadrature is employed. The third variational crime deals with the case in which the trial functions fail to satisfy essential boundary conditions. (This usually occurs in the presence of curved boundaries.) A number of methods for treating the last situation were considered in [1], where optimal L^∞ error estimates were established. In this paper the first two variational crimes are emphasized, although one method for treating the Dirichlet boundary condition based on polygonal approximation of the domain is also considered.

The present paper may now be outlined as follows. In Section 2, a description is given of the second-order linear elliptic boundary value problem under consideration, as

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well as certain fundamental properties of the finite element spaces. Both the Dirichlet and Neumann boundary conditions are considered. Some of the basic results proved in [1] and [2] are recalled. These results are then employed to prove an additional useful result (Corollary 2.1).

In Section 3, the finite element method described in Section 2 is generalized in two ways. The first generalization occurs when the finite element spaces are nonconforming. The second occurs when the finite element equations are perturbed, e.g., due to a perturbation of the bilinear form. In both cases, certain results of Section 2 employed in the proof of L^∞ error estimates are generalized in a manner suitable for application in Sections 4 and 5.

In Section 4, it is demonstrated how to obtain L^∞ error estimates when nonconforming finite element methods are employed. In the specific finite element method treated here, the finite element spaces consist of piecewise linear functions defined on a triangulation of a two-dimensional domain. These functions need not be continuous across interelement boundaries but only at the midpoints of these sides. The error estimate is obtained by combining the results of Sections 2 and 3 with certain polynomial invariances.

In Section 5, L^∞ error estimates are obtained for both the Dirichlet and Neumann problem when numerical quadrature is employed. Again, the results are obtained by combining results of Sections 2 and 3 with certain polynomial invariances. L^∞ estimates were obtained for the Dirichlet problem by Wahlbin in [13] using quadratic triangular isoparametric elements and numerical quadrature.

2. **Preliminaries.** In this section we shall describe some of the results proved in [1] and [2] that will be relevant in the remainder of the paper. The main results are embodied in Theorems 2.1–2.4. A useful consequence of Theorem 2.1 is Corollary 2.1, to be proved later in this section. In order to simplify the statement of our results, we shall impose more restrictive assumptions on both the boundary value problem and finite element spaces than necessary. (See Remark 2.2 below for an indication of some extensions of these results.)

Let Ω denote a bounded open set in R^N, $N = 2$ or 3, with smooth boundary $\partial\Omega$. We shall consider the following boundary value problems:

$$(2.1)(\mathrm{D}) \qquad Au = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

and

$$(2.1)(\mathrm{N}) \qquad Au = f \quad \text{in } \Omega, \qquad Tu = \sum_{i,j=1}^{N} a_{ij} \frac{\partial u}{\partial x_j} \cos(n_i) = 0 \quad \text{on } \partial\Omega,$$

where $\cos(n_i)$ denotes the directional cosine of the outer normal with respect to the x_i-direction, and A is the selfadjoint, uniformly elliptic differential operator given by

$$(2.2) \qquad Au = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij} \frac{\partial u}{\partial x_j}\right) + cu,$$

where $c \geqslant 0$ on $\overline{\Omega}$ ($c > 0$ in (2.1)(N)). For simplicity we assume that $\partial\Omega$ is of class C^∞ and that $c(x) \in C^\infty(\overline{\Omega})$ and $a_{ij}(x) \in C^\infty(\overline{\Omega})$, $i, j = 1, \ldots, N$. The smoothness of

f will be specified implicitly later in terms of u. We shall refer to either of the problems, (2.1)(D) or (2.1)(N), as (2.1) when it is not necessary to specify the boundary condition.

Note that the Dirichlet problem (2.1)(D) has a unique solution under the above assumptions. To ensure unique solvability for the Neumann problem (2.1)(N), we assume that $c > 0$ on $\bar{\Omega}$ in this case. We observe that the results and arguments of this paper hold if the differential operator and the boundary condition are generalized, although some of the technical details become a bit more complicated. In addition, the smoothness assumptions can be considerably relaxed.

We shall employ conventional notation for Sobolev spaces. Suppose that $p \in [1, \infty)$ and M is a nonnegative integer. Set

$$|u|_{W_p^M(\Omega)} = \left(\sum_{|\alpha|=M} \|D^\alpha u\|^p_{L^p(\Omega)} \right)^{1/p},$$

and

$$\|u\|_{W_p^M(\Omega)} = \left(\sum_{s=0}^{M} |u|^p_{W_p^s(\Omega)} \right)^{1/p},$$

where $\alpha = (\alpha_1, \ldots, \alpha_N)$, each α_j denotes a nonnegative integer, and $|\alpha| = \Sigma_{j=1}^N \alpha_j$. Note that $D^\alpha u$ denotes the weak derivative of the real-valued function u. For $p = \infty$, set

$$|u|_{W_\infty^M(\Omega)} = \sum_{|\alpha|=M} \|D^\alpha u\|_{L^\infty(\Omega)},$$

and

$$\|u\|_{W_\infty^M(\Omega)} = \sum_{s=0}^{M} |u|_{W_\infty^s(\Omega)}.$$

Define $W_p^M(\Omega) = \{u\colon \|u\|_{W_p^M(\Omega)} < \infty\}$ and $H^M(\Omega) = W_2^M(\Omega)$. For nonintegral $s > 0$, we may define $H^s(\Omega)$ in the sense of interpolation theory; see, e.g., [14]. $\overset{\circ}{H}{}^s(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ under the norm given by $\| \ \|_{H^s(\Omega)}$. The dual of $\overset{\circ}{H}{}^s(\Omega)$ will be denoted by $H^{-s}(\Omega)$ and is defined as the closure of $C^\infty(\Omega)$ under the dual norm. We shall denote the inner product on $L^2(\Omega)$ by $(\ , \ )$.

A typical finite element method employed to approximately solve (2.1) consists of a family of finite-dimensional spaces, S^h, and bilinear forms, $a^h(\ , \ )$, where $h \in (0, 1]$. In order to define the spaces, S^h, we first define the notion of a quasi-uniform family of triangulations of Ω. By a triangulation, T^h, of Ω we mean a finite number of disjoint, open simplices, t^h, such that $\bar{\Omega} = \bigcup_{t^h \in T^h} \bar{t}^h$. The boundary "simplices" have (possibly) one curved face, whereas all interior faces are flat. We shall say that $\{T^h\colon h \in (0, 1]\}$ is a quasi-uniform family of triangulations of Ω if each simplex t^h contains a ball of radius $c_1 h$ and is contained in a ball of radius $c_2 h$, with $0 < c_1 < c_2$ independent of h and t^h.

We shall assume that our family of finite element spaces, S^h, satisfies the following condition with respect to a given quasi-uniform family of triangulations, $\{T^h\colon h \in (0, 1]\}$.

(A.1) Each linear space, S^h, consists of functions, v, such that v restricted to t^h is a polynomial of degree $< K$ for some fixed integer $K \geq 2$. K is independent of $h \in (0, 1]$, $v \in S^h$, and $t^h \in T^h$. Finally, S^h is generated by a nodal basis of Lagrange type.

For more detailed descriptions of nodal finite element spaces, see [11] or [12]. Note that $S^h \subset L^\infty(\Omega)$. However, functions in S^h are not necessarily continuous in Ω, nor need they satisfy any boundary condition. The following inverse inequality is an important consequence of (A.1):

$$\|v\|_{W_q^s(t^h)} \leq Ch^{t-s-N(1/p-1/q)}\|v\|_{W_p^t(t^h)}, \tag{2.3}$$

where $p, q \in [1, \infty]$, s and t are integers such that $0 \leq t \leq s \leq K - 1$, $t - s - N(1/p - 1/q) \leq 0$, and the constant C is independent of $h \in (0, 1]$, $t^h \in T^h$, and $v \in S^h$. (In this paper, we shall generally use the same letter C to denote different constants when there is no danger of confusion.) For a proof of (2.3) see, e.g., [2].

The finite element approximation, u^h, to the solution of (2.1) will be defined using a family of symmetric bilinear forms, $a^h(\ ,\ )$, defined on $S^h \times S^h$. Since S^h is finite dimensional, $a^h(\ ,\ )$ is a continuous bilinear form for each $h \in (0, 1]$. We shall require the following condition in order to define u^h.

(A.2) If $v_1 \in S^h$ and $a^h(v_1, v) = 0$, for each $v \in S^h$, then $v_1 = 0$.

Condition (A.2) states that the bilinear form, $a^h(\ ,\ )$, is nondegenerate. Using (A.2) and the finite dimensionality of S^h, we may now define our finite element approximation, u^h, by means of the equation

$$a^h(u^h, v) = (f, v), \quad \text{for each } v \in S^h. \tag{2.4}$$

Observe that neither of the above conditions implies a relationship between the bilinear forms, $a^h(\ ,\ )$, and problem (2.1). The following condition states that optimal mean square error estimates hold for the solutions, u and u^h, of (2.1) and (2.4), respectively.

(A.3) If s is an integer such that $0 \leq s \leq K - 2$ and $u \in H^2(\Omega)$, then there exists a constant C independent of $h \in (0, 1]$ and u such that

$$\|u - u^h\|_{H^{-s}(\Omega)} \leq Ch^{s+2}\|u\|_{H^2(\Omega)}.$$

We shall require certain estimates for the Green's function for problem (2.1) and its finite element approximation. Let G_{x_0} denote the Green's function for (2.1) with singularity at the point $x_0 = (x_{0_1}, \ldots, x_{0_N})$ in Ω. The following estimate was proved in [15]:

$$|D_x^\alpha G_{x_0}(x)| \leq \begin{cases} C\,|\ln|x - x_0||, & \text{for } N = 2,\ |\alpha| = 0, \\[2ex] C\,|x - x_0|^{2-N-|\alpha|}, & \text{for } N \geq 3 \text{ or } N = 2,\ |\alpha| > 0, \end{cases} \tag{2.5}$$

where D_x^α denotes a derivative of order α with respect to x, $x \neq x_0$, and C is independent of the points x and x_0 in Ω (up to the boundary). From now on we shall delete the subscript, x_0, when referring to $G = G_{x_0}$.

LEMMA 2.1. *Consider problem* (2.1)(N) ((2.1)(D)). *Suppose that s is an integer such that* $0 \leqslant s \leqslant K-2$, $N = 2$ *or* 3, $h \in (0, 1]$, $x_0 \in t_0^h$ *for some* $t_0^h \in T^h$, *and conditions* (A.1) *and* (A.2) *hold. Then there exist functions* $\widetilde{G} \in C^\infty(\Omega)$, $G^h \in S^h$, *and* $\widetilde{\delta} \in C_0^\infty(\Omega)$, *satisfying the following conditions*:

(i) $A\widetilde{G} = \widetilde{\delta}$, $T\widetilde{G} = 0$ *on* $\partial\Omega$ ($\widetilde{G} = 0$ *on* $\partial\Omega$),

(ii) $a^h(G^h, v) = (\widetilde{\delta}, v) = v(x_0)$, *for each* $v \in S^h$,

(iii) $\max_{x\in\Omega} |D_x^\alpha \widetilde{\delta}(x)| \leqslant C_\alpha h^{-N-|\alpha|}$, *where* $|\alpha| \geqslant 0$,

and

(iv) $\|G - G^h\|_{H^{-s}(\Omega)} \leqslant Ch^{s+2-N/2}$, *provided that* (A.3) *also holds.*

In addition, the support of $\widetilde{\delta}(x)$ *is contained in a sphere of radius* $C^1 h$ *centered at* x_0. *The constants* C_α, C^1, *and C are independent of h and* x_0.

The proof of Lemma 2.1 follows from the same arguments as in [1] or [2] and is based on the Bramble-Hilbert lemma (see, e.g., [11]) and elliptic regularity theory. Observe that Lemma 2.1 holds for nonconforming finite element spaces, S^h. We shall also require the following approximation assumption.

(A.4) Suppose that $p \in [1, \infty]$ and that s is an integer greater than 1 satisfying the condition $N/p < s \leqslant K$ if $p > 1$ and $N \leqslant s \leqslant K$ if $p = 1$. Also, suppose that $u \in W_p^s(\overline{t^h})$ for each $t^h \in T^h$. Then there exists a function u^A in S^h and a constant C, independent of u, $h \in (0, 1]$ and $t^h \in T^h$, such that

$$\sum_{j=0}^{2} h^j |u - u^A|_{W_p^j(t^h)} \leqslant Ch^s |u|_{W_p^s(t^h)},$$

and

$$\|u^A\|_{W_p^s(t^h)} \leqslant C\|u\|_{W_p^s(t^h)}.$$

Furthermore, we have

$$\sum_{j=0}^{1} h^{N/p+j} |u - u^A|_{W_\infty^j(t^h)} \leqslant Ch^s |u|_{W_p^s(t^h)},$$

provided $N/p + 1 < s \leqslant K$ if $p > 1$ and $N + 1 \leqslant s \leqslant K$ if $p = 1$.

Conditions (A.1)–(A.4) may be employed to establish L^∞ error estimates for a variety of finite element methods, as shown in [1] and [2]. We first describe a specific finite element method for solving the Neumann problem, (2.1)(N), and present the main results obtained from [1] and [2] that will be of use in Section 5 below. We shall then do the same thing with regard to the Dirichlet problem.

A natural finite element method for treating problem (2.1)(N) was employed in [2] and consists of a family of bilinear forms given by

$$(2.6) \qquad a^h(u, v) = \int_\Omega \sum_{i,j=1}^{N} \left(a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + cuv\right) dx, \quad \text{for each } u, v \in H^1(\Omega),$$

and a family of finite element spaces, S_N^h, satisfying condition (A.1) with $K \geqslant 2$. We assume that functions in S_N^h are continuous in Ω, so that $S_N^h \subset H^1(\Omega)$. Observe that for the Neumann problem, it is not necessary to impose any boundary condition on functions in S_N^h. We see that the bilinear form, $a^h(\ ,\ )$, given by (2.6), is coercive over $H^1(\Omega) \times H^1(\Omega)$ since $c > 0$. Hence, we have $a^h(v, v) \geqslant C\|v\|_{H^1(\Omega)}^2$, for each $v \in H^1(\Omega)$.

Since $a^h(\ ,\ )$ is coercive over $H^1(\Omega) \times H^1(\Omega)$, it is readily seen that condition (A.2) holds. The approximation assumption, (A.4), follows from the Bramble-Hilbert lemma as in [2]. Furthermore, condition (A.3) follows using a standard duality argument; see, e.g., [16]. It thus follows that Lemma 2.1 holds for this finite element method, where x_0 may now be an arbitrary point in Ω.

The next result is a key ingredient in the proof of L^∞ error estimates. The proof of this result follows from the same arguments as employed in [1] or [2] and hence will not be repeated here.

THEOREM 2.1. *Consider the finite element method for solving problem* (2.1)(N) *with the bilinear form, $a^h(\ )$, defined by* (2.6), *and the finite element space given by S_N^h for each $h \in (0, 1]$. Then there exists a constant C, independent of h and $x_0 \in \Omega$, such that the following estimates hold*:

(a) *If $K \geqslant 3$ and $N = 2$ or 3, we have*

$$\|G - G^h\|_{W_1^1(\Omega)} \leqslant Ch,$$

and

(b) *If $K = N = 2$, we have*

$$\|G - G^h\|_{W_1^1(\Omega)} \leqslant Ch|\log h|.$$

Before proceeding further, let us introduce some additional notation. For each subset, $D \subseteqq \Omega$, set

$$W_p^{lh}(D) = \{u \in L^p(D)\colon \|u\|^h_{W_p^l(D)} < \infty\},$$

where l is a positive integer and

$$\|u\|^h_{W_p^l(D)} = \begin{cases} \left(\displaystyle\sum_{t^h \in T^h; t^h \cap D \neq \varnothing} \|u\|^p_{W_p^l(t^h)} \right)^{1/p}, & 1 \leqslant p < \infty, \\ \displaystyle\max_{t^h \in T^h; t^h \cap D \neq \varnothing} \|u\|_{W_p^l(t^h)}, & p = \infty. \end{cases}$$

Denote $W_2^{lh}(D)$ by $H^{lh}(D)$. We next state and prove the following useful corollary.

COROLLARY 2.1. *Suppose that the hypotheses of Theorem* 2.1(a) *hold and l is an integer such that $2 \leqslant l \leqslant K - 1$. Then there exists a constant C, independent of $h \in (0, 1]$ and $x_0 \in \Omega$, such that the following estimate holds*:

$$\|G^h\|^h_{W_1^l(\Omega)} \leqslant C \begin{cases} h^{-(l-2)} & \text{for } l > 2, \\ |\log h| & \text{for } l = 2. \end{cases}$$

Proof. Set $C_h = \{x\colon |x - x_0| \leqslant h\}$, $\Omega_h = \{\bigcup_{t^h \in T^h}\colon t^h \cap C_h \neq \varnothing\}$, and $\Omega_h' = \Omega - \bar{\Omega}_h$. (It may be seen from the quasi-uniformity of the family of triangulations that the number of simplices, t^h, contained in Ω_h is bounded by a constant independent of h.) Note that

$$\|G^h\|^h_{W_1^l(\Omega)} \leqslant \|G^h\|^h_{W_1^l(\Omega_h)} + \|G^h\|^h_{W_1^l(\Omega_h')}, \tag{2.7}$$

and

$$\|G^h\|^h_{W^l_1(\Omega'_h)} \leqslant \|G - G^h\|^h_{W^l_1(\Omega'_h)} + \|G\|^h_{W^l_1(\Omega'_h)}. \tag{2.8}$$

In view of (2.5), we have

$$\|G\|^h_{W^l_1(\Omega'_h)} \leqslant C \begin{cases} h^{-(l-2)} & \text{for } l > 2, \\ |\log h| & \text{for } l = 2. \end{cases} \tag{2.9}$$

We next consider $\|G - G^h\|^h_{W^l_1(\Omega'_h)}$. Let G' denote a smooth function equal to G in Ω'_h and let $G^A \in S^h$ denote the function obtained from (A.4) with u given by G'. We may now apply (2.3) and (A.4) to obtain the following estimates for each $t^h \subset \Omega'_h$:

$$\begin{aligned} \|G - G^h\|_{W^l_1(t^h)} &\leqslant \|G - G^A\|_{W^l_1(t^h)} + \|G^A - G^h\|_{W^l_1(t^h)} \\ &\leqslant C\|G\|_{W^l_1(t^h)} + Ch^{-(l-1)}\|G^A - G^h\|_{W^1_1(t^h)} \\ &\leqslant C\|G\|_{W^l_1(t^h)} + Ch^{-(l-1)}\|G - G^h\|_{W^1_1(t^h)}. \end{aligned} \tag{2.10}$$

We now sum over each $t^h \subset \Omega'_h$ and apply (2.9), (2.10), and Theorem 2.1 to obtain

$$\|G - G^h\|^h_{W^l_1(\Omega'_h)} \leqslant C \begin{cases} h^{-(l-2)} & \text{for } l > 2, \\ |\log h| & \text{for } l = 2. \end{cases} \tag{2.11}$$

Finally, we consider

$$\|G^h\|^h_{W^l_1(\Omega_h)} \leqslant \|\widetilde{G} - G^h\|^h_{W^l_1(\Omega_h)} + \|\widetilde{G}\|_{W^l_1(\Omega_h)}, \tag{2.12}$$

where $\widetilde{G}$ is defined by Lemma 2.1(i). Employing Lemma 2.1 and elliptic regularity theory, we see that

$$\|\widetilde{G}\|_{W^l_1(\Omega_h)} \leqslant Ch^{N/2}\|\widetilde{G}\|_{H^l(\Omega_h)} \leqslant Ch^{N/2}\|\widetilde{\delta}\|_{H^{l-2}(\Omega_h)} \leqslant Ch^{-(l-2)}. \tag{2.13}$$

Suppose that $t^h \in \Omega_h$ and let $\widetilde{G}^A \in S^h$ denote the approximation to $\widetilde{G}$ given by (A.4). It follows from Lemma 2.1(i) and (ii) that $G^h = \widetilde{G}^h$, the finite element approximation of $\widetilde{G}$. Hence we may apply (2.3), (A.3), and (A.4) to deduce

$$\begin{aligned} \|\widetilde{G} - G^h\|_{W^l_1(t^h)} &\leqslant Ch^{N/2}(\|\widetilde{G} - \widetilde{G}^A\|_{H^l(t^h)} + \|\widetilde{G}^A - G^h\|_{H^l(t^h)}) \\ &\leqslant Ch^{N/2}(\|\widetilde{G}\|_{H^l(t^h)} + h^{-l}\|\widetilde{G}^A - G^h\|_{L^2(t^h)}) \\ &\leqslant Ch^{N/2}(\|\widetilde{G}\|_{H^l(t^h)} + h^{-l}\|\widetilde{G} - \widetilde{G}^A\|_{L^2(t^h)} + h^{-l}\|\widetilde{G} - G^h\|_{L^2(t^h)}) \\ &\leqslant Ch^{N/2}(\|\widetilde{G}\|_{H^l(t^h)} + h^{-(l-2)}\|\widetilde{G}\|_{H^2(t^h)}). \end{aligned} \tag{2.14}$$

We now sum over the finite number of simplices, $t^h \subset \Omega_h$, and apply elliptic regularity theory and Lemma 2.1 to conclude that

$$\begin{aligned}\|\widetilde{G} - G^h\|^h_{W^l_1(\Omega_h)} &\leqslant Ch^{N/2}(\|\widetilde{G}\|_{H^l(\Omega_h)} + h^{-(l-2)}\|\widetilde{G}\|_{H^2(\Omega^h)}) \\ &\leqslant Ch^{N/2}(\|\widetilde{\delta}\|_{H^{l-2}(\Omega_h)} + h^{-(l-2)}\|\widetilde{\delta}\|_{L^2(\Omega_h)}) \leqslant Ch^{-(l-2)},\end{aligned} \tag{2.15}$$

with C again independent of h and x_0. Combining (2.7)–(2.9), (2.11)–(2.13), and (2.15), we have proved the corollary. Q.E.D.

We now apply Theorem 2.1 to obtain the following L^∞ error estimates.

THEOREM 2.2. *Suppose the hypotheses of Theorem* 2.1 *hold, u satisfies* (2.1)(N), *and* $u \in W^K_\infty(\Omega)$. *Then there exists a constant C independent of* $h \in (0, 1]$, *such that the following results hold.*

(a) *If* $K \geqslant 3$ *and* $N = 2$ *or* 3, *we have*

$$\|u - u^h\|_{L^\infty(\Omega)} \leqslant Ch^K \|u\|_{W^K_\infty(\Omega)}.$$

(b) *If* $K = N = 2$, *we have*

$$\|u - u^h\|_{L^\infty(\Omega)} \leqslant Ch^2 \,|\log h|\, \|u\|_{W^2_\infty(\Omega)}.$$

Proof. (a) Suppose that $x_0 \in \Omega$. Employing the properties of the Green's function for problem (2.1)(N), as well as (2.1)(N), (2.4), (2.6), and Lemma 2.1, we obtain

$$u(x_0) - u^h(x_0) = a^h(u - u^h, G) = a^h(u - u^h, G - G^h) = a^h(u - u^A, G - G^h), \tag{2.16}$$

with u^A obtained from (A.4).

We now combine (2.16), (A.4), and Theorem 2.1(a) to deduce

$$\begin{aligned}|u(x_0) - u^h(x_0)| &\leqslant C\|u - u^A\|_{W^1_\infty(\Omega)}\|G - G^h\|_{W^1_1(\Omega)} \\ &\leqslant Ch^{K-1}\|u\|_{W^K_\infty(\Omega)}\|G - G^h\|_{W^1_1(\Omega)} \leqslant Ch^K\|u\|_{W^K_\infty(\Omega)}.\end{aligned}$$

This proves (a).

(b) The proof is almost the same as that of (a) except that Theorem 2.1(b) is employed instead of Theorem 2.1(a). Q.E.D.

We next consider the Dirichlet problem, (2.1)(D), and observe that in order to employ the bilinear forms given by (2.6), it would be necessary to assume that functions in S^h vanish on $\partial\Omega$. Since this is generally not practical for curved boundaries, various alternative finite element methods have been developed to circumvent this difficulty. A few of these methods were analyzed in [1], where optimal L^∞ error estimates were proved.

In this paper, we shall treat problem (2.1)(D) by approximating Ω by a family of polygons, Ω^h, with $h \in (0, 1]$. For simplicity, we assume that Ω is a convex open set in R^2 and each Ω^h is an inscribed convex (open) polygon in Ω. Let S^h_D denote a family of spaces satisfying condition (A.1) with respect to a given quasi-uniform family of triangulations of Ω^h for each $h \in (0, 1]$. Thus, each $t^h \in T^h$ is a triangle with straight edges. We assume that $K = 2$, so that each function in S_D^h is piecewise linear. Furthermore, suppose that functions in S_D^h are continuous in Ω and vanish in $\Omega - \Omega^h$. The bilinear form, $a^h(\ ,\ )$, is now defined by

$$a^h(u, v) = \int_{\Omega^h} \sum_{i,j=1}^{2} \left(a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + cuv\right)dx, \tag{2.17}$$

for each $u, v \in H^1(\Omega)$ and $h \in (0, 1]$.

Remark 2.1. It is not necessary to assume that the polygonal approximations, Ω^h, are as restrictive as given here. See, e.g., the arguments in [13], where the approximating domains, Ω^h, are not assumed to be contained in Ω. However, our present assumptions suffice for the purposes of this paper. We also observe that the present method for treating the Dirichlet problem yields an error of order $O(h^2)$, even if K is greater than 2 in assumption (A.1). This is due to estimate (2.20) below (see the proof of (2.28)). The use of isoparametric elements as in [13] or of nonstandard finite element methods as in [1] yield errors of order $O(h^K)$ with $K > 2$.

It follows as before that assumptions (A.1)–(A.4) again hold with Ω replaced by Ω^h and $K = 2$. Furthermore, Lemma 2.1 holds in this case where x_0 may now be an arbitrary point in Ω^h. We shall next state results analogous to Theorems 2.1 and 2.2 with $K = N = 2$. These results will be applied in Section 5 below. In Section 4, we shall treat a nonconforming finite element method closely related to the conforming method under consideration here.

THEOREM 2.3. *Suppose that Ω is a convex open set in R^2. Consider the finite element method for solving problem* (2.1)(D) *with $a^h(\ ,\ )$ defined by* (2.17) *and the finite element spaces given by S_D^h for each $h \in (0, 1]$. Then there exists a constant C, independent of $h \in (0, 1]$ and $x_0 \in \Omega^h$, such that the following estimate holds*:

$$\|G - G^h\|_{W_1^1(\Omega^h)} \leq Ch|\log h|. \tag{2.18}$$

The proof of Theorem 2.3 follows from the arguments of [1] or [2]. Furthermore, we have the following stronger estimate:

$$\|G - G^h\|_{W_1^1(\Omega)} \leq Ch|\log h|. \tag{2.19}$$

To see this, note that it follows from the definition of Ω^h and S_D^h that

$$\operatorname{dist}(\partial\Omega^h, \partial\Omega) = O(h^2), \tag{2.20}$$

and

$$\|v^h\|_{W_\infty^1(\Omega - \Omega^h)} = 0, \quad \text{for each } v^h \in S^h. \tag{2.21}$$

Employing (2.5) and (2.20), we obtain

$$\|G\|_{W_1^1(\Omega - \Omega^h)} \leq Ch^2. \tag{2.22}$$

We may now combine (2.21) and (2.22) with (2.18) to obtain (2.19).

THEOREM 2.4. *Suppose that the hypotheses of Theorem 2.3 hold, u satisfies* (2.1)(D), *and* $u \in W^2_\infty(\Omega)$. *Then there exists a constant C independent of* $h \in (0, 1]$, *such that the following estimate holds*:

$$\|u - u^h\|_{L^\infty(\Omega)} \leqslant Ch^2 |\log h| \, \|u\|_{W^2_\infty(\Omega)}.$$

Proof. Employing the properties of the Green's function for problem (2.1)(D), as well as (2.1)(D), (2.4), (2.17), and Lemma 2.1, we obtain the following equation for each point $x_0 \in \Omega^h$:

$$\begin{aligned} u(x_0) - u^h(x_0) &= a^h(u - u^A, G - G^h) \\ &\quad + \int_{\Omega - \Omega^h} \sum_{i,j=1}^{2} \left(a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial G}{\partial x_j} + cuG \right) dx, \end{aligned} \tag{2.23}$$

with u^A obtained from (A.4). We see from (A.4) and Theorem 2.3 that

$$|a^h(u - u^A, G - G^h)| \leqslant Ch^2 |\log h| \, \|u\|_{W^2_\infty(\Omega^h)}. \tag{2.24}$$

Furthermore, we may apply (2.22) to deduce

$$\begin{aligned} \left| \int_{\Omega - \Omega^h} \sum_{i,j=1}^{2} \left(a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial G}{\partial x_j} + cuG \right) dx \right| &\leqslant C \|u\|_{W^1_\infty(\Omega)} \|G\|_{W^1_1(\Omega - \Omega^h)} \\ &\leqslant Ch^2 \|u\|_{W^1_\infty(\Omega)}. \end{aligned} \tag{2.25}$$

Combining (2.23)–(2.25), we obtain

$$\|u - u^h\|_{L^\infty(\Omega^h)} \leqslant Ch^2 |\log h| \, \|u\|_{W^2_\infty(\Omega)}. \tag{2.26}$$

It follows from (2.20) that

$$\|v\|_{L^\infty(\Omega - \Omega^h)} \leqslant Ch^2 \|\nabla v\|_{L^\infty(\Omega)}, \quad \text{for each } v \in \mathring{H}^1(\Omega) \cap W^2_\infty(\Omega). \tag{2.27}$$

We now conclude from (2.21) and (2.27) that

$$\|u - u^h\|_{L^\infty(\Omega - \Omega^h)} \leqslant Ch^2 \|u\|_{W^1_\infty(\Omega)}. \tag{2.28}$$

Estimates (2.26) and (2.28) together imply the theorem. Q.E.D.

Remark 2.2. The results of this section may be stated in greater generality than given here. For example, analogous results were proved in [1] for nonselfadjoint differential operators, A, under more general assumptions on the finite element spaces than given in (A.1). However, our assumptions suffice for the purposes of this paper.

3. Variational Crimes. In the remainder of this paper, we shall be concerned with proving results analogous to Theorems 2.2 and 2.4 when the finite element method is altered in either of two ways. The first case we treat may be considered as a perturbation of the finite element subspaces, S^h, and the second as a perturbation of the coefficients of the finite element equations. In order to apply the method of proof described in Section 2, we shall derive, in this section, an extension of Eq. (2.16) under much weaker assumptions than in Section 2. In both cases, the right-hand side will contain additional terms. It will be seen in Sections 4 and 5 how each of these terms may be estimated in specific examples. We again consider the boundary value problem (2.1) ((D) or (N)) with the same assumptions on the coefficients and boundary as in Section 2.

Case I—*Nonconforming Subspaces.* We shall consider the finite element method as described in Section 2, with the exception that functions in S^h need not be continuous across interelement boundaries. Hence, we shall weaken the assumptions of Section 2 as follows. To begin with, suppose that Ω is replaced by an open set, $\Omega^h \subseteqq \Omega$, for each $h \in (0, 1]$. Let S^h denote a family of finite-dimensional spaces satisfying condition (A.1) with respect to a given quasi-uniform family of triangulations, $\{T^h\colon h \in (0, 1]\}$, of Ω^h. We thus assume that $\Omega^h = \bigcup_{t^h \in T^h} t^h$ and observe that Ω^h does not include interior faces.

It is readily seen that $S^h \subset L^\infty(\Omega^h) \cap W_\infty^{l^h}(\Omega^h)$ for each positive integer l, employing the notation of Section 2 for the piecewise Sobolev spaces $W_p^{l^h}(D)$. Note that functions in S^h are only defined on Ω^h, not Ω. We next define a family of bilinear forms, $a^h(\ ,\ )$, acting on $H^{1^h}(\Omega^h) \times H^{1^h}(\Omega^h)$, by

$$a^h(u, v) = \sum_{t^h \in T^h} \int_{t^h} \sum_{i,j=1}^{N} \left(a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + cuv\right) dx, \tag{3.1}$$

where $N = 2$ or 3, $u, v \in H^{1^h}(\Omega^h)$, and $h \in (0, 1]$. We assume that condition (A.2) holds. Hence there exists a unique function, $u^h \in S^h$, satisfying

$$a^h(u^h, v) = (f, v), \quad \text{for each } v \in S^h. \tag{3.2}$$

We now assume that $x_0 \in \Omega^h$, so that $x_0 \in t_0^h$ for some $t_0^h \in T^h$. We observe that Lemma 2.1(i)–(iii) again holds as in Section 2. (Lemma 2.1(iv) would require an additional assumption such as (A.3).) Hence there exists a unique function, $G^h \in S^h$, satisfying

$$a^h(G^h, v) = v(x_0), \quad \text{for each } v \in S^h. \tag{3.3}$$

As before, we denote the Green's function for (2.1) with singularity at x_0 by $G = G_{x_0}$.

THEOREM 3.1. *Suppose that* $u \in W_\infty^1(\Omega)$ *and satisfies* (2.1) ((D) *or* (N)), $x_0 \in \Omega^h$, (A.1) *and* (A.2) *hold,* $a^h(\ ,\ )$ *is defined by* (3.1), u^h *satisfies* (3.2) *and* G^h *satisfies* (3.3). *Then we have*

$$\begin{aligned} u(x_0) - u^h(x_0) &= a^h(G - G^h, u - \chi) + a^h(G^h, u - u^h) \\ &\quad + a^h(G - G^h, \chi) + (u(x_0) - a^h(G, u)), \quad \textit{for each } \chi \in S^h. \end{aligned} \tag{3.4}$$

Proof. We first express the left side of (3.4) as follows:

$$u(x_0) - u^h(x_0) = a^h(G, u - u^h) + \{u(x_0) - u^h(x_0) - a^h(G, u - u^h)\}. \tag{3.5}$$

It is easily seen that, for each $\chi \in S^h$, we have

$$
\begin{aligned}
a^h(G, u - u^h) &= a^h(G - G^h, u - u^h) + a^h(G^h, u - u^h) \\
&= a^h(G - G^h, u - \chi) + a^h(G^h, u - u^h) + a^h(G - G^h, \chi - u^h).
\end{aligned}
\tag{3.6}
$$

We now employ (3.3) to obtain

$$
\begin{aligned}
u(x_0) - u^h(x_0) - a^h(G, u - u^h) &= u(x_0) - a^h(G, u) + a^h(G, u^h) - u^h(x_0) \\
&= u(x_0) - a^h(G, u) + a^h(G - G^h, u^h) = u(x_0) - a^h(G, u) \\
&\quad + a^h(G - G^h, \chi) + a^h(G - G^h, u^h - \chi), \quad \text{for each } \chi \in S^h.
\end{aligned}
\tag{3.7}
$$

Finally, we add (3.6) and (3.7) and then substitute in (3.5) to obtain (3.4). Q.E.D.

Remark 3.1. An analogue of Theorem 3.1 remains valid even if the spaces, S^h, and the boundary value problem, (2.1), are generalized as in Remark 2.2. We also observe that the specific form of $a^h(\ ,\ )$, given by Eq. (3.1), was not necessary for the proof of Theorem 3.1. The main requirements on $a^h(\ ,\ )$ are that condition (A.2) holds and all terms in (3.4) are defined.

In Section 4, we shall apply (3.4) to obtain an L^∞ error estimate for a specific nonconforming finite element method for solving the Dirichlet problem. In addition to the two conditions, (A.1) and (A.2), required here, it will be necessary to apply (A.3) and (A.4), as well as certain polynomial invariances (related to the "patch test"), in order to choose χ appropriately and suitably estimate each term on the right side of (3.4). Note that if $\overline{\Omega} = \overline{\Omega^h}$, the last term in (3.4) is zero. Finally we observe that if $\overline{\Omega} = \overline{\Omega}^h$ and the finite element method is conforming (so that the required continuity and boundary conditions are satisfied by functions in S^h), then the last three terms in (3.4) are zero and (3.4) reduces to (2.16) (with $\chi = u^A$).

Case II—*Perturbation of the Coefficients.* We next consider a conforming finite element method for solving problem (2.1) when one or both sides of Eq. (2.4) are perturbed. Again, suppose that Ω is replaced by an open set, $\Omega^h \subsetneqq \Omega$, for each $h \in (0, 1]$. Let S^h denote a family of finite-dimensional spaces contained in $W^1_\infty(\Omega^h)$ and satisfying condition (A.1) with respect to a quasi-uniform family of triangulations, $\{T^h : h \in (0, 1]\}$, of Ω^h. We define Ω^h as the interior of $\overline{\Omega^h} = \bigcup_{t^h \in T^h} \overline{t^h}$. (Hence Ω^h now includes interior faces.)

We next define a family of bilinear forms, $a^h(\ ,\ )$, as follows:

$$
a^h(u, v) = \int_{\Omega^h} \sum_{i,j=1}^{N} \left(a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + cuv \right) dx, \quad \text{for each } u, v \in H^1(\Omega^h),
\tag{3.8}
$$

where $N = 2$ or 3 and $h \in (0, 1]$. We assume that condition (A.2) holds. As a consequence of (A.1) and (A.2), it follows that Lemma 2.1(i)–(iii) holds for each x_0 in Ω^h.

We now assume that the bilinear form $a^h(\ ,\ )$ and the linear functional $f(\ )$ (given by $f(v) = (f, v)$, for each $v \in L^2(\Omega)$) are perturbed. Let $\widetilde{a}^h(\ ,\ )$ denote the perturbed symmetric bilinear form defined on $S^h \times S^h$, and let $\widetilde{f}^h$ denote the perturbed linear functional defined on S^h. In addition to the solution, $u^h \in S^h$ of (3.2), suppose that there exists a solution, $\widetilde{u}^h \in S^h$, of the equation

$$\tilde{a}^h(\tilde{u}^h, v) = \tilde{f}^h(v), \quad \text{for each } v \in S^h. \tag{3.9}$$

(We shall see in Section 5 that the existence and uniqueness of $\tilde{u}^h$ may be proved under suitable assumptions in the case of numerical quadrature.)

THEOREM 3.2. *Suppose that* $u \in W^1_\infty(\Omega)$, $x_0 \in \Omega^h$, (A.1) *and* (A.2) *hold,* $S^h \subset W^1_\infty(\Omega^h)$, $a^h(\ ,\ )$ *is defined by* (3.8), u^h *satisfies* (3.2), G^h *satisfies* (3.3), *and* $\tilde{u}^h$ *satisfies* (3.9). *Furthermore, suppose that either u satisfies* (2.1)(N) *and* $\Omega = \Omega^h$, *or u satisfies* (2.1)(D) *and* $S^h \subset \mathring{H}^1(\Omega^h)$. *Then we have*

$$\begin{aligned} u(x_0) - \tilde{u}^h(x_0) &= a^h(G - G^h, u - \chi) + (\tilde{a}^h - a^h)(G^h, \tilde{u}^h) \\ &\quad + (f - \tilde{f}^h)(G^h) + (u(x_0) - a^h(G, u)), \quad \text{for each } \chi \in S^h. \end{aligned} \tag{3.10}$$

Proof. We first observe, using the properties of u, (3.2), and the properties of the Green's function, $G = G_{x_0}$, that

$$\begin{aligned} u(x_0) - \tilde{u}^h(x_0) &= u(x_0) - a^h(G, u) + a^h(G, u - \tilde{u}^h) \\ &= u(x_0) - a^h(G, u) + a^h(G, u - u^h) + a^h(G, u^h - \tilde{u}^h) \\ &= u(x_0) - a^h(G, u) + a^h(G - G^h, u - u^h) + a^h(G, u^h - \tilde{u}^h). \end{aligned} \tag{3.11}$$

We next apply Eq. (3.3) to each of the last two terms in (3.11) to obtain

$$\begin{aligned} u(x_0) - \tilde{u}^h(x_0) &= a^h(G - G^h, u - \chi) + a^h(G^h, u^h - \tilde{u}^h) \\ &\quad + (u(x_0) - a^h(G, u)), \qquad \text{for each } \chi \in S^h. \end{aligned} \tag{3.12}$$

Employing (3.2), (3.9), and the symmetry of $a^h(\ ,\ )$ and $\tilde{a}^h(\ ,\ )$, we deduce

$$a^h(G^h, u^h) = a^h(u^h, G^h) = f(G^h), \tag{3.13}$$

and

$$\tilde{a}^h(G^h, \tilde{u}^h) = \tilde{a}^h(\tilde{u}^h, G^h) = \tilde{f}^h(G^h). \tag{3.14}$$

Finally, we combine (3.12)–(3.14) to obtain (3.10). Q.E.D.

Observe that when no perturbation is present and $\Omega = \Omega^h$, Eq. (3.10) reduces to (2.16) (with $\chi = u^A$). In Section 5, we shall apply (3.10) to prove the existence of a function, $\tilde{u}^h$, satisfying (3.9), and also establish L^∞ estimates for $u - \tilde{u}^h$ when the perturbation is due to a sufficiently accurate numerical quadrature scheme. In addition to the assumptions (A.1) and (A.2) required here, it will be necessary in Section 5 to apply conditions (A.3) and (A.4), as well as certain polynomial invariances in order to estimate the right side of (3.10).

4. A Nonconforming Finite Element Method. In this section we consider a specific nonconforming finite element method for solving a boundary value problem in R^2. We shall derive L^∞ error estimates using Theorem 3.1. We begin by describing our boundary value problem and finite element method. Consider the following model problem:

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{4.1}$$

where Ω is a bounded, convex two-dimensional domain and $\partial\Omega$ is of class C^∞.

We shall approximately solve (4.1) using the following nonconforming finite element method. Suppose that for each $h \in (0, 1]$, Ω is replaced by a convex (open) inscribed polygon, Ω^h. Consider a quasi-uniform family of triangulations, T^h, of Ω^h into disjoint open triangles, t^h, of diameter $O(h)$. Thus, we have $\overline{\Omega^h} = \bigcup_{t^h \in T^h} \overline{t^h}$. We assume that two neighboring triangles may only intersect at a vertex or along an entire side.

We construct a family of spaces, S^h, as follows. Suppose first of all that each v^h in S^h vanishes in $\Omega - \overline{\Omega^h}$. Assume that the restriction of v^h to each t^h is linear and that v^h is continuous at the midpoint of each side of t^h. Also suppose that $v^h = 0$ at the midpoint of each side of t^h contained in $\partial\Omega^h$. It is clear that condition (A.1) holds for this family of spaces, S^h.

Observe that in general, functions in S^h are discontinuous and fail to vanish on $\partial\Omega^h$. However, $S^h \subset L^\infty(\Omega)$, and S^h contains the conforming subspace, S_D^h, described in Section 2 and consisting of continuous piecewise linear functions vanishing on $\partial\Omega^h$. It is readily seen using the Bramble-Hilbert lemma, as in [11], that condition (A.4) holds with $K = 2$. Furthermore, u^A may be taken to be the interpolate, u^I, of u with respect to the nodal basis for S_D^h.

The nonconforming space, S^h, was constructed and applied in solving the Stokes problem in [17] and [18], where this space was employed to impose the incompressibility condition, div $v^h = 0$, on functions in S^h. The fact that S^h is roughly three times as large as S_D^h made it possible to impose this additional constraint.

We shall require the piecewise Sobolev spaces, $W_p^{l^h}(D)$ and $H^{l^h}(D)$, defined in Section 2. Furthermore, we define our bilinear form, $a^h(\ ,\ )$, as in Eq. (3.1):

$$a^h(v, w) = \sum_{t^h \in T^h} \int_{t^h} \nabla v \cdot \nabla w \, dx, \quad \text{for each } v, w \in H^{1^h}(\Omega^h) \text{ and } h \in (0, 1]. \tag{4.2}$$

It may be seen, using the argument of [11] (Section 4.2) or [17], that condition (A.2) holds. As a consequence there exists a unique function, $u^h \in S^h$, satisfying the equation

$$a^h(u^h, v) = (f, v), \quad \text{for each } v \in S^h. \tag{4.3}$$

We next consider mean-square error estimates. It may be seen, using the arguments of [17] and [18], that condition (A.3) holds with Ω replaced by Ω^h and $K = 2$. Furthermore, it follows from these arguments that

$$\|u - u^h\|_{H^1(\Omega^h)}^h \leqslant Ch\|u\|_{H^2(\Omega)}, \quad \text{for each } h \in (0, 1]. \tag{4.4}$$

We are now ready to prove the main result of this section.

THEOREM 4.1. *Suppose that* $u \in W_\infty^2(\Omega)$, *u satisfies* (4.1), *and u^h satisfies* (4.3). *Then there exists a constant C, independent of $h \in (0, 1]$, such that the following estimate holds*:

$$\|u - u^h\|_{L^\infty(\Omega)} \leqslant Ch^2 |\log h| \, \|u\|_{W_\infty^2(\Omega)}.$$

Proof. We first note that it follows from (2.21) and (2.27) that

$$\|u - u^h\|_{L^\infty(\Omega - \Omega^h)} \leqslant Ch^2 \|u\|_{W^1_\infty(\Omega)}. \tag{4.5}$$

In view of (4.5), it suffices to prove

$$\|u - u^h\|_{L^\infty(\Omega^h)} \leqslant Ch^2 |\log h| \, \|u\|_{W^2_\infty(\Omega)}. \tag{4.6}$$

Suppose that $x_0 \in t_0^h$ for some triangle $t_0^h \in T^h$. Since conditions (A.1)–(A.3) hold, we observe that Lemma 2.1 holds in the present case. Applying Eq. (3.4) with $\chi = u^A$, we thus obtain

$$\begin{aligned} u(x_0) - u^h(x_0) &= a^h(G - G^h, u - u^A) + a^h(G^h, u - u^h) \\ &\quad + a^h(G - G^h, u^A) + (u(x_0) - a^h(G, u)). \end{aligned} \tag{4.7}$$

Consider the last term in (4.7) and employ (4.2) and the properties of the Green's function, $G = G_{x_0}$, to deduce

$$u(x_0) - a^h(G, u) = \int_{\Omega - \Omega^h} \nabla G \cdot \nabla u \, dx.$$

We may now combine (2.5) and (2.22) to obtain

$$|u(x_0) - a^h(G, u)| \leqslant C |\nabla u|_{L^\infty(\Omega)} \int_{\Omega - \Omega^h} |\nabla G| \, dx \leqslant Ch^2 |u|_{W^1_\infty(\Omega)}. \tag{4.8}$$

We now consider the third term on the right side of (4.7) and note that $u^A = u^I \in \mathring{H}^1(\Omega)$ and is continuous in Ω. Employing integration by parts and the properties of the Green's function, we see that $a^h(G, u^A) = u^A(x_0)$. Furthermore we see from (3.3) that $a^h(G^h, u^A) = u^A(x_0)$. Hence we conclude that

$$a^h(G - G^h, u^A) = 0. \tag{4.9}$$

We next estimate the first term on the right side of (4.7). Since $u^A \in W^1_\infty(\Omega^h)$, we may employ (4.2) and (A.4) to deduce

$$\begin{aligned} |a^h(G - G^h, u - u^A)| &\leqslant C \|G - G^h\|^h_{W^1_1(\Omega^h)} \|u - u^A\|_{W^1_\infty(\Omega^h)} \\ &\leqslant Ch \|u\|_{W^2_\infty(\Omega)} \|G - G^h\|^h_{W^1_1(\Omega^h)}. \end{aligned}$$

In view of this, we thus require the following estimate:

$$\|G - G^h\|^h_{W^1_1(\Omega^h)} \leqslant Ch |\log h|. \tag{4.10}$$

Estimate (4.10) is analogous to the estimate in Theorem 2.3 and follows from the same arguments as in [1] or [2]. It now follows that

$$|a^h(G - G^h, u - u^A)| \leqslant Ch^2 |\log h| \, \|u\|_{W^2_\infty(\Omega)}. \tag{4.11}$$

We are left with estimating the second term on the right side of (4.7). Hence we wish to prove

$$|a^h(G^h, u - u^h)| \leqslant Ch^2 |\log h| \, \|u\|_{W^2_\infty(\Omega)}. \tag{4.12}$$

We shall establish (4.12) with the aid of certain polynomial invariances. We first note that

$$a^h(u - u^h, w^h) = \sum_{t^h \in T^h} D_{t^h}(u, w^h), \quad \text{for each } w^h \in S^h, \tag{4.13}$$

where

$$D_{t^h}(v, w^h) = \sum_{j=1}^{3} \oint_{e_j^h} \frac{\partial v}{\partial n} (w^h - C(w^h, e_j^h))\, ds, \quad \text{for each } v \in W_\infty^2(t^h),\ w^h \in S^h. \tag{4.14}$$

Here, e_j^h, $j = 1, 2, 3$, denote the sides of t^h and $C(w^h, e_j^h)$ denotes the value of w^h at the midpoint of e_j^h. Equations (4.13) and (4.14) follow from (4.1), (4.3), integration by parts, and the fact that functions in S^h are continuous at the midpoint of e_j^h.

We next observe that the following polynomial invariances hold for each $t^h \in T^h$:

$$D_{t^h}(v, p) = 0, \quad \text{for each } v \in W_\infty^2(t^h) \text{ and } p \in P_0(t^h), \tag{4.15}$$

and

$$D_{t^h}(q, w^h) = 0, \quad \text{for each } q \in P_1(t^h) \text{ and } w^h \in S^h, \tag{4.16}$$

where $P_l(t^h)$ denotes the space of polynomials of degree $\leqslant l$ defined on t^h for each integer $l \geqslant 0$. The first invariance, (4.15), follows easily from (4.14). The second invariance was proved in [17]. It follows from the fact that the integral of a linear function over an interval is zero provided the function vanishes at the midpoint of the interval. Equation (4.16) readily implies the "patch test"; see, e.g., [12].

We now define a function, G^A, as the unique function in S_D^h equal to G at all vertices of the partition, T^h, that do not intersect t_0^h or any of its neighboring triangles, and equal to zero at the remaining vertices of T^h. Thus, G^A is zero near x_0 and is equal to the interpolate of G with respect to the conforming subspace, S_D^h, away from x_0. Since G^A is continuous, we readily see with the aid of (4.1), (4.3), and (4.13) that

$$a^h(u - u^h, G^h) = a^h(u - u^h, G^h - G^A) = \sum_{t^h \in T^h} D_{t^h}(u, G^h - G^A). \tag{4.17}$$

In order to estimate the right side of (4.17), we first observe that each $t^h \in T^h$ is affinely equivalent to a reference element, $\hat{t}$. For example, $\hat{t}$ may be taken to be the triangle with vertices at $(0, 0)$, $(1, 0)$, and $(1, 1)$. Thus, $\hat{t}$ may be mapped onto each t^h by means of an invertible affine mapping, F_{t^h}, given by

$$x = F_{t^h}(\hat{x}) = B_{t^h}\hat{x} + b_{t^h}, \quad \text{for each } \hat{x} \in \hat{t}, \tag{4.18}$$

where B_{t^h} is a 2×2 matrix and b_{t^h} is a 2-vector. Furthermore, functions in $W_p^j(t^h)$ are mapped into functions in $W_p^j(\hat{t})$, with $j \geqslant 0$ and $p \in [1, \infty]$, by means of the usual correspondence:

$$\hat{\varphi}(\hat{x}) = \varphi(x), \quad \text{for each } \hat{x} \in \hat{t}. \tag{4.19}$$

Finally, we define P^h to be the restriction of S^h to t^h and let $\hat{P}$ denote the image of P^h with respect to the correspondence given by (4.19).

It is readily seen, using (4.18) and (4.19), that the bilinear form given by (4.14) yields a bilinear form, $D_{\hat{t}}(\ ,\ )$, acting on $W^2_\infty(\hat{t}) \times \hat{P}$ and given by

$$D_{\hat{t}}(\hat{v}, \hat{w}^h) = D_{t^h}(v, w^h), \quad \text{for each } \hat{v} \in W^2_\infty(\hat{t}),\ \hat{w}^h \in \hat{P}. \tag{4.20}$$

Combining (4.20) with (4.15), (4.16), (4.18), and (4.19), we deduce

$$D_{\hat{t}}(\hat{v}, \hat{p}) = 0, \quad \text{for each } \hat{v} \in W^2_\infty(\hat{t}) \text{ and } \hat{p} \in P_0(\hat{t}), \tag{4.21}$$

and

$$D_{\hat{t}}(q, \hat{w}^h) = 0, \quad \text{for each } \hat{q} \in \hat{P}_1(\hat{t}) \text{ and } \hat{w}^h \in \hat{P}. \tag{4.22}$$

We may now combine (4.21) and (4.22) with the Bramble-Hilbert lemma (as in [11, Section 4.2]) to obtain

$$|D_{\hat{t}}(\hat{v}, \hat{w}^h)| \leqslant C|\hat{v}|_{W^2_\infty(\hat{t})}\, |\hat{w}^h|_{W^1_1(\hat{t})}, \quad \text{for each } \hat{v} \in W^2_\infty(\hat{t}),\ \hat{w}^h \in \hat{P}. \tag{4.23}$$

It also follows, as in [11], that

$$|\hat{v}|_{W^j_p(\hat{t})} \leqslant Ch^{j-2/p}|v|_{W^j_p(t^h)}, \quad \text{for each } \hat{v} \in W^j_p(\hat{t}), \tag{4.24}$$

where $j \geqslant 0$ and $p \in [1, \infty]$. Finally, we combine (4.20), (4.23), and (4.24) to conclude that

$$|D_{t^h}(u, G^h - G^A)| \leqslant Ch|u|_{W^2_\infty(t^h)}\, |G^h - G^A|_{W^1_1(t^h)}. \tag{4.25}$$

Substituting (4.25) into (4.17), we obtain

$$|a^h(u - u^h, G^h)| \leqslant Ch|u|_{W^2_\infty(\Omega)}\, |G^h - G^A|^h_{W^1_1(\Omega^h)}. \tag{4.26}$$

We next observe that

$$\begin{aligned} |G^h - G^A|^h_{W^1_1(\Omega^h)} &\leqslant |G - G^h|^h_{W^1_1(\Omega^h)} + |G - G^A|^h_{W^1_1(\Omega^h)} \\ &\leqslant Ch|\log h| + |G - G^A|^h_{W^1_1(\Omega^h)}, \end{aligned} \tag{4.27}$$

using (4.10). It follows from the definition and approximation properties of G^A and (2.5) that

$$|G - G^A|^h_{W^1_1} \leqslant Ch|\log h|. \tag{4.28}$$

Combining (4.26)–(4.28), we thus conclude that (4.12) holds. Combining (4.8), (4.9), (4.11), and (4.12) with (4.7), we finally obtain

$$|u(x_0) - u^h(x_0)| \leqslant Ch^2\,|\log h|\ \|u\|_{W^2_\infty(\Omega)}, \tag{4.29}$$

where C is independent of x_0. We have thus proved (4.6) and hence Theorem 4.1. Q.E.D.

Remark 4.1. The arguments employed above to prove L^∞ error estimates are applicable to other nonconforming finite element methods and other boundary value problems. For a discussion of mean-square error estimates in connection with a variety of nonconforming methods, see [11] and the references cited there.

5. Numerical Quadrature. In this section we consider a conforming finite element method for solving problem (2.1). However, we assume that the integrations necessary to obtain the finite element equations, (2.4), are replaced by numerical quadrature. We shall derive L^∞ error estimates for both the Dirichlet and Neumann problems with the aid of Theorem 3.2. Let us observe that L^∞ estimates were established for the Dirichlet problem in the presence of numerical quadrature by Wahlbin in [13]. (See Remark 5.1 below for a brief description of his results.)

We first consider the boundary value problem given by (2.1)(D), assuming that Ω is a bounded open subset of R^2 and that the boundary of Ω and coefficients of A are smooth. For the sake of simplicity, we assume that $c = 0$ in (2.2). As in Section 2, we may treat this problem employing the family of finite element spaces, $S^h = S^h_D$, and bilinear forms, $a^h(\ ,\ )$, defined by (2.17). To this end, assume for simplicity that Ω is convex and is replaced by a convex (open) inscribed polygon, Ω^h, for each $h \in (0, 1]$. Recall that S^h satisfies assumption (A.1) with respect to a quasi-uniform family of triangulations of Ω^h for each $h \in (0, 1]$. Furthermore, S^h consists of continuous piecewise linear functions vanishing in $\Omega - \Omega^h$. The bilinear forms are defined by

$$(5.1)\qquad a^h(u, v) = \int_{\Omega^h} \sum_{i,j=1}^{2} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j}\, dx, \quad \text{for each } u, v \in H^1(\Omega^h), h \in (0, 1].$$

As we remarked in Section 2, it may be seen that conditions (A.2)–(A.4) hold, as well as Lemma 2.1, Theorem 2.3, and Theorem 2.4 (with Ω replaced by Ω^h when necessary). In particular, there exists a unique function, $u^h \in S^h$, satisfying the equation

$$(5.2)\qquad a^h(u^h, v) = (f, v), \quad \text{for each } v \in S^h.$$

In addition, it follows from the arguments of [11] that

$$(5.3)\qquad \|u - u^h\|_{H^1(\Omega^h)} \leqslant Ch\|u\|_{H^2(\Omega)}.$$

We see from Lemma 2.1 that there exists a unique function, $G^h_{x_0} = G^h \in S^h$, such that

$$(5.4)\qquad a^h(G^h, v) = v(x_0), \quad \text{for each } v \in S^h,$$

where x_0 is a fixed point in Ω^h.

Before defining our perturbed bilinear forms, $\tilde{a}^h(\ ,\ )$, and linear functionals, $\tilde{f}^h(\ )$, we shall define our numerical quadrature scheme over each t^h in the same way as described in [11] (Section 4.1). To this end, we first observe, as in the previous section, that there is a reference element, $\hat{t}$, such that $\hat{t}$ may be mapped onto each t^h by means of an invertible affine mapping, F_{t^h}, given by

$$(5.5)\qquad x = F_{t^h}(\hat{x}) = B_{t^h}\hat{x} + b_{t^h},$$

where B_{t^h} is a 2×2 matrix and b_{t^h} is a 2-vector. The determinant of B_{t^h} is denoted by $\det(B_{t^h})$. It is clear that, for any integrable function $\varphi(x)$ defined on t^h, we have

$$\int_{t^h} \varphi(x)\,dx = \det(B_{t^h}) \int_{\hat{t}} \hat{\varphi}(\hat{x})\,d\hat{x}, \tag{5.6}$$

where φ and $\hat{\varphi}$ are in the usual correspondence:

$$\varphi(x) = \hat{\varphi}(\hat{x}), \quad \text{for each } x = F_{t^h}(\hat{x}), \hat{x} \in \hat{t}. \tag{5.7}$$

We define our quadrature scheme over the reference element, $\hat{t}$, by means of a formula of the following form:

$$I_{\hat{t}}(\hat{\varphi}) = \sum_{j=1}^{J} \hat{w}_j \hat{\varphi}(\hat{b}_j). \tag{5.8}$$

We assume that the nodes, $\hat{b}_j$, belong to $\hat{t}$. In accordance with (5.6) and (5.7), we define our quadrature scheme over each t^h by means of the formula:

$$I_{t^h}(\varphi) = \sum_{j=1}^{J} w_{j,t^h} \varphi(b_{j,t^h}), \tag{5.9}$$

where $w_{j,t^h} = \det(B_{t^h})\hat{w}_j$, and φ is defined at each node $b_{j,t^h} = F_{t^h}(\hat{b}_j)$, $1 \leqslant j \leqslant J$. Set

$$E_{t^h}(\varphi) = \int_{t^h} \varphi(x)\,dx - I_{t^h}(\varphi), \tag{5.10}$$

and define $E_{\hat{t}}(\hat{\varphi})$ similarly. Note that $E_{t^h}(\varphi) = \det(B_{t^h}) E_{\hat{t}}(\hat{\varphi})$.

We now define, $\tilde{a}^h(\ ,\ )$, by replacing the integral on the right side of (5.1) by numerical quadrature. Hence we have

$$\tilde{a}^h(u, v) = \sum_{t^h \in T^h} I_{t^h}\left(\sum_{i,j=1}^{2} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} \right), \tag{5.11}$$

provided $u, v \in C^1(\overline{t^h})$, for each $t^h \in T^h$. Furthermore, we denote $\int_\Omega fv$ by $f(v)$ and set

$$\tilde{f}^h(v) = \sum_{t^h \in T^h} I_{t^h}(fv), \tag{5.12}$$

provided $f, v \in C^0(\overline{t^h})$, for each $t^h \in T^h$. Let us assume for the time being that condition (A.2) holds with respect to $\tilde{a}^h(\ ,\ )$. We may now define our approximate solution, $\tilde{u}^h$, of (2.1)(D) as follows:

$$\tilde{a}^h(\tilde{u}^h, v) = \tilde{f}^h(v), \quad \text{for each } v \in S^h. \tag{5.13}$$

We shall now show that, when the quadrature scheme is sufficiently accurate, condition (A.2) holds with respect to $\tilde{a}^h(\ ,\ )$. Furthermore, an analogue of Theorem 2.4 holds with u^h replaced by $\tilde{u}^h$. For this purpose, we recall that $P_l(D)$ consists of all polynomials of degree not greater than l defined on the set D.

THEOREM 5.1. *Suppose that* $N = K = 2$, u *satisfies* (2.1)(D), *and* $f \in W^2_{p_0}(\Omega)$ *for some* $p_0 > 1$. *Finally, suppose that*

$$E_{\hat{t}}(\hat{\varphi}) = 0, \quad \text{for each } \hat{\varphi} \in P_1(\hat{t}). \tag{5.14}$$

Then for $h > 0$ *sufficiently small, there exists a unique function,* $\tilde{u}^h \in S^h$, *satisfying* (5.13). *Furthermore, we have*

$$\|u - \tilde{u}^h\|_{L_\infty(\Omega)} \leqslant Ch^2 |\log h| \, \|f\|_{W^2_{p_0}(\Omega)},$$

where C *is independent of* h.

Proof. To begin with, assume that there exists a function, $\tilde{u}^h \in S^h$, satisfying (5.13). Using elliptic regularity theory and Sobolev's inequality, we conclude from the hypothesis on f that $u \in W^2_\infty(\Omega)$. We observe that it follows, exactly as in the proof of Theorem 4.1, that

$$\|u - \tilde{u}^h\|_{L_\infty(\Omega - \Omega^h)} \leqslant Ch^2 \|u\|_{W^1_\infty(\Omega)}. \tag{5.15}$$

Now suppose that x_0 is an arbitrary point in Ω^h, so that $x_0 \in \overline{t_0^h}$ for some triangle $t_0^h \in T^h$. We apply Eq. (3.10) with χ replaced by u^A (obtained from (A.4)) to deduce

$$\begin{aligned} u(x_0) - u^h(x_0) = a^h(G - G^h, u - u^A) + (\tilde{a}^h - a^h)(G^h, \tilde{u}^h) \\ + (f - \tilde{f}^h)(G^h) + (u(x_0) - a^h(G, u)). \end{aligned} \tag{5.16}$$

We may estimate the last term in (5.16) using the same argument as in the proof of Theorem 4.1. Hence we have

$$|u(x_0) - a^h(G, u)| \leqslant Ch^2 \|u\|_{W^1_\infty(\Omega)}. \tag{5.17}$$

Now consider the first term on the right side of (5.16). We may apply (A.4) and Theorem 2.3 to obtain

$$\begin{aligned} |a^h(G - G^h, u - u^A)| &\leqslant C\|G - G^h\|_{W^1_1(\Omega^h)} \|u - u^A\|_{W^1_\infty(\Omega^h)} \\ &\leqslant Ch^2 |\log h| \, \|u\|_{W^2_\infty(\Omega)}. \end{aligned} \tag{5.18}$$

We next consider the second term on the right side of (5.16). It follows from (5.1), (5.10), and (5.11) that

$$(\tilde{a}^h - a^h)(v, w) = -\sum_{t^h \in T^h} E_{t^h}\left(\sum_{i,j=1}^{2} a_{ij} \frac{\partial v}{\partial x_i} \frac{\partial w}{\partial x_j}\right), \quad \text{for each } v, w \in S^h. \tag{5.19}$$

Suppose that $t^h \in T^h$ and recall that $\hat{t}$ is mapped onto t^h by means of the affine mapping given by (5.5). It may be readily seen using the arguments of [11] that

$$|\det(B_{t^h})| \leqslant Ch^2, \tag{5.20}$$

and

$$(5.21)\qquad \left|E_{t^h}\left(\sum_{i,j=1}^{2} a_{ij}\frac{\partial v}{\partial x_i}\frac{\partial w}{\partial x_j}\right)\right| \leqslant C\left|E_{\hat t}\left(\sum_{i,j=1}^{2}\hat a_{ij}\frac{\partial\hat v}{\partial\hat x_i}\frac{\partial\hat w}{\partial\hat x_j}\right)\right|.$$

We may now apply the Bramble-Hilbert lemma and the hypothesis, (5.14), to obtain

$$(5.22)\qquad |E_{\hat t}(\hat\varphi)| \leqslant C|\hat\varphi|_{W^2_\infty(\hat t)}, \quad \text{for each } \hat\varphi\in W^2_\infty(\hat t).$$

Set

$$\hat\varphi = \sum_{i,j=1}^{2}\hat a_{ij}\frac{\partial\hat v}{\partial\hat x_i}\frac{\partial\hat w}{\partial\hat x_j}$$

and substitute in (5.22). Since $\hat v$ and $\hat w$ are linear, we deduce

$$(5.23)\qquad \left|E_{\hat t}\left(\hat a_{ij}\frac{\partial\hat v}{\partial\hat x_i}\frac{\partial\hat w}{\partial\hat x_j}\right)\right| \leqslant \sum_{i,j=1}^{2}|\hat a_{ij}|_{W^2_\infty(\hat t)}|\hat v|_{W^1_\infty(\hat t)}|\hat w|_{W^1_\infty(\hat t)}.$$

We now map $\hat t$ back onto t^h and combine (2.3) and (4.24) with (5.21) and (5.23) to obtain

$$(5.24)\qquad \begin{aligned}&\left|E_{t^h}\left(\sum_{i,j=1}^{2} a_{ij}\frac{\partial v}{\partial x_i}\frac{\partial w}{\partial x_j}\right)\right|\\ &\qquad\leqslant C\sum_{i,j=1}^{2} h^2|a_{ij}|_{W^2_\infty(t^h)}|v|_{W^1_\infty(t^h)}\|w\|_{W^1_1(t^h)}\\ &\qquad\leqslant Ch^2\max_{i,j=1,2}\|a_{ij}\|_{W^2_\infty(t^h)}\|v\|_{W^1_\infty(t^h)}\|w\|_{W^1_1(t^h)}.\end{aligned}$$

Set $v = \tilde u^h$ and $w = G^h$ and then combine (5.19) and (5.24) to conclude that

$$(5.25)\qquad |(\tilde a^h - a^h)(\tilde u^h, G^h)| \leqslant Ch^2\|\tilde u^h\|_{W^1_\infty(\Omega^h)}\|G^h\|_{W^1_1(\Omega^h)}.$$

It follows from (2.5) and Theorem 2.3 that

$$(5.26)\qquad \begin{aligned}\|G^h\|_{W^1_1(\Omega^h)} &\leqslant \|G - G^h\|_{W^1_1(\Omega^h)} + \|G\|_{W^1_1(\Omega^h)} \leqslant Ch|\log h|\\ &\quad + \|G\|_{W^1_1(\Omega^h)} \leqslant C\end{aligned}$$

for h sufficiently small. We now employ (5.25) and (5.26) to deduce

$$(5.27)\qquad |(\tilde a^h - a^h)(\tilde u^h, G^h)| \leqslant Ch^2\|\tilde u^h\|_{W^1_\infty(\Omega^h)}.$$

We next estimate the third term on the right side of (5.16). Suppose that $t^h\in T^h$ and apply (5.5), (5.14), (5.20), Sobolev's inequality, and the Bramble-Hilbert lemma to obtain

$$(5.28)\qquad |E_{t^h}(fG^h)| \leqslant Ch^2|E_{\hat t}(\hat f\hat G^h)| \leqslant Ch^2|\hat f\hat G^h|_{W^2_1(\hat t)}.$$

Choose $q_0 < \infty$ such that $1/q_0 + 1/p_0 = 1$. Furthermore, suppose that $p = 2 + \epsilon$ with $\epsilon > 0$ small and choose $q < 2$ such that $1/p + 1/q = 1$. Note that $\hat{G}^h$ is linear in $\hat{t}$ and apply (4.24) and Hölder's inequality to deduce

$$
(5.29)\qquad
\begin{aligned}
|\hat{f}\hat{G}^h|_{W_1^2(\hat{t})} &\leq C(|f|_{W_{p_0}^2(\hat{t})}|\hat{G}^h|_{L^{q_0}(\hat{t})} + |\hat{f}|_{W_p^1(\hat{t})}|\hat{G}^h|_{W_q^1(\hat{t})}) \\
&\leq C(|f|_{W_{p_0}^2(t^h)}|G^h|_{L^{q_0}(t^h)} + |f|_{W_p^1(t^h)}|G^h|_{W_q^1(t^h)}).
\end{aligned}
$$

We next sum over all $t^h \in T^h$ and combine (5.28) and (5.29) with Sobolev's inequality to deduce

$$
(5.30)\qquad |(f - \tilde{f}^h)(G^h)| \leq Ch^2 \|f\|_{W_{p_0}^2(\Omega)} \|G^h\|_{W_q^1(\Omega^h)}.
$$

Applying Lemma 2.1 and (5.3), we see that

$$
(5.31)\qquad
\begin{aligned}
\|G^h\|_{W_q^1(\Omega^h)} &\leq \|\widetilde{G} - G^h\|_{H^1(\Omega^h)} + \|\widetilde{G}\|_{W_q^1(\Omega^h)} \\
&\leq Ch\|\widetilde{G}\|_{H^2(\Omega)} + \|\widetilde{G}\|_{W_q^1(\Omega)} \leq C + \|\widetilde{G}\|_{W_q^1(\Omega)}.
\end{aligned}
$$

In order to estimate the last term in (5.31), we write $\Omega = D_0^h \cup D^h$, where D_0^h is defined as the intersection of Ω with a sphere of radius $O(h)$ centered at x_0 and $D^h = \Omega - D_0^h$. We choose D^h such that $\operatorname{dist}(D^h, \operatorname{supp}(\widetilde{\delta})) \geq Ch$. We now apply Sobolev's inequality to deduce

$$
(5.32)\qquad \|\widetilde{G}\|_{W_q^1(\Omega)} \leq C\|\widetilde{G}\|_{W_1^2(\Omega)} \leq C(\|\widetilde{G}\|_{W_1^2(D_0^h)} + \|\widetilde{G}\|_{W_1^2(D^h)}).
$$

Employing Lemma 2.1 and elliptic regularity theory, we obtain

$$
\|\widetilde{G}\|_{W_1^2(D_0^h)} \leq Ch\|\widetilde{G}\|_{H^2(D_0^h)} \leq Ch\|\widetilde{\delta}\|_{L^2(\Omega)} \leq C.
$$

To estimate the last term in (5.32), we note with the aid of Lemma 2.1 that $\widetilde{G}(x) = \int_\Omega G_x(x')\widetilde{\delta}(x')dx'$, for each $x \in \Omega$. We may thus combine (2.5) and Lemma 2.1 with the definition of D^h to see that

$$
\|\widetilde{G}\|_{W_1^2(D^h)} \leq C|\log h|.
$$

Combining the last two inequalities with (5.32), we obtain

$$
(5.33)\qquad \|\widetilde{G}\|_{W_q^1(\Omega)} \leq C|\log h|.
$$

Combining (5.30), (5.31), and (5.33), we conclude that

$$
(5.34)\qquad |(f - \tilde{f}^h)(G^h)| \leq Ch^2|\log h|\ \|f\|_{W_{p_0}^2(\Omega)}.
$$

We now combine (5.16)–(5.18), (5.27), and (5.34) to conclude that there exists an $h_0 \in (0, 1]$ such that, for each $h \in (0, h_0]$ and each point $x_0 \in \Omega^h$, we have

$$
(5.35)\quad |u(x_0) - \tilde{u}^h(x_0)| \leq Ch^2|\log h|\ (\|u\|_{W_\infty^2(\Omega)} + \|f\|_{W_{p_0}^2(\Omega)}) + Ch^2\|\tilde{u}^h\|_{W_\infty^1(\Omega^h)},
$$

where C is independent of x_0 and h. In order to estimate the last term in (5.35), we employ a "kickback argument" as in [13]. We first observe that

$$\|\tilde{u}^h\|_{W^1_\infty(\Omega^h)} \leqslant \|u^h - \tilde{u}^h\|_{W^1_\infty(\Omega^h)} + \|u^h\|_{W^1_\infty(\Omega^h)}, \tag{5.36}$$

where u^h satisfies Eq. (5.2). It follows from the "inverse inequalities", (2.3), that

$$\begin{aligned} \|u^h - \tilde{u}^h\|_{W^1_\infty(\Omega^h)} &\leqslant Ch^{-1}\|u^h - \tilde{u}^h\|_{L^\infty(\Omega^h)} \\ &\leqslant Ch^{-1}(\|u - u^h\|_{L^\infty(\Omega^h)} + \|u - \tilde{u}^h\|_{L^\infty(\Omega^h)}). \end{aligned} \tag{5.37}$$

We readily see using (A.4) and (2.3) that

$$\begin{aligned} \|u^h\|_{W^1_\infty(\Omega^h)} &\leqslant \|u^h - u^A\|_{W^1_\infty(\Omega^h)} + \|u^A - u\|_{W^1_\infty(\Omega^h)} \\ &\quad + \|u\|_{W^1_\infty(\Omega^h)} \leqslant Ch^{-1}(\|u - u^A\|_{L^\infty(\Omega^h)} + \|u - u^h\|_{L^\infty(\Omega^h)}) \\ &\quad + \|u - u^A\|_{W^1_\infty(\Omega^h)} + \|u\|_{W^1_\infty(\Omega)} \leqslant Ch\|u\|_{W^2_\infty(\Omega)} \\ &\quad + C\|u\|_{W^1_\infty(\Omega)} + Ch^{-1}\|u - u^h\|_{L^\infty(\Omega^h)}. \end{aligned} \tag{5.38}$$

In view of the definiton of u^h and Theorem 2.4, we readily obtain

$$\|u - u^h\|_{L^\infty(\Omega^h)} \leqslant Ch^2|\log h|\ \|u\|_{W^2_\infty(\Omega)}. \tag{5.39}$$

We may thus combine (5.36)–(5.39) to deduce

$$\begin{aligned} \|\tilde{u}^h\|_{W^1_\infty(\Omega^h)} &\leqslant Ch|\log h|\ \|u\|_{W^2_\infty(\Omega)} \\ &\quad + C\|u\|_{W^1_\infty(\Omega)} + Ch^{-1}\|u - \tilde{u}^h\|_{L^\infty(\Omega^h)}. \end{aligned} \tag{5.40}$$

We may now combine (5.35) with (5.40) to conclude that the following estimate holds for h sufficiently small:

$$\begin{aligned} \|u - \tilde{u}^h\|_{L^\infty(\Omega^h)} &\leqslant Ch^2|\log h|\ (\|u\|_{W^2_\infty(\Omega)} + \|f\|_{W^2_{p_0}(\Omega)}) \\ &\quad + Ch\|u - \tilde{u}^h\|_{L^\infty(\Omega^h)}. \end{aligned} \tag{5.41}$$

It follows, from the unique solvability of (2.1)(D), estimate (5.41), and the finite dimensionality of S^h, that $\tilde{u}^h$ exists and is unique for h sufficiently small. Combining (5.15) and (5.41) with Sobolev's inequality and elliptic regularity theory, we have proved the theorem. Q.E.D.

Remark 5.1. The arguments employed in the proof of estimates (5.25), (5.30), and (5.40) follow along lines similar to those employed by Wahlbin in [13]. In [13], L^∞ error estimates were established for a second-order linear elliptic boundary value problem in R^2 using a finite element method based on quadratic triangular isoparametric elements. It was assumed there that $f \in W_1^3(\Omega)$ and the quadrature scheme was sufficiently accurate that $E_{\hat{t}}(\hat{\varphi}) = 0$, for each $\varphi \in P_2(\hat{t})$. It was then shown that

$$\|u - \tilde{u}^h\|_{L^\infty(\Omega)} \leqslant C_\epsilon h^{3-\epsilon} \|f\|_{W_1^3(\Omega)} \tag{5.42}$$

for arbitrary $\epsilon > 0$. (Note that $C_\epsilon \longrightarrow \infty$ as $\epsilon \longrightarrow 0$.) The arguments of [13] may also be shown to yield the estimate

$$\|u - \tilde{u}^h\|_{L^\infty(\Omega)} \leqslant C_\epsilon h^{2-\epsilon} \|f\|_{W_1^2(\Omega)}, \tag{5.43}$$

where $\tilde{u}^h$ satisfies (5.13), $f \in W_1^2(\Omega)$, and the remaining assumptions of Theorem 5.1 hold. Note that, in Theorem 5.1, we were able to eliminate the factor $h^{-\epsilon}$ present in (5.43) under the slightly stronger assumption that $\|f\|_{W^2_{1+\epsilon}(\Omega)} < \infty$ for some $\epsilon > 0$.

Remark 5.2. The arguments of Theorem 5.1 may be extended to the case in which $K > 2$ and $N = 2$ or 3. Let us consider each term on the right side of Eq. (5.16). The first term may be estimated in the same way as before. The two middle terms are due to the presence of numerical quadrature. Hence it is necessary to employ a sufficiently accurate quadrature scheme in order to suitably estimate these terms. We shall demonstrate this in the proof of Theorem 5.2 below. The last term in (5.16) is due to the difference between the polygonal domain, Ω^h, and the given domain, Ω. For this reason, estimate (5.17), as well as (5.15), remain the same regardless of K. This estimate may be improved using alternative methods for treating the Dirichlet problem. For example, isoparametric elements may be employed as in [13] (with $N = 2$ and $K = 3$).

We next consider the Neumann problem, (2.1)(N), employing the finite element method described in Section 2 to solve this problem. Hence, our family of finite element spaces, $S^h = S^h_N$, satisfies condition (A.1) and consists of continuous functions defined on $\bar{\Omega} = \bigcup_{t^h \in T^h} \overline{t^h}$. Recall that functions in S^h need not satisfy any boundary condition on $\partial\Omega$ and that boundary elements may have one curved face. We may assume that $K > 2$ and $N = 2$ or 3 since the case $N = K = 2$ may be treated using the arguments of Theorem 5.1. The bilinear form, $a^h(\ ,\ )$, is now given by

$$a^h(u, v) = \sum_{i,j=1}^{N} \int_\Omega \left(a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + cuv \right) dx, \tag{5.44}$$

for each $u, v \in H^1(\Omega)$, $h \in (0, 1]$.

We again observe that $a^h(\ ,\ )$ is coercive over $H^1(\Omega) \times H^1(\Omega)$ since $c > 0$.

Suppose that we are given a quadrature scheme, I_{t^h}, defined on each element $t^h \in T^h$. We also assume that

$$E_{t^h}(\varphi) = 0, \quad \text{for each } \varphi \in P_{2K-4}(t^h), \text{ and } t^h \in T^h, \tag{5.45}$$

where $E_{t^h}(\varphi)$ is defined as in (5.10). We define $\tilde{a}^h(\ ,\ )$ as a perturbation of $a^h(\ ,\ )$ as before, where $a^h(\ ,\ )$ is now defined by (5.44). We are now ready to prove a result analogous to Theorem 5.1. As we observed above, the second and third terms on the right side of (5.16) describe the effect of numerical quadrature. For the sake of simplicity, we shall only estimate the second term. Thus we may assume that $f = \tilde{f}^h$ in Theorem 5.2. If $f \neq \tilde{f}^h$, we may estimate the third term on the right side of (5.16) using an argument similar to that employed in Theorem 5.1. In this case, we would again require additonal smoothness assumptions on f in order to obtain an estimate analogous to (5.30).

THEOREM 5.2. *Suppose that $K > 2$, $N = 2$ or 3, u satisfies* (2.1)(N), *and $u \in W_\infty^K(\Omega)$. In addition, suppose that Eq.* (5.45) *holds. Then for $h > 0$ sufficiently small, there exists a unique function, $\tilde{u}^h \in S^h$, satisfying*

$$\tilde{a}^h(\tilde{u}^h, v) = (f, v) = f(v), \quad \text{for each } v \in S^h. \tag{5.46}$$

Furthermore, we have

$$\|u - u^h\|_{L^\infty(\Omega)} \leqslant Ch^K |\log h|^{\delta_{K3}} \|u\|_{W_\infty^K(\Omega)}, \tag{5.47}$$

where C is independent of h and

$$\delta_{K3} = \begin{cases} 1 & \text{for } K = 3, \\ 0 & \text{for } K > 3. \end{cases}$$

Proof. Assume, for the time being, that there exists a function, $\tilde{u}^h \in S^h$, satisfying (5.46). Now suppose that $x_0 \in \overline{t_0^h}$ for some simplex, $t_0^h \in T^h$, and apply (3.10) (with $f = \tilde{f}^h$ and χ replaced by u^A) to obtain

$$u(x_0) - \tilde{u}^h(x_0) = a^h(G - G^h, u - u^A) + (\tilde{a}^h - a^h)(G^h, \tilde{u}^h). \tag{5.48}$$

Here, we used the fact that $u(x_0) = a^h(G, u)$. The first term on the right side of (5.48) may be estimated using the same argument as in Theorem 5.1. We thus obtain

$$|a^h(G - G^h, u - u^A)| \leqslant Ch^K \|u\|_{W_\infty^K(\Omega)}. \tag{5.49}$$

We next estimate the last term in (5.48) using a version of the Bramble-Hilbert lemma established in [19]. Suppose that $t^h \in T^h$ and observe that $|E_{t^h}(\varphi)| \leqslant C|\varphi|_{L^\infty(t^h)}$ for each $\varphi \in C^0(t^h)$. It now follows from (5.45) and [19] that there exists a constant C independent of h, $t^h \in T^h$, and $v, w \in S^h$ such that the following estimate holds:

$$\begin{aligned} \left| E_{t^h}\left(\sum_{i,j=1}^N a_{ij} \frac{\partial v}{\partial x_i} \frac{\partial w}{\partial x_j} \right) \right| &\leqslant Ch^{2K-3} \left| \sum_{i,j=1}^N a_{ij} \frac{\partial v}{\partial x_i} \frac{\partial w}{\partial x_j} \right|_{W_1^{2K-3}(t^h)} \\ &\leqslant Ch^{2K-3} \sum_{i,j=1}^N \sum_{\substack{\alpha+\beta+\gamma=2K-3,\\ \beta+1\leqslant K-1,\\ \gamma+1\leqslant K-1}} |a_{ij}|_{W_\infty^\alpha(t^h)} |v|_{W_\infty^{\beta+1}(t^h)} |w|_{W_1^{\gamma+1}(t^h)} \\ &\leqslant Ch^{2K-3} \max_{i,j=1,\dots,N} \|a_{ij}\|_{W_\infty^{2K-3}(t^h)} \|v\|_{W_\infty^{K-1}(t^h)} \|w\|_{W_1^{K-1}(t^h)}. \end{aligned} \tag{5.50}$$

Similarly, we obtain the following estimate:

$$(5.51) \qquad |E_{t^h}(cvw)| \leqslant Ch^{2K-3} \|c\|_{W_\infty^{2K-3}(t^h)} \|v\|_{W_\infty^{K-1}(t^h)} \|w\|_{W_1^{K-1}(t^h)}.$$

Set $v = \widetilde{u}^h$ and $w = G^h$, apply (5.50) and (5.51), and sum over each $t^h \in T^h$ to obtain

$$(5.52) \qquad |(\widetilde{a}^h - a^h)(\widetilde{u}^h, G^h)| \leqslant Ch^{2K-3} \|\widetilde{u}^h\|^h_{W_\infty^{K-1}(\Omega)} \|G^h\|^h_{W_1^{K-1}(\Omega)}.$$

We see from Corollary 2.1 that

$$(5.53) \qquad \|G^h\|^h_{W_1^{K-1}(\Omega)} \leqslant Ch^{-(K-3)} |\log h|^{\delta_{3K}}.$$

We may now apply an argument analogous to that in Theorem 5.1 to deduce

$$(5.54) \qquad \|\widetilde{u}^h\|^h_{W_\infty^{K-1}(\Omega)} \leqslant C(\|u\|_{W_\infty^K(\Omega)} + h^{-(K-1)} \|u - \widetilde{u}^h\|_{L^\infty(\Omega)}).$$

Substituting (5.49) and (5.52)–(5.54) into (5.48), we conclude that

$$|u(x_0) - \widetilde{u}^h(x_0)| \leqslant Ch^K (|\log h|^{\delta_{3K}} \|u\|_{W_\infty^K(\Omega)} + h |\log h|^{\delta_{3K}} \|u - \widetilde{u}^h\|_{L^\infty(\Omega)}),$$

where C is independent of x_0 and h. We thus obtain the following estimate for h sufficiently small:

$$(5.55) \qquad \|u - \widetilde{u}^h\|_{L^\infty(\Omega)} \leqslant Ch^K |\log h|^{\delta_{3K}} \|u\|_{W_\infty^K(\Omega)}.$$

It follows, from the unique solvability of (2.1)(N), estimate (5.55), and the finite dimensionality of S^h, that $\widetilde{u}^h$ exists and is unique for h sufficiently small. The theorem now follows from (5.55). Q.E.D.

Remark 5.3. We see from (5.14) that the hypothesis, (5.45), of Theorem 5.2 must be strengthened when $K = 2$. This is also the case for L^2 error estimates. (See [11] and the references cited there for the proof of mean square estimates in the presence of numerical integration.) We also observe from Theorem 5.2 that there is an additional $|\log h|$ present when $K = 3$. This may be removed, e.g., by improving the quadrature for $K = 3$, so that (5.45) is replaced by the equation $E_{\hat{t}}(\hat{\varphi}) = 0$, for each $\hat{\varphi} \in P_3(\hat{t})$.

Remark 5.4. The results of this section may be proved under more general assumptions on the boundary value problem, as well as for more general finite element spaces. (See Remark 2.2 for an analogous observation regarding the results of Section 2.) For example, lower-order terms may be present in the differential operator given by (2.2) and A need not be symmetric. Furthermore, the smoothness assumptions on the boundary and coefficients may be relaxed.

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