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On Some Orthogonal Polynomial Integrals

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Abstract. The modified moments of the weight functions $w(x) = x^{\rho}(1-x)^{\alpha}\ln(1/x)$, on $[0, 1]$, with respect to the shifted Jacobi polynomials $P_n^{*(\alpha,\beta)}(x) = P_n^{(\alpha,\beta)}(2x-1)$, and $w_p(x) = x^{\rho}e^{-x}(\ln x)^p$, $p = 1, 2$, on $[0, \infty)$, with respect to the generalized Laguerre polynomials $L_n^{(\alpha)}(x)$, are explicitly evaluated.

1. A Jacobi Polynomial Integral. In a recent paper, Gautschi [3], generalizing a result of Blue [2], has considered and explicitly evaluated the modified moments of the weight function

$$w(x) = x^{\mu}\ln(1/x), \qquad \mu > -1,$$

on $[0, 1]$, with respect to the shifted Legendre polynomials $P_n^*(x) = P_n(2x-1)$.

We further generalize these results by considering the weight function

$$w(x) = x^{\rho}(1-x)^{\alpha}\ln(1/x), \qquad \alpha, \rho > -1, \tag{1.1}$$

and evaluating its modified moments on $[0, 1]$ with respect to the shifted Jacobi polynomials $P_n^{*(\alpha,\beta)}(x) = P_n^{(\alpha,\beta)}(2x-1)$.

It is convenient from now on to replace ρ by $\beta + \mu$; thus, the modified moments we have to examine assume the form

$$\nu_n^{(\alpha,\beta)}(\mu) = \int_0^1 x^{\beta+\mu}(1-x)^{\alpha}\ln(1/x)P_n^{*(\alpha,\beta)}(x)\,dx, \qquad \alpha, \beta, \beta+\mu > -1, n = 0, 1, 2, \ldots. \tag{1.2}$$

We easily see that

$$\begin{aligned}\nu_n^{(\alpha,\beta)}(\mu) &= -2^{-(\alpha+\beta+\mu+1)}\int_{-1}^1 (1-t)^{\alpha}(1+t)^{\beta+\mu}\ln(\tfrac{1}{2}(1+t))P_n^{(\alpha,\beta)}(t)\,dt \\ &= -2^{-(\alpha+\beta+\mu+1)}\left\{\int_{-1}^1 (1-t)^{\alpha}(1+t)^{\beta+\mu}\ln(1+t)P_n^{(\alpha,\beta)}(t)\,dt\right. \\ &\qquad \left. -\ln 2\cdot\int_{-1}^1 (1-t)^{\alpha}(1+t)^{\beta+\mu}P_n^{(\alpha,\beta)}(t)\,dt\right\},\end{aligned}$$

hence, by putting

$$I_n^{(\alpha,\beta)}(\mu) = \int_{-1}^1 (1-t)^{\alpha}(1+t)^{\beta+\mu}P_n^{(\alpha,\beta)}(t)\,dt, \qquad \alpha, \beta, \beta+\mu > -1, n = 0, 1, 2, \ldots, \tag{1.3}$$

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we obtain

$$\nu_n^{(\alpha,\beta)}(\mu) = 2^{-(\alpha+\beta+\mu+1)}\left\{I_n^{(\alpha,\beta)}(\mu)\ln 2 - \frac{d}{d\mu} I_n^{(\alpha,\beta)}(\mu)\right\}. \tag{1.4}$$

The following expression for (1.3),

$$I_n^{(\alpha,\beta)}(\mu) = 2^{\alpha+\beta+\mu+1} \frac{\Gamma(\mu+1)}{n!\Gamma(\mu-n+1)} \frac{\Gamma(\beta+\mu+1)\Gamma(n+\alpha+1)}{\Gamma(n+\alpha+\beta+\mu+2)}, \tag{1.5}$$

is known ([1], [4, p. 256]). Indeed, (1.5) is easily obtained, multiplying on both sides of Rodrigues' formula,

$$(1-t)^\alpha(1+t)^\beta P_n^{(\alpha,\beta)}(t) = \frac{(-1)^n}{2^n n!} \frac{d^n}{dt^n}\{(1-t)^{n+\alpha}(1+t)^{n+\beta}\},$$

by $(1+t)^\mu$, integrating from -1 to 1 and carrying out n partial integrations on the right-hand side.

Differentiating (1.5) with respect to μ gives

$$\begin{aligned}\frac{d}{d\mu} I_n^{(\alpha,\beta)}(\mu) = I_n^{(\alpha,\beta)}(\mu)\{\ln 2 + \psi(\mu+1) + \psi(\beta+\mu+1) \\ -\psi(\mu-n+1) - \psi(n+\alpha+\beta+\mu+2)\},\end{aligned} \tag{1.6}$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$ is the logarithmic derivative of the gamma function, and, if μ coincides with an integer $m < n$, $m \geqslant 0$, the right-hand member must be replaced by its limit as $\mu \longrightarrow m$.

We first consider the case where $\mu \neq 0, 1, 2, \ldots, n-1$, whenever $n \geqslant 1$. By inserting (1.5) and (1.6) in (1.4), we obtain

$$\begin{aligned}\nu_n^{(\alpha,\beta)}(\mu) = \frac{\Gamma(\mu+1)\Gamma(\beta+\mu+1)\Gamma(n+\alpha+1)}{n!\Gamma(\mu-n+1)\Gamma(n+\alpha+\beta+\mu+2)} \\ \cdot\{\psi(\mu-n+1) + \psi(n+\alpha+\beta+\mu+2) - \psi(\mu+1) - \psi(\beta+\mu+1)\},\end{aligned} \tag{1.7}$$

with $\alpha, \beta, \beta+\mu > -1$, $n = 0, 1, 2, \ldots$ and $\mu \neq 0, 1, 2, \ldots, n-1$ if $n \geqslant 1$.

Taking into account the recurrence relations $\Gamma(x+1) = x\Gamma(x)$ and $\psi(x+1) = \psi(x) + 1/x$, we may derive a useful algorithm for the computation of the modified moments $\nu_n^{(\alpha,\beta)}(\mu)$. Indeed, it is easily seen that, if we put

$$a_0^{(\alpha,\beta)}(\mu) = \frac{\Gamma(\alpha+1)\Gamma(\beta+\mu+1)}{\Gamma(\alpha+\beta+\mu+2)},$$

$$b_0^{(\alpha,\beta)}(\mu) = \psi(\alpha+\beta+\mu+2) - \psi(\beta+\mu+1),$$

and we construct the two sequences $\{a_n^{(\alpha,\beta)}(\mu)\}$ and $\{b_n^{(\alpha,\beta)}(\mu)\}$, defined by the recurrence relationships

$$a_n^{(\alpha,\beta)}(\mu) = a_{n-1}^{(\alpha,\beta)}(\mu) \frac{(\alpha+n)(\mu-n+1)}{n(\alpha+\beta+\mu+n+1)},$$

$$b_n^{(\alpha,\beta)}(\mu) = b_{n-1}^{(\alpha,\beta)}(\mu) + \frac{1}{\alpha+\beta+\mu+1+n} - \frac{1}{\mu+1-n},$$

we have

$$\nu_n^{(\alpha,\beta)}(\mu) = a_n^{(\alpha,\beta)}(\mu) b_n^{(\alpha,\beta)}(\mu).$$

Therefore, this last expression also shows that (1.7) can be written in the following rational form with respect to n

$$\nu_n^{(\alpha,\beta)}(\mu) = \frac{\Gamma(\alpha+1)\Gamma(\beta+\mu+1)}{\Gamma(\alpha+\beta+\mu+2)} \Bigg\{ \psi(\alpha+\beta+\mu+2) - \psi(\beta+\mu+1)$$

$$+ \sum_{k=1}^{n} \left(\frac{1}{\alpha+\beta+\mu+1+k} - \frac{1}{\mu+1-k} \right) \Bigg\} \tag{1.8}$$

$$\cdot \prod_{k=1}^{n} \frac{(\alpha+k)(\mu+1-k)}{k(\alpha+\beta+\mu+1+k)},$$

where α, β, and μ satisfy the above-mentioned conditions.

To examine the remaining case $n \geqslant 1$ and $\mu = m = 0, 1, \ldots, n-1$, we recall that for any integer $r \geqslant 0$,

$$\lim_{\epsilon \to 0} \frac{\psi(-r+\epsilon)}{\Gamma(-r+\epsilon)} = (-1)^{r-1} r!.$$

Then, from (1.7), we obtain

$$\nu_n^{(\alpha,\beta)}(m) = \lim_{\mu \to m} \nu_n^{(\alpha,\beta)}(\mu)$$

$$= \frac{\Gamma(n+\alpha+1)\Gamma(m+1)\Gamma(\beta+m+1)}{n!\,\Gamma(n+\alpha+\beta+m+2)} \lim_{\epsilon \to 0} \frac{\psi(m+\epsilon-n+1)}{\Gamma(m+\epsilon-n+1)},$$

and finally

$$\nu_n^{(\alpha,\beta)}(m) = (-1)^{n-m} \frac{m!\,(n-m-1)!}{n!} \frac{\Gamma(n+\alpha+1)\Gamma(\beta+m+1)}{\Gamma(n+\alpha+\beta+m+2)},$$
$$\alpha, \beta > -1,\ m = 0, 1, 2, \ldots, n-1,\ n \geqslant 1. \tag{1.9}$$

This completes the evaluation of the integrals (1.2). Integrals of the form

$$\int_0^1 x^{\beta+\mu}(1-x)^{\alpha}(\ln(1/x))^{p} P_n^{*(\alpha,\beta)}(x)\,dx,$$

may be similarly evaluated by repeatedly differentiating (1.7) with respect to μ.

2. **Some Examples.** The results derived in the previous section show that if one has to evaluate modified moments of a given weight function of type (1.1) for given values of ρ and α, then one may choose as polynomial basis the Jacobi polynomials

$P_n^{(\alpha,\beta)}(2x-1)$, with β being a free parameter. For instance, in the case of the weight function

$$w(x) = x^\rho \ln(1/x), \qquad \rho > -1,$$

we can construct the modified moments associated with the basis $\{P_n^{(0,\beta)}(2x-1)\}$ instead of the particular one, $\{P_n^{(0,0)}(2x-1)\}$ considered by Gautschi [3].

It may be of some interest to note that the choice $\rho = \beta$ yields very simple expressions for the corresponding modified moments,

$$(2.1)\quad \nu_n^{(0,\beta)}(0) = \int_0^1 x^\beta \ln(1/x) P_n^{(0,\beta)}(2x-1)\,dx = \begin{cases} 1/(\beta+1)^2, & n = 0, \\ \dfrac{(-1)^n \Gamma(\beta+1)(n-1)!}{\Gamma(n+\beta+2)}, & n \geqslant 1. \end{cases}$$

Also, in the case of the more general weight functions (1.1), the formulas we obtain are particularly simple when we let $\rho = \beta$,

$$(2.2)\quad \begin{aligned} &\int_0^1 x^\beta (1-x)^\alpha \ln(1/x) P_n^{(\alpha,\beta)}(2x-1)\,dx \\ &= \begin{cases} \dfrac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)} \{\psi(\alpha+\beta+2) - \psi(\beta+1)\}, & n = 0, \\ (-1)^n \dfrac{\Gamma(n+\alpha+1)\Gamma(\beta+1)}{n\Gamma(n+\alpha+\beta+2)}, & n \geqslant 1. \end{cases} \end{aligned}$$

An example of (1.1), with $\alpha \neq 0$, could be the weight function

$$w(x) = x^\rho (1-x)^{-1/2} \ln(1/x), \qquad \rho > -1,$$

for which, recalling that [4, p. 60]

$$P_n^{*(-1/2,-1/2)}(x) = T_n^*(x) \prod_{k=1}^n \frac{2k-1}{2k},$$

where $T_n^*(x) = T_n(2x-1)$ is the shifted Chebyshev polynomial of degree n. Setting

$$\tau_n(\rho) = \int_0^1 x^\rho (1-x)^{-1/2} \ln(1/x) T_n^*(x)\,dx, \qquad \rho > -1,$$

and applying (1.9) and (1.8), we have

$$(2.3)\quad \tau_n(\rho) = \begin{cases} (-1)^{n-m} \dfrac{m!\,(n-m-1)!}{(n+m)!} \pi \displaystyle\prod_{k=1}^m \frac{2k-1}{2}, \\ \qquad\qquad \rho + 1/2 = m < n, \ m \geqslant 0 \text{ an integer}, \\ \dfrac{\sqrt{\pi}\,\Gamma(\rho+1)}{\Gamma(\rho+3/2)} \left\{ \psi(\rho+3/2) - \psi(\rho+1) \right. \\ \qquad \left. + \displaystyle\sum_{k=1}^n \left(\frac{1}{\rho+1/2+k} - \frac{1}{\rho+3/2-k} \right) \right\} \\ \qquad \cdot \displaystyle\prod_{k=1}^n \frac{\rho+3/2-k}{\rho+1/2+k}, \qquad \text{otherwise}. \end{cases}$$

3. Two Laguerre Polynomial Integrals. In this section we consider the problem of evaluating the modified moments of the weight functions

$$w_p(x) = e^{-x}x^\rho(\ln x)^p, \qquad \rho > -1, p = 1, 2, \ldots, \tag{3.1}$$

on $[0, \infty)$, with respect to the generalized Laguerre polynomials $L_n^{(\alpha)}(x)$.

We first examine the case $p = 1$, which is relative to a weight function of mixed sign. By introducing, for notational convenience, a new parameter μ such that $\rho = \alpha + \mu$, we refer to the integrals

$$N_{1,n}^{(\alpha)}(\mu) = \int_0^\infty e^{-x}x^{\alpha+\mu}\ln x\, L_n^{(\alpha)}(x)\,dx, \quad \alpha, \alpha + \mu > -1, n = 0, 1, 2, \ldots, \tag{3.2}$$

for which, if we set

$$I_n^{(\alpha)}(\mu) = \int_0^\infty e^{-x}x^{\alpha+\mu}L_n^{(\alpha)}(x)\,dx, \tag{3.3}$$

we have

$$N_{1,n}^{(\alpha)}(\mu) = \frac{d}{d\mu} I_n^{(\alpha)}(\mu). \tag{3.4}$$

The evaluation of $I_n^{(\alpha)}(\mu)$, hence of $N_{1,n}^{(\alpha)}(\mu)$, can be carried out in a way much similar to that concerning the Jacobi polynomials described in Section 1. To this end, we need to recall the Rodrigues formula for Laguerre polynomials

$$e^{-x}x^\alpha L_n^{(\alpha)}(x) = \frac{1}{n!}\frac{d^n}{dx^n}(e^{-x}x^{n+\alpha}).$$

Indeed, inserting this last formula in (3.3) and integrating by parts n times, we obtain

$$I_n^{(\alpha)}(\mu) = \frac{(-1)^n}{n!}\frac{\Gamma(\mu + 1)\Gamma(\mu + \alpha + 1)}{\Gamma(\mu - n + 1)}.$$

At this point it is not difficult to derive from (3.4) the following two expressions

$$\begin{aligned} N_{1,n}^{(\alpha)}(\mu) &= \frac{(-1)^n}{n!}\frac{\Gamma(\mu + 1)\Gamma(\mu + \alpha + 1)}{\Gamma(\mu - n + 1)} \\ &\quad \cdot \{\psi(\mu + 1) + \psi(\mu + \alpha + 1) - \psi(\mu - n + 1)\}, \end{aligned} \tag{3.5}$$

with $\alpha, \alpha + \mu > -1, n = 0, 1, 2, \ldots$ and $\mu \neq 0, 1, \ldots, n - 1$ if $n \geqslant 1$, and

$$N_{1,n}^{(\alpha)}(m) = (-1)^{m-1}\frac{m!\,(n - m - 1)!}{n!}\Gamma(m + \alpha + 1), \qquad m = 0, 1, \ldots, n - 1, n \geqslant 1; \tag{3.6}$$

the second being obtained from the first by taking its limit as $\mu \longrightarrow m$.

Finally, we remark that (3.5) may be put in the following form

$$N_{1,n}^{(\alpha)}(\mu) = \Gamma(\mu+\alpha+1)\left\{\psi(\mu+\alpha+1) - \sum_{k=1}^{n}\frac{1}{k-\mu-1}\right\}\prod_{k=1}^{n}\frac{k-\mu-1}{k}. \tag{3.7}$$

The evaluation of the modified moments

$$N_{p,n}^{(\alpha)}(\mu) = \int_0^\infty e^{-x}x^{\alpha+\mu}(\ln x)^p L_n^{(\alpha)}(x)\,dx, \qquad p \geqslant 2,$$

associated to the weight functions (3.1), can be obtained by repeatedly differentiating (3.5) with respect to μ. We shall only examine, with some details, the case $p = 2$.

Differentiating (3.5) once, with respect to μ, gives

$$\begin{aligned} N_{2,n}^{(\alpha)}(\mu) = {}& \frac{(-1)^n}{n!}\,\frac{\Gamma(\mu+1)\Gamma(\mu+\alpha+1)}{\Gamma(\mu-n+1)} \\ & \cdot\{(\psi(\mu+1)+\psi(\mu+\alpha+1)-\psi(\mu-n+1))^2 \\ & + \psi'(\mu+1)+\psi'(\mu+\alpha+1)-\psi'(\mu-n+1)\}, \end{aligned} \tag{3.8}$$

which holds for all $n \geqslant 0$ with $\mu \neq 0, 1, 2, \ldots, n-1$, when $n \geqslant 1$.

A more convenient form of (3.8), obtained by using the previously recalled properties of the functions $\Gamma(x)$ and $\psi(x)$, together with the recurrence relation $\psi'(x+1) = \psi'(x) - 1/x^2$, is

$$\begin{aligned} N_{2,n}^{(\alpha)}(\mu) = \Gamma(\mu+\alpha+1) & \left\{\left(\psi(\mu+\alpha+1) - \sum_{k=1}^{n}\frac{1}{k-\mu-1}\right)^2\right. \\ & \left. + \psi'(\mu+\alpha+1) - \sum_{k=1}^{n}\frac{1}{(k-\mu-1)^2}\right\}\prod_{k=1}^{n}\frac{k-\mu-1}{k}. \end{aligned} \tag{3.9}$$

If $\mu = m = 0, 1, 2, \ldots, n-1, n \geqslant 1$, from (3.8) we have

$$N_{2,n}^{(\alpha)}(m) = \lim_{\mu\to m} N_{2,n}^{(\alpha)}(\mu) = \frac{(-1)^n}{n!}\Gamma(m+1)\Gamma(m+\alpha+1)\{A_n(m) - 2B_n(m)\}, \tag{3.10}$$

where

$$A_n(m) = \lim_{\epsilon\to 0}\frac{\psi^2(m+\epsilon-n+1) - \psi'(m+\epsilon-n+1)}{\Gamma(m+\epsilon-n+1)},$$

$$B_n(m) = \lim_{\epsilon\to 0}\{\psi(m+\epsilon+1)+\psi(m+\epsilon+\alpha+1)\}\frac{\psi(m+\epsilon-n+1)}{\Gamma(m+\epsilon-n+1)}.$$

By means of the two series expansions

$$\begin{aligned} \Gamma(x) &= \frac{(-1)^r}{r!}\,\frac{1}{x+r} + \sum_{k=0}^{\infty} a_k(x+r)^k, \\ & \qquad\qquad\qquad\qquad\qquad\qquad r = 0, 1, 2, \ldots, \\ \psi(x) &= \frac{-1}{x+r} + \psi(1+r) + \sum_{k=0}^{\infty} b_k(x+r)^k, \end{aligned}$$

which are valid for $|x + r| < 1$, it is easily seen that

$$A_n(m) = (-1)^{n-m} 2(n - m - 1)!\, \psi(n - m),$$
$$B_n(m) = (-1)^{n-m}(n - m - 1)!\{\psi(m + 1) + \psi(m + \alpha + 1)\}.$$

Hence, substituting these last two expressions into (3.10), we obtain the final result

$$\begin{aligned} N_{2,n}^{(\alpha)}(m) = (-1)^m 2\, &\frac{m!\,(n - m - 1)!}{n!}\, \Gamma(m + \alpha + 1) \\ &\cdot \{\psi(n - m) - \psi(m + 1) - \psi(m + \alpha + 1)\}, \end{aligned} \tag{3.11}$$
$$m = 0, 1, \ldots, n - 1, n \geqslant 1.$$

4. Some Particular Cases. The results derived in Section 3 assume a very simple form when $\mu = 0$, that is in the cases where the weight functions

$$e^{-x}x^\alpha \ln x \quad \text{and} \quad e^{-x}x^\alpha (\ln x)^2, \qquad \alpha > -1,$$

and the polynomials $L_n^{(\alpha)}(x)$ have the same parameter α.

For the first weight function, applying (3.7) and (3.6), we find

$$\int_0^\infty e^{-x}x^\alpha \ln x\, L_n^{(\alpha)}(x)\, dx = \begin{cases} \Gamma(\alpha + 1)\psi(\alpha + 1), & n = 0, \\ -\Gamma(\alpha + 1)/n, & n \geqslant 1, \end{cases} \tag{4.1}$$

which may be regarded as a generalization of the well-known integral representation

$$\gamma = -\int_0^\infty e^{-x} \ln x\, dx,$$

of the Euler-Mascheroni constant $\gamma = -\psi(1) = .57721\ 56649\ldots$.

For the second weight function, by using (3.9) and (3.11), we obtain

$$\int_0^\infty e^{-x}x^\alpha (\ln x)^2 L_n^{(\alpha)}(x)\, dx = \begin{cases} \Gamma(\alpha + 1)\{\psi^2(\alpha + 1) + \psi'(\alpha + 1)\}, & n = 0, \\ \dfrac{2}{n}\,\Gamma(\alpha + 1)\left\{\displaystyle\sum_{k=1}^{n-1} \frac{1}{k} - \psi(\alpha + 1)\right\}, & n \geqslant 1. \end{cases} \tag{4.2}$$

Two other cases of interest, which may be used, for instance, in constructing the modified moments of the weight functions

$$\exp(-x^2)\ln|x| \quad \text{and} \quad \exp(-x^2)(\ln|x|)^2,$$

on $(-\infty, \infty)$, with respect to the Hermite polynomials $H_n(x)$, are obtained by means of Szegö's relationships [4, p. 106] between Hermite and Laguerre polynomials,

$$H_{2n}(x) = (-1)^n 2^{2n} n!\, L_n^{(-1/2)}(x^2), \qquad H_{2n+1}(x) = (-1)^n 2^{2n+1} n!\, x L_n^{(1/2)}(x^2). \tag{4.3}$$

Indeed, setting $x = t^2$ in the integral (3.2), with $\alpha = -1/2$ and $\mu = 0$, from (3.7) and (3.6) we obtain

$$4\int_0^\infty \exp(-t^2) \ln t\, L_n^{(-1/2)}(t^2)\, dt = \begin{cases} \sqrt{\pi}\psi(1/2) = -\sqrt{\pi}(\gamma + 2\ln 2), & n = 0, \\ -\sqrt{\pi}/n, \quad n \geqslant 1, \end{cases}$$

hence, by applying the first relation in (4.3),

$$\int_0^\infty \exp(-x^2)\ln x\, H_{2n}(x)\,dx = \begin{cases} \dfrac{-\sqrt{\pi}}{4}(\gamma + 2\ln 2), & n = 0, \\ (-1)^{n-1}(n-1)!\, 2^{2(n-1)}\sqrt{\pi}, & n \geqslant 1. \end{cases} \tag{4.4}$$

Similarly, if we assume $\alpha = ½$ and $\mu = -½$ in (3.2), then (3.7), together with the second relation in (4.3), gives

$$\begin{aligned} &\int_0^\infty \exp(-x^2)\ln x\, H_{2n+1}(x)\,dx \\ &\qquad = (-1)^{n-1}2^{n-1}\left(\gamma + 2\sum_{k=1}^{n}\frac{1}{2k-1}\right)\prod_{k=1}^{n}(2k-1), \\ &\qquad\qquad n = 0, 1, 2, \ldots . \end{aligned} \tag{4.5}$$

The integrals involving the weight function $\exp(-x^2)(\ln x)^2$ can be dealt with in the same way. Recalling that $\psi'(½) = \pi^2/2$ and $\psi'(1) = \pi^2/6$, by use of (3.9) and (3.11), this leads to the following results,

$$\begin{aligned} &\int_0^\infty \exp(-x^2)(\ln x)^2 H_{2n}(x) \\ &= \begin{cases} \dfrac{\sqrt{\pi}}{8}\{\psi^2(½) + \psi'(½)\} = 1.94752\,21803\ldots, & n = 0, \\ (-1)^n 2^{2(n-1)}(n-1)!\sqrt{\pi}\left(\gamma + 2\ln 2 + \displaystyle\sum_{k=1}^{n-1}\frac{1}{k}\right), & n \geqslant 1, \end{cases} \end{aligned} \tag{4.6}$$

and

$$\begin{aligned} &\int_0^\infty \exp(-x^2)(\ln x)^2 H_{2n+1}(x)\,dx \\ &= (-1)^n 2^{n-2}\left\{\left(\gamma + \sum_{k=1}^{n}\frac{2}{2k-1}\right)^2 + \frac{\pi^2}{6} - \sum_{k=1}^{n}\frac{4}{(2k-1)^2}\right\}\prod_{k=1}^{n}(2k-1), \\ &\qquad\qquad n = 0, 1, 2, \ldots . \end{aligned} \tag{4.7}$$

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