On Some Trigonometric Integrals

By Henry E. Fettis

Abstract. Expressions are obtained for the integrals

$$I_{\lambda}^{(p)} = \int_0^{\pi/2} \left(\frac{\sin \lambda \theta}{\sin \theta} \right)^p d\theta, \quad J_{\lambda}^{(p)} = \int_0^{\pi/2} \left(\frac{1 - \cos \lambda \theta}{\sin \theta} \right)^p d\theta$$

for arbitrary real values of " λ ", and $p = 1, 2$.

1. The integrals

$$(1) \quad I_{\lambda}^{(p)} = \int_0^{\pi/2} \left(\frac{\sin \lambda \theta}{\sin \theta} \right)^p d\theta$$

and

$$(2) \quad J_{\lambda}^{(p)} = \int_0^{\pi/2} \left(\frac{1 - \cos \lambda \theta}{\sin \theta} \right)^p d\theta, \quad p = 1, 2,$$

are of a sufficiently general, standard type that one would expect to find them in almost any table of integrals or other reference work (e.g., [1], [2], [3]). However, a comprehensive search by the author has disclosed that (with the exception of (1) for integer " λ "), these integrals are conspicuously absent from the literature. Should any one of the above integrals arise in a practical problem (as well they might) one would, in the absence of a closed-form expression, be inclined to evaluate it either numerically or in series. However, as will be shown in the following sections, this is not necessary, as such a closed form does exist, and can in fact be found in terms of the logarithmic derivative of the gamma function, $\Psi(z)$.

2. It is evident that $I_{\lambda}^{(1)}$ and $J_{\lambda}^{(1)}$ satisfy the recurrence relations

$$(3) \quad I_{\lambda+1}^{(1)} - I_{\lambda-1}^{(1)} = \frac{2}{\lambda} \sin \frac{\pi}{2} \lambda,$$

$$(4) \quad J_{\lambda+1}^{(1)} - J_{\lambda-1}^{(1)} = \frac{2}{\lambda} \left(1 - \cos \frac{\pi}{2} \lambda \right),$$

with $I_0^{(1)} = J_0^{(1)} = 0$, $I_1^{(1)} = \pi/2$, $J_1^{(1)} = \ln(2)$, from which we easily obtain, by induction, that for integer values of $\lambda = n > 0$,

$$(5) \quad I_{2n}^{(1)} = 2 \sum_{k=0}^{n-1} \frac{(-1)^k}{(2k+1)}, \quad I_{2n+1}^{(1)} = \frac{\pi}{2},$$

Received February 5, 1980.

1980 *Mathematics Subject Classification.* Primary 33A15, 33A10.

Key words and phrases. Integrals, definite integrals, trigonometric integrals, Gamma function, Psi function.

© 1980 American Mathematical Society
0025-5718/80/0000-0174/\$02.25

$$(6) \quad J_{2n}^{(1)} = 2 \sum_{k=0}^{n-1} \frac{1}{(2k+1)}, \quad J_{4n-1}^{(1)} = J_{4n+1}^{(1)} = \ln(2) + 2 \sum_{k=0}^{n-1} \frac{1}{(2k+1)}.$$

The results (5) are well known (see [1, Eqs. 3.612–3]), while those of (6) have been previously found by the author (*Math. Mag.*, v. 39, no. 5, 1966, p. 281) but do not appear in any of the standard references, such as [1], [2], [3].

3. For $p = 1$ and noninteger values of “ λ ”, the integrals (1) and (2) can be readily evaluated by complex integration of the function

$$(7) \quad f(z) = \frac{1 - z^\lambda}{z^2 - 1}$$

in the $z = \rho e^{i\theta}$ plane around the contour consisting of

I—The quadrant of the unit circle $z = e^{i\theta}$ ($0 \leq \theta \leq \pi/2$).

II—The imaginary axis extending from $z = i$ to $z = 0$, and the real axis from $z = 0$ to $z = 1$.

The integral around this contour must be zero, since the integrand has no singularities within it, and is single valued throughout.

On the arc constituting part I of the contour, $z = e^{i\theta}$, and integration gives

$$(8) \quad i \int_0^{\pi/2} \frac{1 - e^{i\lambda\theta}}{e^{i\theta} - e^{-i\theta}} d\theta = \frac{1}{2} \int_0^{\pi/2} \frac{1 - e^{i\lambda\theta}}{\sin \theta} d\theta = \frac{1}{2} (J_\lambda^{(1)} - iI_\lambda^{(1)}).$$

On the portion from $z = i$ to $z = 0$, $z = ye^{i\pi/2}$ ($0 \leq y \leq 1$), and hence integration along this segment gives

$$(9) \quad + e^{i\pi/2} \int_0^1 \frac{1 - e^{i\pi\lambda/2} y^\lambda}{1 + y^2} dy = \frac{1}{2} e^{i\pi/2} \int_0^1 \frac{t^{-1/2} (1 - e^{i\pi\lambda/2} t^{\lambda/2})}{1 + t} dt,$$

while on the segment from $x = 0$ to $x = 1$, the contribution is

$$(10) \quad \int_0^1 \frac{1 - x^\lambda}{x^2 - 1} dx = -\frac{1}{2} \int_0^1 \frac{t^{-1/2} (1 - t^{\lambda/2})}{1 - t} dt.$$

Both of the integrals (9) and (10) can be expressed in terms of the function $\Psi(z) = d[\ln \Gamma(z)]/dz$, using formulae 5 and 6 from [2, Chapter I, p. 9]. Finally, when the total integral is set equal to zero, and real and imaginary parts separated, we obtain

$$(11) \quad \begin{aligned} J_\lambda^{(1)} &= \Psi\left(\frac{\lambda+1}{2}\right) - \Psi\left(\frac{1}{2}\right) - \frac{1}{2} \sin\left(\frac{\pi}{2}\lambda\right) \left[\Psi\left(\frac{\lambda+3}{4}\right) - \Psi\left(\frac{\lambda+1}{4}\right) \right], \\ I_\lambda^{(1)} &= \frac{\pi}{2} - \frac{1}{2} \cos\left(\frac{\pi}{2}\lambda\right) \left[\Psi\left(\frac{\lambda+3}{4}\right) - \Psi\left(\frac{\lambda+1}{4}\right) \right]. \end{aligned}$$

4. The results (11) can be found alternatively in the following way. We have

$$\begin{aligned} I_{\lambda+1}^{(1)} + I_{\lambda-1}^{(1)} &= 2 \int_0^{\pi/2} \sin \lambda \theta \cot \theta \, d\theta = \int_0^{\pi} \sin \frac{\lambda}{2} \theta \cot \left(\frac{1}{2} \theta \right) \, d\theta = G_1(\lambda/2), \\ (12) \quad J_{\lambda+1}^{(1)} + J_{\lambda-1}^{(1)} &= 2 \ln(2) + \int_0^{\pi/2} \left(1 - \cos \frac{\lambda}{2} \theta \right) \cot \frac{1}{2} \theta \, d\theta \\ &= 2 \ln(2) + G_2(\lambda/2), \end{aligned}$$

where the integrals

$$(13) \quad G_1(\alpha) = \int_0^{\pi} \sin \alpha \theta \cot \frac{1}{2} \theta \, d\theta \quad \text{and} \quad G_2(\alpha) = \int_0^{\pi} (1 - \cos \alpha \theta) \cot \frac{1}{2} \theta \, d\theta$$

can be evaluated by a limiting process from the known result [2, Chapter I, p. 8]:

$$(14) \quad P(x, y) + iQ(x, y) = \int_0^{\pi} \sin^x t e^{iyt} dt = \frac{\pi}{2^x} \left[\frac{\Gamma(1+x) e^{i\pi y/2}}{\Gamma\left(1 + \frac{x+y}{2}\right) \Gamma\left(1 + \frac{x-y}{2}\right)} \right].$$

We illustrate the procedure for G_1 , which, since $\cot \frac{1}{2} \theta = (1 + \cos \theta)/\sin \theta$, can be written

$$\begin{aligned} (15) \quad G_1(\alpha) &= \frac{1}{2} \int_0^{\pi} \frac{2 \sin \alpha \theta + \sin(\alpha+1)\theta + \sin(\alpha-1)\theta}{\sin \theta} \, d\theta \\ &= \frac{1}{2} \lim_{x \rightarrow -1} [2Q(x, \alpha) + Q(x, \alpha+1) + Q(x, \alpha-1)]. \end{aligned}$$

After substituting from (14), combining, and simplifying with the aid of known Γ -function identities, this becomes

$$\begin{aligned} (16) \quad G_1(\alpha) &= \lim_{x \rightarrow -1} \pi(1+x)^{-1} \left[\frac{2 \sin \frac{\pi}{2} \alpha}{\Gamma\left(1 + \frac{x+\alpha}{2}\right) \Gamma\left(1 + \frac{x-\alpha}{2}\right)} \right. \\ &\quad \left. - \frac{\alpha \cos \frac{\pi}{2} \alpha}{\Gamma\left(\frac{3}{2} + \frac{x+\alpha}{2}\right) \Gamma\left(\frac{3}{2} + \frac{x-\alpha}{2}\right)} \right]. \end{aligned}$$

The limit is easily evaluated with the aid of l'Hospital's rule, and, upon making use of various familiar identities relating to the Γ - and Ψ -functions, we get ultimately

$$(17) \quad G_1(\alpha) = \pi - \frac{\sin \pi \alpha}{\alpha} \left\{ 1 + \alpha \left[\Psi\left(\frac{1+\alpha}{2}\right) - \Psi\left(1 + \frac{\alpha}{2}\right) \right] \right\}.$$

In an exactly analogous way we find*

$$(18) \quad G_2(\alpha) = -2\Psi\left(\frac{1}{2}\right) - \frac{1 - \cos \pi \alpha}{\alpha} + 2 \left[\cos^2 \frac{\pi}{2} \alpha \Psi\left(\frac{1+\alpha}{2}\right) + \sin^2 \frac{\pi}{2} \alpha \Psi\left(1 + \frac{\alpha}{2}\right) \right],$$

and, when (17) and (18) are substituted into (12), the results combined with (3) and (4), and certain obvious simplifications made, expressions identical to (11) are obtained.

* The author was unable to find either (17) or (18) in the literature.

As an exercise, the reader may verify that, when λ is an integer, (11) reduces to the previously obtained relations (5) and (6). In this regard, the numerous identities dealing with the Ψ -function of rational argument found in [3, Chapter XXIV] will prove helpful.

5. The Integrals $I_\lambda^{(2)} = \int_0^{\pi/2} (\sin \lambda \theta / \sin \theta)^2 d\theta$, $J_\lambda^{(2)} = \int_0^{\pi/2} ((1 - \cos \lambda \theta) / \sin \theta)^2 d\theta$. The first of these integrals, according to [1, formula 3.624(6)] is equal to

$$(19) \quad \frac{\lambda \pi}{2}$$

Although it is not so stated in the above reference, this result only holds if λ is an integer. For noninteger values of λ , the integral can be evaluated in the following way:

We have

$$(20) \quad I_\lambda^{(2)} \equiv \int_0^{\pi/2} \left(\frac{\sin \lambda \theta}{\sin \theta} \right)^2 d\theta = \frac{1}{2} \int_0^\pi \frac{1 - \cos \lambda \phi}{1 - \cos \phi} d\phi.$$

Consider the more general integral

$$(21) \quad \int_0^\pi \frac{1 - \cos \lambda \phi}{\cosh \alpha - \cos \phi} d\phi = \frac{\pi}{\sinh \alpha} - \int_0^\pi \frac{\cos \lambda \phi}{\cosh \alpha - \cos \phi} d\phi,$$

and let

$$(22) \quad y(\lambda, \alpha) = \sinh \alpha \int_0^\pi \frac{\cos \lambda \phi}{\cosh \alpha - \cos \phi} d\phi,$$

so that

$$(23) \quad \int_0^\pi \frac{1 - \cos \lambda \phi}{\cosh \alpha - \cos \phi} d\phi = \frac{\pi - y(\lambda, \alpha)}{\sinh \alpha}.$$

Now, differentiation of (22) with respect to “ α ”, followed by integration by parts, gives

$$(24) \quad \frac{\partial y}{\partial \alpha} = -\lambda \int_0^\pi \frac{\sin \lambda \phi \sin \phi}{\cosh \alpha - \cos \phi} d\phi.$$

Hence, since

$$(25) \quad \int_0^\pi \frac{1 - \cos \lambda \phi}{1 - \cos \phi} d\phi = \lim_{\alpha \rightarrow 0} \left(\frac{1 - y(\lambda, \alpha)}{\sinh \alpha} \right),$$

application of l'Hospital's rule gives

$$(26) \quad \int_0^\pi \frac{1 - \cos \lambda \phi}{1 - \cos \phi} d\phi = \lambda \int_0^\pi \frac{\sin \lambda \phi \sin \phi}{1 - \cos \phi} d\phi = \lambda G_1(\lambda).$$

Substituting the expression for $G_1(\lambda)$ from (17):

$$(27) \quad \int_0^{\pi/2} \left(\frac{\sin \lambda \theta}{\sin \theta} \right)^2 d\theta = \frac{\lambda \pi}{2} - \frac{\sin \lambda \pi}{2} \left\{ 1 + \lambda \left[\Psi \left(\frac{1 + \lambda}{2} \right) - \Psi \left(1 + \frac{\lambda}{2} \right) \right] \right\},$$

and since

$$(28) \quad \int_0^{\pi/2} \left(\frac{1 - \cos \lambda \theta}{\sin \theta} \right)^2 d\theta = 4 \int_0^{\pi/2} \left(\frac{\sin \frac{\lambda}{2} \theta}{\sin \theta} \right)^2 d\theta - \int_0^{\pi/2} \left(\frac{\sin \lambda \theta}{\sin \theta} \right)^2 d\theta,$$

this integral can be expressed in a similar fashion.

6. As a final observation, it may be mentioned that integrals of the type

$$(29) \quad \int_0^{\pi/2N} \frac{\sin \lambda \theta}{\sin \theta} d\theta$$

and

$$(30) \quad \int_0^{\pi/2N} \frac{1 - \cos \lambda \theta}{\sin \theta} d\theta,$$

where N is an integer, can be expressed as finite combinations of the integrals $I_{\lambda}^{(1)}$ and $J_{\lambda}^{(1)}$ with the aid of the relations

$$(31) \quad \frac{\sin 2M\theta}{\sin \theta} = 2 \sum_{n=1}^M \cos(2n-1)\theta, \quad \frac{\sin(2M+1)\theta}{\sin \theta} = 1 + 2 \sum_{n=1}^M \cos 2n\theta,$$

after making the change of variable θ to ϕ/N , giving

$$(32) \quad \begin{aligned} \int_0^{\pi/2N} \frac{\sin \lambda \theta}{\sin \theta} d\theta &= \frac{2}{N} \int_0^{\pi/2} \frac{\sin(\lambda \phi/N)}{\sin \phi} \left[\sum_{n=1}^{N/2} \cos \frac{(2n-1)\phi}{N} \right] d\phi, \quad N \text{ even}, \\ &= \frac{1}{N} \int_0^{\pi/2} \frac{\sin(\lambda \phi/N)}{\sin \phi} \left[1 + 2 \sum_{n=1}^{(N-1)/2} \cos \frac{2n\phi}{N} \right] d\phi, \quad N \text{ odd}, \end{aligned}$$

with similar expressions resulting for

$$(34) \quad \int_0^{\pi/2N} \frac{1 - \cos \lambda \theta}{\sin \theta} d\theta.$$

1885 California, Apt. 62
Mountain View, California 94041

1. I. S. GRADSHTEYN & I. M. RYZHIK, *Table of Integrals, Series and Products*, Academic Press, New York, 1965.
2. W. MAGNUS & F. OBERHETTINGER, *Formeln und Sätze für die Speziellen Funktionen der Mathematischen Physik*, Springer-Verlag, Berlin, 1948.
3. J. EDWARDS, *A Treatise on the Integral Calculus*, Vol. II, Macmillan, New York, 1922; reprinted by Chelsea, New York, 1977.