

Reciprocal Polynomials Having Small Measure

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Abstract. The measure of a monic polynomial is the product of the absolute value of the roots which lie outside and on the unit circle. We describe an algorithm, based on the root-squaring method of Graeffe, for finding all polynomials with integer coefficients whose measures and degrees are smaller than some previously given bounds. Using the algorithm, we find all such polynomials of degree at most 16 whose measures are at most 1.3. We also find all polynomials of height 1 and degree at most 26 whose measures satisfy this bound. Our results lend some support to Lehmer's conjecture. In particular, we find no noncyclotomic polynomial whose measure is less than the degree 10 example given by Lehmer in 1933.

1. Introduction. The *measure* $M(P)$ of a polynomial $P(x) = a_0x^n + \cdots + a_n$ with $a_0 \neq 0$ with zeros $\alpha_1, \dots, \alpha_n$ is defined by [8, p. 5]

$$M(P) = |a_0| \prod_{i=1}^n \max(|\alpha_i|, 1) = \exp \left\{ \int_0^1 \log |P(e^{2\pi it})| dt \right\}.$$

If P has integer coefficients, as we assume unless otherwise stated, Kronecker's theorem tells us that if $M(P) = 1$ and $a_n \neq 0$, then P is cyclotomic. Lehmer [7] raised the question of whether there might be a constant $\epsilon_0 > 0$ independent of n so that, if P is not cyclotomic, then $M(P) \geq 1 + \epsilon_0$.

Smyth [12] has proved that if P is nonreciprocal, then $M(P) \geq \theta_0$, where $\theta_0 = 1.3247 \dots$ is the smallest Pisot-Vijayaraghavan number. For reciprocal P , the question is still open, although Dobrowolski [5] has recently proved that

$$M(P) \geq 1 + \frac{1}{1200} \left(\frac{\log \log n}{\log n} \right)^3,$$

unless P is cyclotomic. Surveys of related results have been given by Stewart [13] and the author [3]. In [3], we used Lehmer's notation $\Omega(P)$ for $M(P)$ and (unfortunately) $M(P)$ for $\max |\alpha_i|$. The notation adopted here is due to Mahler [8].

In [7, p. 477], Lehmer states that, "we have not made an examination of all 10th degree symmetric polynomials, but a rather intensive search has failed to reveal a better polynomial than

$$x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1, \Omega = 1.176280821.$$

All efforts to find a better equation of degree 12 and 14 have been unsuccessful".

Received May 21, 1979; revised March 24, 1980.

1980 *Mathematics Subject Classification.* Primary 12A20; Secondary 12-04, 26C05, 12A15.

*Supported in part by a grant from the Natural Sciences and Engineering Research Council of Canada.

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0025-5718/80/0000-0178/\$05.25

In view of the current interest in Lehmer's conjecture and urged on by C. L. Stewart, we recently carried out an exhaustive search of polynomials of small degree ($n \leq 16$) for those of small measure (generally $M \leq 1.3$), and in addition a search of all polynomials of height 1 up to $n = 26$. (In this paper, the height $H(P)$ is defined by $H(P) = \max |a_i|$.) We also investigated a number of families of polynomials as described in Section 5. In spite of this extensive investigation, we found no better polynomial than Lehmer's 1933 example.

Because of Smyth's result, we were able to restrict consideration to reciprocal polynomials, although it should be pointed out that the structure of the set $\{M(P): P \text{ nonreciprocal}\}$ is far from being well understood. With $M(P) < 2$, we have $|a_0| = 1$ and hence we may assume that $a_0 = a_n = 1$. Using the symmetry $x \rightarrow -x$, we may also assume that $a_1 \geq 0$. If $P(x) = Q(x^s)$, then $M(P) = M(Q)$, so we can omit P of this form with $s > 1$.

For any fixed M and n , the set of P with degree n and $M(P) \leq M$ is clearly finite. For example [8, p. 7],

$$(2) \quad |a_k| \leq \binom{n}{k} M.$$

However, the number of reciprocal P satisfying (2) with, say, $n = 10$, $M = 1.3$, $a_0 = 1$, and $a_1 \geq 0$ is about 1.5×10^{11} so a direct calculation of $M(P)$ for all such P is out of the question. Using Newton's identities, a smaller set results, as described in Section 3(c), but even for relatively small n a more elaborate algorithm is needed.

Our approach is based on the root-squaring method of Graeffe, described in most numerical analysis texts and in complete detail by Bareiss [1]. The efficiency of our algorithm depends to a considerable extent on certain improvements of (2) given in Section 2. However, the basic idea can be illustrated using (2), and the following consequence of (1) and (2), [8, pp. 7–8]: Let $L(P) = \sum_{k=0}^n |a_k|$ be the *length* of P , then

$$(3) \quad M(P) \leq L(P) \leq 2^n M(P).$$

Let $P_m(x) = a(0, m)x^n + \cdots + a(n, m)$ be the monic polynomial whose roots are the 2^m th powers of the roots of P . As is well known,

$$(4) \quad \begin{aligned} a(k, m) &= (-1)^k a(k, m-1)^2 \\ &+ 2 \sum_{j=1}^{\min(k, n-k)} (-1)^{j+k} a(k-j, m-1) a(k+j, m-1), \end{aligned}$$

so P_1, P_2, \dots are easily generated recursively. Since evidently $M(P_m) = M(P)^{2^m}$, the requirement $M(P) \leq M$ and (2) imply that

$$(5) \quad |a(k, m)| \leq \binom{n}{k} M^{2^m}.$$

Thus, if one begins with an initial set R_0 of polynomials (all those satisfying (2), say), one obtains in succession sets $R_0 \supset R_1 \supset R_2 \supset \cdots$, where R_m consists of those polynomials in R_{m-1} satisfying (5). Now (3) implies that

$$(6) \quad M(P) = M(P_m)^{2^{-m}} \leq L(P_m)^{2^{-m}} \leq 2^{n/2^m} M(P),$$

so that R_m contains only polynomials with measures $\leq 2^{n/2^m}M$. Thus $R_\infty = \bigcap R_m$ is exactly the set $R_0 \cap \{M(P) \leq M\}$.

In fact one stops the process for a moderately small value of $m = m^*$ and simply computes $M(P)$ for all P in R_{m^*} from (1). Although (6) shows that $\lim L(P_m)^{2^{-m}} = M(P)$, we have found that the direct computation of $M(P)$ by finding the zeros of P and using (1) is a more effective way of calculating an accurate value of $M(P)$. The advantage of (6) is that it gives reasonably good bounds on $M(P)$ by a small amount of computation.

An additional feature of this approach is that if P is cyclotomic, then $P_m = P_{m+1}$ as soon as $2^{m-1} > n$ (see Section 3(b)). Thus the cyclotomic polynomials may be detected in a small number of steps.

Timings are given in Section 4. Initially we hoped to extend our exhaustive search to $n = 20$ but eventually settled for $n \leq 16$. The case $n = 18$ is feasible, but potentially expensive, so this has been deferred in the hope of future theoretical progress or else an improved algorithm.

The search among polynomials with $H(P) = \max |a_i| = 1$ is considerably less expensive. It is known that, for any P with $M(P) < 2$, there is a Q with integer coefficients with $H(PQ) = 1$. It has been our experience that Q can be taken to be cyclotomic and of fairly small degree relative to n . The proof of a quantitative result of this nature would greatly enhance the value of the lists in Section 6 for $18 \leq n \leq 26$.

2. The Basic Inequalities. In this section, P is any polynomial with complex coefficients. We shall derive a number of inequalities for $|a_k|$ which depend on $M(P)$.

We begin by recalling the notion of Schur convexity [10, pp. 167–168]. Suppose that $x = (x_1, \dots, x_n)$ is a vector with real components and that x_1^*, \dots, x_n^* denote x_1, \dots, x_n arranged in decreasing order. If $y = (y_1, \dots, y_n)$, we write $x < y$ provided that

$$\sum_{i=1}^k x_i^* \leq \sum_{i=1}^k y_i^*, \quad k = 1, \dots, n,$$

with equality holding if $k = n$. A function $F(x)$ is said to be *Schur convex* if $x < y$ implies $F(x) \leq F(y)$.

LEMMA 1 (SCHUR, OSTROWSKI [10, p. 168]). *If F is differentiable, then it is Schur convex if and only if $(x_i - x_j)(F_i - F_j) \geq 0$, for all i and j , where F_i denotes the partial derivative with respect to x_i .*

LEMMA 2. *Let $\sigma_k(\alpha_1, \dots, \alpha_n)$ be the k th elementary symmetric function of $\alpha_1, \dots, \alpha_n$. Then $F(x) = \sigma_k(e^{x_1}, \dots, e^{x_n})$ is a Schur convex function of x .*

Proof. $F_i - F_j = (e^{x_i} - e^{x_j})A$, where A is a nonnegative function of the remaining variables.

LEMMA 3. *Let $P(x) = a_0x^n + \dots + a_n$ satisfy $M(P) \leq M$. Then*

$$(7) \quad |a_k| \leq \binom{n-2}{k-1} M + \binom{n-2}{k-2} |a_n| + \binom{n-2}{k} |a_0| + \binom{n-2}{k-1} |a_0 a_n| M^{-1}.$$

Proof. Assume for the moment that $a_n \neq 0$. Number the roots of P so that $|\alpha_1| \geq \cdots \geq |\alpha_m| \geq 1 > |\alpha_{m+1}| \geq \cdots \geq |\alpha_n| > 0$. Define $x_i = \log |\alpha_i|$. Then

$$(8) \quad x_1 + \cdots + x_m = \log(M(P)/|a_0|) \leq \log(M/|a_0|),$$

$$(9) \quad x_{m+1} + \cdots + x_n = \log(|a_n|/M(P)) \geq \log(|a_n|/M).$$

If we define $y_1 = \log(M/|a_0|)$, $y_n = \log(|a_n|/M)$, and $y_i = 0$ otherwise, then it is easy to check that $x < y$. Hence, by Lemma 2,

$$(10) \quad \sigma_k(|\alpha_1|, \dots, |\alpha_n|) \leq \sigma_k(M/|a_0|, 1, \dots, 1, |a_n|/M),$$

from which (7) is immediate.

Using continuity, the assumption $a_n \neq 0$ can now be dropped.

Remarks. 1. The inequality (7) is uninformative if $k = 0$ or n , but in these cases $|a_0| \leq M$ and $|a_n| \leq M$. For our purposes $a_0 = a_n = 1$, so (7) improves (2) considerably, especially for large M . Note that (2) is sharp only if $M = 1$, while (7) is sharp for all M , with equality for $P(x) = (x + M)(x + 1)^{n-2}(x + M^{-1})$.

2. It is often possible to obtain information concerning $\max |\alpha_i|$ and $\min |\alpha_i|$, (see e.g. Lemma 6 or [9, Chapter 7]). Lemma 2 allows us to use this information very easily, as the next result illustrates:

LEMMA 4. Let P satisfy $M(P) \leq M$. Suppose in addition that $|\alpha_1| = \max |\alpha_i| \leq a < M/|a_0|$, and that $|\alpha_n| = \min |\alpha_i| \geq c > a_n/M$. Define integers s, t and real numbers b, d with $1 \leq b < a$, $1 \leq d < c$ by

$$a^s b = M/|a_0| \quad \text{and} \quad c^t d = |a_n|/M.$$

Then

$$(11) \quad |a_k| \leq |a_0| \sigma_k(a, \dots, a, b, 1, \dots, 1, d, c, \dots, c),$$

where a, c , and 1 are repeated respectively s, t , and $n - s - t - 2$ times.

Proof. Analogous to Lemma 3 except that y is defined by $y_1 = \cdots = y_s = \log a$, $y_{s+1} = \log b$, $y_{n-t} = \log d$, $y_{n-t+1} = \cdots = y_n = \log c$, and $y_i = 0$ otherwise. Obvious modifications are of course made if $n - s - t - 2 < 0$.

LEMMA 5. If P is a reciprocal polynomial with $a_0 = a_n = 1$, for which $M(P) \leq M$ and $\max |\alpha_i| \leq M^{1/2}$, then

$$(12) \quad |a_k| \leq (M + 4 + M^{-1}) \binom{n-4}{k-2} + 2(M^{1/2} + M^{-1/2}) \binom{n-4}{k-3} + \binom{n-4}{k-1} \\ + \binom{n-4}{k-4} + \binom{n-4}{k}.$$

Proof. A special case of Lemma 4.

LEMMA 6. Let P be a reciprocal polynomial with real coefficients and $a_0 = a_n = 1$. Suppose in addition that all real roots of P are positive. If $a_1 \geq n - 4$, then $|\alpha_1| = \max |\alpha_i| \leq M(P)^{1/2}$.

Proof. If α_1 is not real, then $\bar{\alpha}_1$ is also a root of P so $|\alpha_1|^2 \leq M$. Thus we may assume α_1 is real, hence positive, and that $\alpha_1 > M^{1/2}$. Let $\beta_1, \dots, \beta_{2s}$ be the complex roots of P . Then

$$\prod_{i=1}^{2s} \max(|\beta_i|, 1) \leq M/\alpha_1 < M^{1/2},$$

so that $|\beta_i| < M^{1/4}$ for all i . Hence, by Lemma 4,

$$\sum \beta_i \leq \sum |\beta_i| < 2(M^{1/4} + M^{-1/4}) + n - 6.$$

Since P is reciprocal, α_1^{-1} is also a root of P , so

$$\sum_{\alpha_i \text{ real}} \alpha_i \geq \alpha_1 + \alpha_1^{-1} > M^{1/2} + M^{-1/2}.$$

Combining these inequalities,

$$(13) \quad -a_1 = \sum_{i=1}^n \alpha_i > M^{1/2} + M^{-1/2} - 2M^{1/4} - 2M^{-1/4} - (n - 6).$$

The right member of (13) is an increasing function of M for $M \geq 1$, hence (13) implies $a_1 < n - 4$. Thus $a_1 \geq n - 4$ implies $|\alpha_1| \leq M^{1/2}$.

Remarks. 3. The assumption that the real roots are positive is valid for P_m if $m \geq 1$.

4. A corresponding result is true for nonreciprocal P , with $n - 4$ replaced by $n - 2$.

LEMMA 7. Let $\alpha_1, \dots, \alpha_n$ be a set of complex numbers closed under complex conjugation. Then

$$(14) \quad \frac{2}{n} \left(\sum_{k=1}^n \alpha_k \right)^2 \leq \sum_{k=1}^n \alpha_k^2 + \sum_{k=1}^n |\alpha_k|^2.$$

Proof. Let $\alpha_k = \beta_k e^{i\theta_k}$, $\beta_k \geq 0$. Then

$$\sum \alpha_k^2 + \sum |\alpha_k|^2 = \sum \beta_k^2 (\cos(2\theta_k) + 1) = 2 \sum \beta_k^2 \cos^2 \theta_k.$$

Thus (14) follows from the Cauchy-Schwarz inequality.

LEMMA 8. Let $L_2^2(P) = |a_0|^2 + \dots + |a_n|^2$. Then

$$(15) \quad M(P) \leq L_2(P) \leq \binom{2n}{n}^{1/2} M(P),$$

$$(16) \quad M(P)^2 + |a_0 a_n|^2 M(P)^{-2} \leq L_2^2(P).$$

Proof. The left inequality of (15) is due to Specht. Using the second definition in (1), it is easily seen to be just the arithmetic-geometric inequality for integrals, together with Parseval's relation [9, p. 129]. The right-hand inequality of (15) follows from (2), as observed by Duncan [6]. The inequality (16) is due to Gonçalves, a more direct proof being given by Ostrowski [11].

3. The Algorithm. We have outlined our basic approach in Section 1. However, there are a number of details to be considered before a workable algorithm is obtained. We shall deal in turn with (a) the root-squaring process, (b) cyclotomic factors, and (c) selection of the initial set R_0 for exhaustive search.

For the remainder of the paper, P will denote a monic reciprocal polynomial of even degree $n = 2h$ with integer coefficients. Given a set of such polynomials R_0 , we are seeking those which have $M(P) \leq M$, where M is a parameter to be specified.

According to Lemma 6, if $a(1, m) \geq n - 4$ for any $m \geq 1$, then $|\alpha_1| \leq M(P)^{1/2}$, in which case (12) provides improved estimates over (7). We thus find it useful to associate with P a parameter $s = s(P)$ taking on one of the values 1 or 2 to indicate that $|\alpha_1| \leq M(P)^{1/s}$. Initially $s = 1$, but if at some point in the processing of P we determine that $|\alpha_1| \leq M(P)^{1/2}$, then we set $s = 2$.

3(a) *Root-Squaring.* We regard n and M as fixed and R_0 as given. Given P in R_0 , we wish to reject P from consideration if $M(P) > M$.

If $M(P) \leq M$ and $|\alpha_1| \leq M(P)^{1/s}$, the coefficients $a(k, m)$ of P_m satisfy

$$(17) \quad |a(k, m)| \leq b(k, m, s),$$

where the quantities $b(k, m, s)$ are given by (7) and (12) as

$$(18) \quad b(k, m, 1) = \binom{n-2}{k-1} (M^{2^m} + M^{-2^m}) + \binom{n-2}{k-2} + \binom{n-2}{k},$$

$$(19) \quad b(k, m, 2) = \binom{n-4}{k-2} (M^{2^m} + 4 + M^{-2^m}) + 2 \left(\binom{n-4}{k-3} + \binom{n-2}{k-1} \right) \\ \cdot (M^{2^{m-1}} + M^{-2^{m-1}}) + \binom{n-4}{k-4} + \binom{n-4}{k}.$$

These can be computed and tabulated once n and M have been specified.

We can then successively compute $a(1, m), \dots, a(k, m)$, for $m = 1, 2, \dots, m^*$, using (4). If (17) is violated for any (k, m) , we reject P . If $a(1, m) \geq n - 4$ for any $m \geq 1$, we set $s = 2$. The advantage of this is fairly evident when one considers the behavior of $b(k, m, 2)$ versus $b(k, m, 1)$ as $m \rightarrow \infty$.

In addition to this "rejection criterion", we include, for $m \geq m_1$, an "acceptance criterion" based on (15). Thus, if $L_2(P_m) \leq M^{2^m}$, then $M(P) \leq M$, so we need no further root-squarings. The cost of calculating $L_2^2(P_m)$ is the same as that of computing $a(m+1, h)$. The choice of m_1 is discussed in 3(b).

We also accept P if it survives (17) for all $m \leq m^*$, even though there is the possibility that $M(P) > M$. The best choice of m^* depends on the relative sizes of the sets R_m , which are initially unknown, and on the time needed to compute P_m from P_{m-1} relative to the time needed to calculate $M(P)$. To see this, consider the relative advantages of computing $M(P)$ for all P in R_{m-1} versus computing P_m for all P in R_{m-1} and then computing $M(P)$ for all P in R_m .

The choice of m^* affects the size of the $a(k, m)$, so we settled on a choice which would allow the $a(k, m)$ to be represented exactly as double-precision reals. This restricted $m^* \leq 7$ in most cases, but fortunately this proved to be close to the optimal choice for $n \leq 16$.

This is an appropriate point to discuss whether it is really necessary to compute $a(k, m)$ exactly, or whether one could not simply increase the $b(k, m, s)$ by a certain amount to compensate for errors in the $a(k, m)$. The answer hinges on how sensitive $M(P)$ is to small perturbations in the coefficients of P , so the following example is perhaps instructive:

Example. Let $P(x) = (x + 1)^{2h}$, and $Q(x) = (x + 1)^{2h} - x^h$. Then $M(P) = 1$ and P has large coefficients, indeed as large as possible according to (2). Q thus represents a small perturbation of P . If $h = 30$, for example, the middle coefficient of P is $\binom{60}{30} = 1.18 \times 10^{17} > 2^{56}$, so P and Q would be identically represented in double precision on the machine used.

On the other hand, it is easy to see that

$$\begin{aligned} \log M(Q) &= 2 \int_0^{1/2} \log |(z + 1)^{2h} - z^h| dt \quad (z = \exp(2\pi it)) \\ &= 2 \int_0^{1/2} \log |(z^{1/2} + z^{-1/2})^n - 1| dt, \quad \text{writing } n = 2h, \end{aligned}$$

so

$$\log M(Q) \sim 2 \int_E \log |(2 \cos \pi t)^n| dt = n \log \beta,$$

where $E = \{0 \leq t \leq 1/2 : 2 \cos \pi t > 1\} = [0, 1/3)$, and

$$\beta = \exp \left\{ 2 \int_0^{1/3} \log(2 \cos \pi t) dt \right\} = 1.381356444.$$

Thus $M(Q) \sim \beta^n$. Coincidentally $\beta = \lim_{n \rightarrow \infty} M(z^n + z + 1)$ [3], and so β was calculated in [4]. Using estimates given in [4], it is easy to obtain the more precise result that $M(Q) = \beta^n e^{-\epsilon_n}$, where $0 < \epsilon_n < 4/n$.

For example, if $n = 64$, then $M(Q) \sim 9.54 \times 10^8$, contrasting markedly with $M(P) = 1$. Of course P is very "ill-conditioned" because of the multiple root at -1 . However, observe that if $P_0 = x^{64} + 1$, which is a reasonably well-conditioned polynomial, then $P_6 = (x + 1)^{64}$. Even if we were in a situation where $Q = Q_6$ for some initial Q_0 , we would still have $M(Q_0) \sim \beta = 1.38 \dots$ which is still not very satisfactory.

3(b) *Cyclotomic Factors.* It is obvious that many of the polynomials P of degree n with $M(P) \leq M$ will be reducible since if S, T have $M(S) \leq M, M(T) = 1$, and $\deg S + \deg T = n$, then $P = ST$ has $M(P) \leq M$. Assuming that we have already computed $M(S)$, it would be inefficient to again compute $M(P)$. Similar remarks apply if P is cyclotomic.

The method of recognizing those P with cyclotomic factors depends to a certain extent on R_0 . Let us begin by observing the effect of root-squaring on cyclotomic polynomials. Suppose that $Q = F_{2^k q}$ is the irreducible cyclotomic polynomial whose roots are the primitive $2^k q$ th roots of unity, with q odd, so $\deg Q = 2^{k-1} \phi(q)$, if $k \geq 1$, and $= \phi(q)$, if $k = 0$. Clearly $Q_m = F_r^s$, where $r = 2^{k-m} q$ and $s = 2^m$ if $m \leq k - 1$, while $r = q$ and $s = 2^{k-1}$ if $m \geq k$. Hence, after $m \geq k$ root-squarings, a cyclotomic factor $F_{2^k q}$ of P stabilizes as a factor F_q of P_m with multiplicity 2^{k-1} . Since $\deg F_{2^k q} \leq \deg P = n$, we have $2^{k-1} \leq n$, so after at most $m_1 = [\log_2 n] + 1$

root-squarings, all cyclotomic factors are of the form F_q with q odd.

As a consequence, if P is cyclotomic, then $P_m = P_{m+1}$ for some $m \leq m_1$, and hence P is detectable by at most $m_1 + 1$ root-squarings.

If R_0 is chosen for exhaustive search, then we can afford to ignore any reducible polynomials in R_0 , since we naturally intend to search in order of increasing n . Cyclotomic factors can be detected simply by testing P_{m_1} for factors F_q with q odd. Since cyclotomic P are detected by the method of the previous paragraph, we can assume that P is not cyclotomic and hence that $\deg F_q \leq n - n_0$, where n_0 is the smallest degree for which there is a polynomial Q having $M(Q) \leq M$. For example, if $M = 1.3$, then $n_0 = 8$, and this fact is known before we attempt $n = 10$.

In our implementation, we decided to check only for the factors F_1 and F_p where p is prime, since in this case there is a very easy criterion for F_p to divide P . We begin by computing $P_{m_1}(1) = t$. If $t = 0$, then F_{2^k} divides P for some k , so P is discarded. Since $F_p(1) = p$, a necessary condition for $F_{2^k p}$ to divide P is that $p \mid t$. If this holds, then we test whether F_p divides P_{m_1} using the following:

LEMMA 9. *Let $P = a_0 x^n + \cdots + a_n$ and let p be prime. Then F_p divides P if and only if $c_0 = c_1 = \cdots = c_{p-1}$, where*

$$(20) \quad c_i = \sum_{k \equiv i \pmod{p}} a_k.$$

Proof. F_p divides P if and only if $x^p - 1$ divides $(x - 1)P$.

Since $a_k = a_{n-k}$, it follows that $c_i = c_{n-i}$, so only about half of the c_i need be calculated. For example, if $n = 16$ and $p = 5$, then $c_0 = c_{16} = c_1$ and $c_2 = c_4$, so one need only check that $c_0 = c_2 = c_3$.

This approach is not appropriate when R_0 consists of all P of degree n and $H(P) = 1$, since it would reject those P with a factor P_0 having $H(P_0) > 1$. Instead, we adopted the following method. Note that if P has exactly $\nu = \nu(P)$ roots outside the unit circle, then $|a(\nu, m)|2^{-m} \rightarrow M(P)$. In fact, $a(\nu, m)$ is the trace of the polynomial of degree $\binom{n}{\nu}$ whose roots are $(\alpha_{i_1} \cdots \alpha_{i_\nu})2^m$, so

$$|a(\nu, m) - M(P)2^m| \leq \left(\binom{n}{\nu} - 1 \right) M^{2^m} |\alpha_\nu|^{-2^m}.$$

If P is in R_m , then $|a(\nu, m)|2^{-m}$ provides an approximation to $M(P)$. Of course we do not usually know $\nu(P)$. Suppose, however, that for some $M_1 > M$, we maintain a list of all previously discovered P_0 with $M(P_0) \leq M_1$ and $\deg P_0 < n$, arranged in order of $\nu(P_0)$, then $M(P_0)$. Then, for $m = m^*$, we need only search the list (by binary search) for a P_0 with $\nu(P_0) = k$ and $M(P_0) \simeq |a(k, m)|2^{-m}$, for $k = s, \dots, h$. If a suitable P_0 is found, with $P_0(x) = Q(x^r)$, then we need only test whether $Q(\pm x^r)$ divides P .

If this is successful, we avoid solving $P(x) = 0$. If $\nu(P) = 1$ or $\nu(P) = 2$ and the roots outside the unit circle are complex conjugates, then the approximation to $M(P)$ is sufficiently accurate that P_0 is usually located. In other cases, the search may fail and then $P(x) = 0$ is solved needlessly, but on balance this test has proved to be worthwhile.

3(c) *Selection of R_0 for Exhaustive Search.* Recall Newton's identities:

$$(21) \quad S_k + a_1 S_{k-1} + \cdots + a_{k-1} S_1 + k a_k = 0,$$

where $S_k = \alpha_1^k + \cdots + \alpha_n^k$. By Lemmas 3 and 5, we have

$$(22) \quad S_k \leq [c(k, s)],$$

where

$$(23) \quad c(k, 1) = M^k + n - 2 + M^{-k},$$

$$(24) \quad c(k, 2) = 2M^{k/2} + n - 4 + 2M^{-k/2}.$$

Thus, if a_1, \dots, a_{k-1} are specified, then (21) shows that a_k is restricted to an interval of length $2[c(k, s)]/k$. For example, if $n = 10$, $M = 1.3$, we find from (22) and (21) that we need only examine at most 32560 polynomials, a considerable improvement over the number 1.5×10^{11} suggested before.

Applying Lemma 6 to the polynomial with roots $\alpha_1^k, \dots, \alpha_n^k$, which is valid if k is even, shows that $-S_k \geq n - 4$ for k even implies $s = 2$, a fact which can be used in all further processing of P . Lemma 7 provides the additional bounds

$$(25) \quad -S_{2k} \leq -2S_k^2/n + c(2k, s),$$

where $[c(2k, s)]$ is not used since $\sum |\alpha_i|^{2k}$ need not be an integer. This bound is very restrictive if $|S_k|$ is near n .

Since P is reciprocal, the conditions (22) for $k > h$ provide extra conditions on a_k , $k \leq h$. It was found to be advantageous to use one of these explicitly: using $a_{h-1} = a_{h+1}$ and combining (21) for $k = h$ and $k = h + 1$, we find

$$(26) \quad \begin{aligned} (h+1)a_1 a_h &= -a_1(a_1 S_{h-1} + \cdots + a_{h-1} S_1) \\ &\quad + (a_2 S_{h-1} + \cdots + a_{h-1} S_2) + S_{h+1}, \end{aligned}$$

which, if a_1, \dots, a_{h-1} are given, and $a_1 \neq 0$, confines a_h to an interval of length $2[c(h+1, s)]/(h+1)a_1$, often determining a_h uniquely if $a_1 \geq 2$.

In addition to all the above, the bound $|a_k| \leq b(k, 0, s)$ is occasionally a further restriction.

It should be clear how R_0 is constructed. In principle, one constructs each P in R_0 individually then processes it as in 3(a) and 3(b). We in fact allowed the processes of 3(c) and 3(a) to interact somewhat in that the $a(k, m)$ were calculated as soon as enough information was available to determine them. For example, if $n = 16$, then a_1, \dots, a_6 already determine $a(k, 1)$ for $k = 1, 2, 3$ and $a(1, 2)$ independent of a_7 and a_8 . Thus it may be possible to reject all P with a certain initial segment of coefficients.

4. Discussion of the Polynomials Found. For even $n \leq 16$, all reciprocal polynomials of degree n with measures $\leq M$ were found, where $M = 2$ for $n = 4, 6, 8$ and 10 , $M = 1.5$ for $n = 12$ and 14 , and $M = 1.3$ for $n = 16$. An idea of the computation

involved is given by the CPU time in seconds on the AMDAHL 470 V/6-II at the University of British Columbia:

n	M	time	number found
10	1.3	1.235	7
10	2.0	11.694	185
12	1.5	16.108	43
14	1.5	191.245	93
16	1.3	827.884	14

For $n = 10, \dots, 26$, all polynomials with $H(P) = 1$ and $M(P) \leq 1.3$ were found. The times in seconds for $n = 16, 18, 20, 22, 24$, and 26 are 4.496, 13.799, 42.010, 131.236, 396.384, and 1232.219 respectively.

The tables in Section 6 contain the noncyclotomic factors of a subset of the polynomials just mentioned, normalized so that the first nonzero a_k with k odd has $a_k > 0$. The column "number found" in the above table gives the total number of polynomials normalized in this manner which would appear in a complete list.

Using the fact that $M(P_1 P_2) = M(P_1) M(P_2)$, and the fact that we know all reciprocal polynomials of degree at most 16 and of sufficiently small measure, it is easy to decide whether any of the polynomials are reducible. From our data, we can rule out possible factorizations of the form $P = P_1 P_2$ with P_1 and P_2 being reciprocal and noncyclotomic. Using Smyth's result on nonreciprocal polynomials, we can rule out nonreciprocal factors P_0 of P , provided $M(P) > \theta_0^2 = 1.7548 \dots$, since if a nonreciprocal P_0 divides P , then so does its reciprocal. It turns out that, after cyclotomic factors have been removed, the only reducible polynomials which remain have $\nu(P) = n/2$ and factor in the form $P_0 P_0^*$ where P_0 is nonreciprocal and P_0^* is the reciprocal of P_0 normalized to be monic.

Among the P of small degree there are a number of remarkable occurrences. For example, the following polynomials, all with $\nu(P) = 2$, have the same measures:

- (i) $M = 1.7467934983$, $P_1 = 1 \ 2 \ 2 \ 1 \ 2 \ 2 \ 1$, $\deg P_1 = 6$,
 $P_2 = 1 \ 1 \ -1 \ -1 \ 1 \ -1 \ -1 \ 1 \ 1$, $\deg P_2 = 8$,
- (ii) $M = 1.7709445842$, $P_1 = 1 \ 1 \ 0 \ -2 \ 0 \ 1 \ 1$, $\deg P_1 = 6$,
 $P_2 = 1 \ 0 \ 1 \ 2 \ 1 \ 2 \ 1 \ 0 \ 1$,
 $P_3 = 1 \ 2 \ 1 \ -2 \ -3 \ -2 \ 1 \ 2 \ 1$,
 $\deg P_2 = \deg P_3 = 8$,
- (iii) $M = 1.8789962005$, $P_1 = 1 \ 1 \ -1 \ -3 \ 1 \ 3 \ 1 \ -3 \ -1 \ 1 \ 1$,
 $P_2 = 1 \ 1 \ 2 \ 0 \ -2 \ -3 \ -2 \ 0 \ 2 \ 1 \ 1$,
 $P_3 = 1 \ 2 \ 2 \ 3 \ 4 \ 3 \ 4 \ 3 \ 2 \ 2 \ 1$,
all of degree 10,

$$\begin{aligned}
 \text{(iv)} \quad M &= 1.4986652983, \quad P_1 = 1 \ 1 \ -1 \ -1 \ 2 \ 0 \ -3 \ 0 \ 2 \ -1 \ -1 \ 1 \ 1, \\
 P_3 &= 1 \ 2 \ 2 \ 1 \ -1 \ -3 \ -3 \ -3 \ -1 \ 1 \ 2 \ 2 \ 1, \\
 P_2 &= 1 \ 1 \ 2 \ 2 \ 2 \ 3 \ 3 \ 3 \ 2 \ 2 \ 2 \ 1 \ 1, \\
 &\text{all of degree 12,}
 \end{aligned}$$

where we have used a notation whose meaning should be obvious; or see [2].

The triples of the same degree in (iii) and (iv) are most easily explained. If one writes $P_{(3)}$ for the polynomial whose roots are the cubes of the roots of P , then it can be verified that, in either (iii) or (iv), each P_k has the same $P_{(3)}$ up to the symmetry $x \rightarrow -x$. In each case, if ω is a primitive cube root of unity, then

$$(27) \quad P_1(\omega x)P_1(\omega^2 x) = P_2(x)P_3(-x),$$

so the surprise is simply that the left member of (27) is reducible over the rationals.

The explanation of (i) and (ii) is somewhat more involved. We first observe that if $\nu(P) = 2$, then $M(P)$ is the largest root of the polynomial Q whose roots are $\alpha_i \alpha_j$, with $i < j$, so $\deg Q = \binom{n}{2}$. When P is reciprocal, $h = n/2$ of the roots of Q are equal to $\alpha_i \alpha_i^{-1} = 1$, hence $M(P)$ is a root of $P^{[2]}(x) = Q(x)/(x-1)^h$, the 2nd compound of P , which has degree $n(n-2)/2$. Note that for $n = 6, 8$ the degree of $P^{[2]}$ is 12 or 24, respectively.

A simple calculation then shows that, in example (i),

$$(28) \quad P_2^{[2]}(x) = P_1^{[2]}(x)P_1(x)^2,$$

while, in example (ii),

$$(29) \quad P_2^{[2]}(x) = P_3^{[2]}(x) = P_1^{[2]}(x)P_1(ix)P_1(-ix).$$

In example (ii), there is additionally

$$(30) \quad P_2(x)P_2(-x) = P_3(ix)P_3(-ix).$$

One moral of the above is that $M(P)$ does not determine P up to the obvious transformations $P(x) \rightarrow P(\pm x^s)$. In fact, it is quite possible for a nonreciprocal P_1 to have the same measure as a reciprocal P_2 . For example, if

$$P_1(x) = (x - \alpha)(x - \bar{\alpha})(x - \beta)(x - \bar{\beta})$$

has $|\alpha| > 1 > |\beta| = |\alpha|^{-1}$, but is nonreciprocal, so $\beta \neq \alpha^{-1}$, $\bar{\alpha}^{-1}$, then $M(P_1) = |\alpha|^2$ is the unique root outside the unit circle of a reciprocal polynomial $P_1^{[2]}$ of degree $\binom{4}{2} = 6$. For example, if $P_1 = 1 \ 0 \ 0 \ 1 \ 1$ with $M(P_1) = 1.401268368$, then $P_1^{[2]} = 1 \ 0 \ -1 \ -1 \ -1 \ 0 \ 1$, so that $P_2(x) = P_1^{[2]}(-x)$ has $M(P_2) = M(P_1)$ as we verify from our list. This explains the following coincidence of measures:

$$(v) \quad M = 1.9635530390 = (1.4012683679)^2, \text{ attained for}$$

$$P_3 = 1 \ 2 \ -1 \ -3 \ -1 \ 2 \ 1, \quad n = 6, \nu(P_3) = 1, \text{ and}$$

$$P_4 = 1 \ 1 \ 0 \ 1 \ 3 \ 1 \ 0 \ 1 \ 1, \quad n = 8, \nu(P_4) = 4.$$

For here we have $P_3(x) = P_2(x^{1/2})P_2(-x^{1/2})$, while $P_4(x) = P_1(x)P_1^*(x)$.

5. Nonexhaustive Search. As was mentioned in [3], the set $\{M(P): P \text{ non-reciprocal}\}$ has at least two limit points smaller than $\theta_0 = 1.3247 \dots$, namely

$$\begin{aligned}\lambda_1 &= \exp \left\{ \int_0^1 \int_0^1 \log |\zeta + \zeta z^{-1} + z + 1 + z^{-1} + \zeta^{-1}z + \zeta^{-1}| ds dt \right\} \\ &= 1.255425 \dots \quad (\zeta = e^{2\pi is}, z = e^{2\pi it}),\end{aligned}$$

and

$$\lambda_2 = \exp \left\{ \int_0^1 \int_0^1 \log |\zeta + z + 1 + z^{-1} + \zeta^{-1}| ds dt \right\} = 1.285734 \dots$$

These are the unique limit points of the following sets of polynomials:

$$P_1(x) = x^{2h} + ax^{2h-m} + bx^{h+m} + cx^h + bx^{h-m} + ax^h + 1,$$

and

$$P_2(x) = x^{2h} + ax^{h+m} + bx^h + ax^{h-m} + 1,$$

respectively, where m and h are relatively prime and a, b, c are either 1 or -1 .

Thus all but a finite number of such P_1 and P_2 will have $M(P) < 1.3$, and it is of interest to examine those with, say, $M(P) \leq 1.25$.

We examined all of these for $h \leq 25$. None had measures less than σ_1 . Among the P_1 have been found the smallest known values of $M(P)$ for fixed $\nu(P) \geq 4$. For example, the following degree 36 polynomial has $\nu(P) = 4$, and $M(P) = 1.2294828102$:

$$\begin{aligned}P = & 1 \ 1 \ 1 \ 0 \ 0 \ -1 \ -1 \ -1 \ 0 \ 1 \ 1 \ 1 \ 0 \ -1 \ -2 \ -1 \ 0 \ 1 \ 1 \ 1 \ 0 \\ & -1 \ -2 \ -1 \ 0 \ 1 \ 1 \ 1 \ 0 \ -1 \ -1 \ -1 \ 0 \ 0 \ 1 \ 1 \ 1.\end{aligned}$$

This is a factor of $x^{44} + x^{35} - x^{31} - x^{22} - x^{13} + x^9 + 1$.

Another more extensive family which was investigated for $h \leq 20$ is the set of polynomials of the form

$$P(x) = (x^{2h+1} - x^{2h-k} - x^{h+s+1} + x^{h-s} + x^{k+1} - 1)/(x - 1).$$

Since $L_2(P(x)(x-1)) = 6^{1/2}$, these all have $M(P) \leq 1 + 2^{1/2}$ by Gonçalves' inequality (16). All known small Salem numbers ($\nu(P) = 1$) are measures of members of this family [2].

6. Lists of Polynomials. The composition of the following tables is described in Section 4. For degrees ≥ 18 , only the first half of the polynomial is listed, as in [2]. Only one of the polynomials listed is reducible, namely $1 \ 1 \ -1 \ -3 \ -1 \ 1 \ 1 = (1 \ 0 \ -1 \ -1)(1 \ 1 \ 0 \ -1)$, as discussed in Section 4. In case $n \geq 18$, $M(P) \leq 1.27$ and $H(P) > 1$, the number in parentheses following the degree is the smallest degree of a reciprocal Q of even degree with $H(Q) = 1$ of which P is a factor.

TABLE 1

*All irreducible reciprocal polynomials P of degree n
and measure $M(P) \leq M_0$, for $4 \leq n \leq 16$.*

$n = 4, M_0 = 2$

Measure	ν	Polynomial
1.7220838057	1	1 1-1 1 1
1.8832035059	1	1 2 1 2 1

$n = 6, M_0 = 2$

Measure	ν	Polynomial
1.4012683679	1	1 0-1 1-1 C 1
1.5061356796	1	1 1 0 1 0 1 1
1.5560301913	1	1 1-1-1-1 1 1
1.5823471837	1	1 0-1 2-1 0 1
1.6355731299	1	1 2 2 3 2 2 1
1.7467934983	2	1 2 2 1 2 2 1
1.7548776662	3	1 1-1-3-1 1 1
1.7709445842	2	1 1 0-2 0 1 1
1.7816435986	1	1 1-1 0-1 1 1
1.7942797448	2	1 0 2 1 2 0 1
1.8310758251	1	1 2 0-1 0 2 1
1.8392867552	2	1 1 2 3 2 1 1
1.8948181172	2	1 0 1 2 1 C 1
1.9468562683	1	1 1-1 1-1 1 1
1.9635530390	1	1 2-1-3-1 2 1
1.9748187083	1	1 2 1 2 1 2 1
1.9877931668	1	1 0-2 3-2 0 1
1.9962080000	2	1 1 1-1 1 1 1

$n = 8, M_0 = 1.75$

Measure	ν	Polynomial
1.2806381563	1	1 C 0 1-1 1 0 0 1
1.3599997117	1	1 1 1 2 1 2 1 1
1.3672228038	2	1 0 0 1 1 1 0 0 1
1.3703868743	2	1 1 1 0-1 0 1 1 1
1.4250052678	1	1 1 0 1 1 1 0 1 1
1.4575874796	1	1 C-1 1 0 1-1 0 1
1.4762823965	2	1 2 2 1 1 1 2 2 1
1.4874875383	2	1 1 1 C 1 0 1 1 1
1.5230602489	1	1 1-1 C 1 C-1 1 1
1.5356181220	2	1 1 2 3 3 3 2 1 1
1.5471965656	1	1 2 2 3 3 3 2 2 1
1.5682443656	2	1 1 1 2 3 2 1 1 1
1.6054458288	1	1 2 1 C-1 C 1 2 1
1.6185305986	2	1 1 2 1 1 1 2 1 1
1.6241475659	2	1 1 0-1 0-1 0 1 1
1.6403340873	1	1 0-2 1 1 1-2 0 1
1.6575497181	2	1 0 1 1 2 1 1 0 1
1.6610477620	1	1 2 1 1 1 1 1 2 1
1.6849101527	1	1 1-1 0 0 0-1 1 1
1.6850319934	2	1 1 1-1-2-1 1 1 1
1.6935073826	1	1 1 0 1-1 1 0 1 1
1.7042442644	2	1 0 1 1-1 1 1 0 1
1.7467934583	2	1 1-1-1 1-1-1 1 1

TABLE 1 (*continued*) $n = 10, M_0 = 1.5$

Measure	ν	Polynomial
1.17628C8183	1	1 1 0 -1 -1 -1 -1 0 1 1
1.2162916611	1	1 0 0 0 -1 1 -1 0 0 1
1.2303914344	1	1 0 0 1 0 1 0 1 0 0 1
1.2612309611	1	1 0 -1 0 0 1 0 0 -1 0 1
1.2672338594	2	1 0 1 1 0 1 0 1 1 0 1
1.28358236C6	2	1 1 0 0 0 -1 0 0 0 1 1
1.2934859531	1	1 0 -1 1 0 -1 0 1 -1 0 1
1.3044637617	2	1 1 0 -1 0 0 0 -1 0 1 1
1.3373132102	1	1 1 0 0 0 1 0 0 0 1 1
1.35098C3377	1	1 1 0 0 -1 -1 -1 0 0 1 1
1.3619551623	2	1 0 0 0 1 1 1 0 0 0 1
1.3689986768	2	1 0 0 1 0 -1 0 1 0 0 1
1.3836365634	1	1 1 0 1 1 1 1 1 0 1 1
1.3984561816	2	1 1 2 2 2 3 2 2 2 1 1
1.42599251C7	2	1 2 2 1 0 -1 0 1 2 2 1
1.43100C9591	1	1 1 -1 -1 0 1 0 -1 -1 1 1
1.4314046376	2	1 1 1 1 0 -1 0 1 1 1 1
1.448423C402	1	1 2 2 2 1 1 1 2 2 2 1
1.4723531176	1	1 1 0 0 -1 0 -1 0 0 1 1
1.4723872015	2	1 0 1 1 1 0 1 1 1 0 1
1.4796086733	1	1 0 -2 2 1 -3 1 2 -2 C 1
1.4804088C56	2	1 1 2 3 3 3 3 3 2 1 1
1.4824156382	2	1 1 1 C 1 C 1 0 1 1 1
1.4938778456	2	1 0 -1 1 1 -2 1 1 -1 C 1

 $n = 12, M_0 = 1.4$

Measure	ν	Polynomial
1.2277855587	2	1 1 1 0 -1 -1 -1 -1 0 1 1 1
1.2407264237	1	1 1 1 1 0 0 -1 0 0 1 1 1
1.2510466172	2	1 1 0 0 0 -1 -1 -1 0 0 0 1
1.2643938547	2	1 0 1 0 0 1 -1 1 0 0 1 0 1
1.2728183651	2	1 0 1 1 1 2 1 2 1 1 1 0 1
1.3019549435	2	1 1 0 -1 0 1 1 1 0 -1 0 1 1
1.3022688C51	1	1 1 0 0 0 1 1 1 0 0 0 1 1
1.3044335627	2	1 0 0 1 -1 -1 1 -1 1 1 0 0 1
1.3101213237	2	1 2 2 1 1 1 1 1 1 2 2 1
1.3159144319	1	1 0 0 0 -1 1 -1 1 -1 0 0 0 1
1.3221661921	2	1 0 0 0 0 1 1 1 0 0 0 0 1
1.3349522567	2	1 1 0 0 0 0 1 0 0 0 0 1 1
1.3434948981	2	1 0 -1 1 0 -1 1 -1 0 1 -1 0 1
1.3702684012	1	1 0 -1 1 0 0 0 0 0 1 -1 0 1
1.3776747894	1	1 0 -1 0 0 1 -1 1 0 0 -1 0 1
1.3877955034	2	1 0 1 0 1 1 0 1 1 0 1 0 1
1.3887530310	2	1 0 0 0 1 1 -1 1 1 0 0 0 1
1.3977398823	2	1 2 2 2 3 3 3 3 3 2 2 2 1

TABLE 1 (continued)

 $n = 14, M_0 = 1.33$

Measure	ν	Polynomial
1.2000265240	1	1 0 0 1-1 C 0-1 0 0-1 1 0 0 1
1.2026167437	1	1 0-1 0 0 0 0 1 0 0 0 0-1 0 1
1.2550935168	1	1 C-1 1 0-1 0 1 0-1 0 1-1 0 1
1.2651220724	2	1 0 0 1 1 1 1 1 1 1 1 0 0 1
1.2672964425	1	1 1 0 0 0 C-1-1-1 0 0 0 0 1 1
1.2780430161	2	1 1 1 0 0-1-1-1-1-1 0 0 1 1 1
1.2801230273	2	1 1 0-1 0 0 0-1 0 0 0-1 0 1 1
1.2901780368	2	1 0 0 0 0 1 0-1 0 1 C 0 0 1
1.2915627600	3	1 1 1 0-1-1-1-1-1-1 0 1 1 1
1.2930728883	3	1 C 0 0-1 C 0 1 0 C-1 0 0 0 1
1.2973983168	2	1 1 1 1 1 0-1-1-1 0 1 1 1 1 1
1.30001931434	2	1 1 0 0 0 C 1 1 1 0 0 0 0 1 1
1.3078649611	3	1 1 0 0 0-1-1-1-1-1 0 0 0 1 1
1.3122008728	2	1 0-1 0 1 0 0 1 0 0 1 0-1 0 1
1.3126595949	2	1 0 0 1 C-1 C 0 0-1 0 1 0 0 1
1.3128725655	3	1 0 0 0 0 1 0 1 0 1 0 0 0 1
1.3175755953	3	1 0 1 0 0 1-1 1-1 1 0 C 1 0 1
1.3181975044	1	1 1 0 1 1 0 0 1 0 C 1 1 0 1 1
1.3190992671	3	1 0 0 C 0 C-1 1-1 0 0 0 0 0 1
1.3199391059	2	1 0 0 1 0 0 0-1 0 0 0 1 0 0 1
1.3211018483	1	1 1 0 0-1 0 1 1 1 0-1 0 0 1 1
1.3243396769	3	1 1 0-1-1-1 0 1 0-1-1-1 0 1 1
1.3254675791	2	1 0 0 1 1 0 0 1 0 0 1 1 0 0 1
1.3254927589	2	1 0 1 0 0 1 0 1 0 1 0 0 1 0 1

 $n = 16, M_0 = 1.3$

Measure	ν	Polynomial
1.2242789072	2	1 1 0-1-1 C 1 1 1 1 1 0-1-1 0 1 1
1.2352567056	2	1 1 0 0 C-1 0 0-1 0 0-1 0 0 0 1 1
1.2363175318	1	1 1 0 0 0 C 0 0-1 C 0 0 0 0 0 1 1
1.2434776187	2	1 1 0-1-1-1 0 1 1 1 C-1-1-1 0 1 1
1.2480136083	2	1 1 1 1 0-1-1-1-1-1-1 0 1 1 1 1
1.2517324113	2	1 1 1 0 0 0 0 0-1 0 0 0 0 0 1 1 1
1.2528286630	2	1 C 1 1 C 1 0 0 1 0 0 1 0 1 1 0 1
1.2573166256	2	1 1 0-1 0 1 1 C-1 0 1 1 0-1 0 1 1
1.2666775265	2	1 1 1 1 1 0-1-2-2-2-1 0 1 1 1 1 1
1.2813411294	2	1 1 0-1 0 C-1 0 1 0-1 0 0-1 0 1 1
1.2895901023	2	1 1 0-1-1 0 1 0 0 0 1 0-1-1 0 1 1
1.2934931125	2	1 1 1 0 0 0 1 0 0 0 1 0 0 0 1 1 1
1.2970647745	2	1 0 0 1 0 0 0 0 1 0 0 0 0 1 0 0 1
1.2980322543	4	1 0 0 1-1 0 0-1 1-1 0 0-1 1 0 0 1

TABLE 2

*All irreducible reciprocal polynomials of degree n , $18 \leq n \leq 26$,
which divide a reciprocal polynomial of height 1
and even degree at most 26, having measure at most 1.27.*

n	Measure	v	First half of polynomial
18	1.1883681475	1	1 1 1 1 0 0-1-1-1-1
18	1.2013961862	2	1 1 1 0 0-1 0-1 0-1
18	1.2194468759	3	1 1 1 0-1-1-1 0 0 1
18	1.2197208590	1	1 1 0 C C C 0 0-1-1
18	1.2255034241	2	1 1 0 0 1 0-1 0 0-1
18	1.2313427700	3	1 C 0 C-1 1 0 0 0-1
18	1.2407706350	2	1 0 1 0 1 0 0 1-1 1
18	1.2446170590	3	1 0-1 0 1 C-1 0 0 1
18	1.2527759374	1	1 0 0 0 0 0-1 1-1 1
18	1.2562211544	1	1 1 0 0-1-1 0 0 0 1
18	1.2663343250	3	1 1 0 0 0-1-1 0 0-1
20	1.2128241810	2	1 1 0 0 1 1 0-1-1-1-1
20	1.2149957008	2	1 1 1 1 1 C-1-1-1-1-1
20	1.2183963625	2	1 1 0-1 0 0 0-1 0 0 1
20(22)	1.2269927582	2	1 1-1-2 0 2 1-1-1 0 1
20	1.2326135486	1	1 1 0 C 0 1 1 0 0 1 1
20	1.2461081427	2	1 1 1 0 0-1-1-1 0 0 1
20	1.2468762715	2	1 1 1 1 0 0 C-1-1-1-1
20	1.2533306502	1	1 0-1 0 0 1 0 0 0 0 0
20	1.2536355657	2	1 1 1 1 0 0-1-1-1-1-1
20	1.2547004271	2	1 0 1 0 0 1 0 1 1 0 1
20	1.2562936626	2	1 0 1 0 0 0-1 0 0 1 1
20	1.2598494325	2	1 0 0 1 0 C 0 0 0 0 1
20	1.2613609976	2	1 0 0 0 0 1-1 0 0-1 1
20	1.2625097625	2	1 C-1 1 1-1 0 0 0 0-1
20(24)	1.2625647407	4	1 2 2 1-1-3-3-2 0 2 3
20	1.2639458564	3	1 0 0 1 0 0 0 1 0 0 1
20	1.2687376481	2	1 0 0 1 0 0 0-1 0 0-1
20(22)	1.2693314288	2	1 2 2 1 0-1-1-1-1-1-1
22	1.2050198542	2	1 0 1 0 0 1-1 1 0 0 1-1
22	1.2295664566	3	1 1 0-1-1-1-1-1 0 1 1 1
22	1.2356645804	1	1 0-1 1 0 C 0-1 1 0-1 1
22	1.2446024455	2	1 0 0 1 0 0 0 0 0 0-1
22	1.2451811363	3	1 1 0-1-1 C 1 1 0-1-1-1
22	1.2456023171	2	1 1 1 1 0-1-1-1-1 C 1 1
22	1.2486111657	3	1 1 0 0 0-1-1-1-1 0 1 1
22	1.2504618711	3	1 0-1 0 0 0 0 1 0 C 0-1
22	1.2527210201	2	1 0 0 1 C C 0 0 1 0 0 1
22	1.2541425192	3	1 0-1 0 0 C C 0 0 1 0-1
22(24)	1.2572590685	3	1 1 1 0-1-2-2-1 0 1 1 1
22	1.2594336068	2	1 C-1 C 1 C 0 0 0 0 0 1
22	1.2602835589	3	1 0 0 1-1 0 0-1 0 0 0 1
22	1.2602842369	1	1 1 0 1 1 C 0 0-1-1-1-1
22	1.2615262661	3	1 1 0-1 0 1 0 0-1-1 0 1
22	1.2624605759	3	1 1 1 0 0-1-1-1-1 0 0 1
22	1.2640511961	3	1 0 0 0 0 0 0 0 0 0-1 1

TABLE 2 (continued)

n	Measure	v	First half of polynomial
24	1.21885515C3	2	1 0 0 C 0 1 0-1 0 0 0 0-1
24	1.2190575078	2	1 1 0-1-1-1-1-1 0 1 1 1 1
24	1.2344438349	2	1 0 0 0-1 0-1 1 0 C 1-1 1
24	1.2537437612	2	1 1 1 1 1 C 0 0-1-1-1-1-1
24	1.2541043056	2	1 1 0-1-1 0 0 0 1 1 0-1-1
24	1.2544917819	2	1 0 0 1-1 0 0-1 0 0 0 0 1
24	1.2562017318	2	1 1 1 0 0 C 0 0 0 C-1-1-1
24	1.2576061050	4	1 1 1 0-1-1-1 0 0 0 0 0 1
24	1.2601C354C4	1	1 1 0 0-1-1 0 1 1 1 0-1-1
24	1.26162239C4	2	1 1 1 1 0 0-1-1-1-1 0 C 1
24	1.264833C8C3	2	1 C-1 C C 1 0-1 0 0 1 0-1
24	1.2665387000	2	1 1 0 0 1 1 0-1-1 0 1 0-1
26	1.2237774549	3	1 1 1 0 0-1-1-1-1 0 0 1 0 1
26	1.226092E945	2	1 0 0 1 0 C 0 C 0 0 1 0 0 1
26	1.2345003369	3	1 0-1 1 0-1 1 0-1 1 0-1 0 1
26	1.2375C46212	1	1 C-1 0 C 1 0 C-1 C 1 0 0-1
26	1.2402541787	3	1 1 0-1-1 0 0 0-1 0 1 1 0-1
26	1.2476C87C26	2	1 0 1 C 1 0 0 0 0 1 0 1 0 1
26	1.2488187537	3	1 1 1 1 0 C 0 0 0 0 0-1-1-1-1
26	1.2494993332	3	1 1 0 0 0 0 0 0 0 0 0 0-1-1
26	1.2573832167	3	1 0 0 0 C C-1 0 0 0 0 0 0 1
26	1.26C7650282	2	1 1 1 0 0-1 0 0 1 1 1 0-1-1
26	1.2613475293	2	1 1 0 C C-1 0 0 0 0 0 0 0-1
26	1.2630381399	1	1 1 0 0 0 0-1 0 0 0 0 0 0-1
26	1.26523C67C5	3	1 1 0-1-1 0 0-1-1 0 1 1 0-1
26	1.2683338900	3	1 0 0 0 C 1 0 C 0 0 0 0 0 1

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