Computation of Pólya Polynomials of Primitive Permutation Groups

By Rudolf Land

Abstract. An almost complete list of Pólya polynomials of all primitive permutation groups up to degree 20 has been computed.

The number-theoretical interpretation of Pólya polynomials and van der Waerden's test make this a good tool to find safe conjectures for determining the group of an equation.

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1. Definitions. Let $G \leq \mathfrak{S}_n$ be a permutation group of degree $n \in \mathbb{N}$. Then, any permutation $\sigma \in G$ has a unique factorization into a product of cycles. If this factorization consists of t_1 1-cycles, t_2 2-cycles, $\dots$ and t_n n -cycles, we call $\mathbf{t} := (t_1, \dots, t_n) \in \mathbb{N}_0^n$ the *type of the permutation* σ , which will be denoted by $\mathbf{t}(\sigma) = (t_1(\sigma), \dots, t_n(\sigma))$. Obviously we have

$$(1) \quad \sum_{i=1}^n it_i = \sum_{i=1}^n it_i(\sigma) = n.$$

For the permutation group G , we define the *Pólya polynomial* as

$$(2) \quad P_G(z_1, \dots, z_n) := \frac{1}{\#G} \sum_{\sigma \in G} \mathbf{z}^{\mathbf{t}(\sigma)} \in \mathbb{Q}[z_1, \dots, z_n],$$

where

$$(3) \quad \mathbf{z} := (z_1, \dots, z_n), \quad \mathbf{t}(\sigma) = \text{type of } \sigma, \quad \mathbf{z}^{\mathbf{t}} := \prod_{i=1}^n z_i^{t_i}.$$

For $\mathbf{t} \in \mathbb{N}^n$ satisfying (1), let

$$(4) \quad a_G(\mathbf{t}) := \#\{\sigma \in G / \mathbf{t}(\sigma) = \mathbf{t}\}.$$

Then

$$(5) \quad P_G(z_1, \dots, z_n) = \frac{1}{\#G} \sum_{\mathbf{t} \in S_n} a_G(\mathbf{t}) \mathbf{z}^{\mathbf{t}},$$

where

$$(6) \quad S_n := \left\{ \mathbf{t} \in \mathbb{N}_0^n / \sum_{i=1}^n it_i = n \right\}.$$

2. Simple Examples. For certain special series of groups, the Pólya polynomials are generally computable; cf. de Bruijn [2, Section 1]

(a) $G = \{e\} \Rightarrow P_G(z_1, \dots, z_n) = z_1^n$ for any $n \in \mathbb{N}$,

(b) $G = \mathfrak{S}_n$.

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For $t \in S_n$ (cf. (6)), we find with combinatorial considerations

$$a_{\mathfrak{S}_n}(t) = \frac{n!}{\prod_{i=1}^n t_i! i^{t_i}}$$

and hence

$$(7) \quad P_{\mathfrak{S}_n}(z_1, \dots, z_n) = \sum_{t \in S_n} \left(\prod_{i=1}^n \frac{1}{t_i!} \left(\frac{z_i}{i} \right)^{t_i} \right),$$

for instance

$$n = 1: \quad P_{\mathfrak{S}_1} = z_1,$$

$$n = 2: \quad P_{\mathfrak{S}_2} = \frac{1}{2}(z_1^2 + z_2),$$

$$n = 3: \quad P_{\mathfrak{S}_3} = \frac{1}{6}(z_1^3 + 3z_1z_2 + 2z_3),$$

$$n = 4: \quad P_{\mathfrak{S}_4} = \frac{1}{24}(z_1^4 + 6z_1^2z_2 + 8z_1z_3 + 6z_4 + 3z_2^2).$$

(c) $G = \mathfrak{A}_n$.

For $\sigma \in \mathfrak{S}_n$, $t := t(\sigma)$, $m := \lfloor n/2 \rfloor$, we have

$$\begin{aligned} \sigma \in \mathfrak{A}_n &\Leftrightarrow \text{sign } \sigma = 1 \Leftrightarrow (-1)^{t_2+t_4+\dots+t_{2m}} = 1 \\ &\Leftrightarrow t_2 + t_4 + \dots + t_{2m} \in 2\mathbb{N}_0. \end{aligned}$$

Furthermore,

$$\bigwedge_{\sigma \in \mathfrak{A}_n} a_{\mathfrak{S}_n}(t(\sigma)) = a_{\mathfrak{A}_n}(t(\sigma))$$

and therefore, with $\#\mathfrak{A}_n = \#\mathfrak{S}_n/2$ and (7),

$$P_{\mathfrak{A}_n}(z_1, \dots, z_n) = 2 \sum_{t \in A_n} \left(\prod_{i=1}^n \frac{1}{t_i!} \left(\frac{z_i}{i} \right)^{t_i} \right),$$

where

$$A_n := \left\{ t \in S_n / \sum_{i=1}^m t_{2i} \in 2\mathbb{N}_0 \right\}.$$

Hence, we find the Pólya polynomial of the alternating group $\mathfrak{A}_n$ by taking the Pólya polynomial of the symmetric group $\mathfrak{S}_n$, cancelling the terms with $t_2 + t_4 + \dots + t_{2m} \in 2\mathbb{N}_0$, and doubling all other coefficients. For instance:

$$n = 3: \quad P_{\mathfrak{A}_3} = \frac{1}{3}(z_1^3 + 2z_3), \quad n = 4: \quad P_{\mathfrak{A}_4} = \frac{1}{12}(z_1^4 + 8z_1z_3 + 3z_2^2).$$

3. Remarks on the Number-Theoretical Interpretation of Pólya polynomials. Any separable polynomial f of degree n over a ground field k determines a Galois group $G = G_f$ as the permutation group of the roots of f . Let $\alpha_1, \dots, \alpha_n$ be the roots of f .

$$G_f \begin{cases} \bullet & N := k(\alpha_1, \dots, \alpha_n) \\ \bullet & K := k(\alpha_1) \\ \bullet & k \end{cases} \quad \mathcal{P}_k \text{ will denote the set of all prime ideals in } k.$$

Let $\mathfrak{p} \in \mathcal{P}_k$ be unramified in K and have

$$\begin{array}{ll} t_1 & \text{prime divisors in } K \text{ of degree } 1, \\ t_2 & \text{prime divisors in } K \text{ of degree } 2, \\ \vdots & \\ t_n & \text{prime divisors in } K \text{ of degree } n. \end{array}$$

It is well known, that $\sum_{i=1}^n i t_i = n$.

$\mathbf{t} := (t_1, \dots, t_n)$ is called the *type of $\mathfrak{p}$ in K* , which is denoted by $\mathbf{t} = \mathbf{t}(\mathfrak{p}) = \mathbf{t}_{K|k}(\mathfrak{p})$.

Already Artin [1] (“Hilfssatz” in Abschnitt 2) and Hurwitz [4] (“Satz von Frobenius”) proved that for $\mathfrak{p} \in \mathcal{P}_k$, which are unramified in N ,

$$\mathbf{t}_{N|k}(\mathfrak{p}) = (t_1, \dots, t_n) \Leftrightarrow \text{any } \sigma \in (N|k/\mathfrak{p}) \text{ (as a permutation of the roots of } f) \\ \text{has the type } (t_1, \dots, t_n):$$

$$\bigwedge_{\sigma \in (N|k/\mathfrak{p})} \mathbf{t}(\sigma) = (t_1, \dots, t_n),$$

where $(N|k/\mathfrak{p})$ denotes the Frobenius symbol.

Using Čebotarev’s density theorem, we get

$$(8) \quad \delta(\{\mathfrak{p} \in \mathcal{P}_k / \mathbf{t}(\mathfrak{p}) = (t_1, \dots, t_n)\}) = \frac{1}{\#G_f} a_{G_f}(t_1, \dots, t_n),$$

where δ denotes the Dirichlet-density (cf. (4)). But the right-hand side is a coefficient of the Pólya polynomial P_{G_f} and can be determined purely group-theoretically.

4. Results for Primitive Permutation Groups Up to Degree 20. I made use of an old version of the program system GROUP, developed by J. Neubüser, Aachen, and J. J. Cannon [3], to compute Pólya polynomials for given permutation groups. I coded and implemented a change in the algorithm determining all elements, so that, in principle, one is able to determine the Pólya polynomial of any given permutation group with the computer. The limitations are naturally memory-usage and time-consumption. Input for the program are generating elements, from which the program collects all occurring permutation types, and their frequencies, during the determination of all elements, so that the output can be the coefficients of the Pólya polynomial.

All primitive permutation groups up to degree 20 (determined by C. C. Sims [5]) were used as input to the program. The results (and thus the program) have been checked by hand for small group-orders and by comparison with theoretical results for $\mathfrak{A}_n$, $\mathfrak{S}_n$.

To identify the groups I used the notation of Sims [5]; for instance, 7.2 denotes the second group of degree 7 in Sims’ list, and 11.6 the sixth group of degree 11 (which is the Mathieu group M_{11}).

The program only failed for the groups 16.19, 16.20, $\mathfrak{A}_n$ for $n = 9, 10, \dots, 20$, $\mathfrak{S}_n$ for $n = 8, 9, 10, \dots, 20$, and for 12.4, the Mathieu group M_{12} . All these groups have orders exceeding 40000, for which the memory of the CDC 7600 is too small. But, since for $\mathfrak{A}_n$, $\mathfrak{S}_n$ we have theoretical results, only the groups 16.19, 16.20, and $12.4 = M_{12}$ are still to be examined.

Table 1 gives a list of all occurring permutation types and Table 2 gives the frequencies of their occurrence in the actual group (omitted types in Table 2 have frequency 0).

Examples. (a) To determine the Pólya-Polynomial of group $3.2 = \mathfrak{S}_3$:

<i>occurring type</i>	<i>frequency</i> ($= a_{3,2}(\text{type})$, cf. (4))
$a = (3, 0, 0)$	1
$b = (1, 1, 0)$	3
$c = (0, 0, 1)$	2
	$\frac{2}{6} = \text{order of group}$

According to (5),

$$\begin{aligned}
 P_{\mathfrak{S}_3} &= \frac{1}{6} (1 \cdot (z_1, z_2, z_3)^{(3,0,0)} + 3 \cdot (z_1, z_2, z_3)^{(1,1,0)} + 2 \cdot (z_1, z_2, z_3)^{(0,0,1)}) \\
 &= \frac{1}{6} (z_1^3 + 3z_1z_2 + 2z_3).
 \end{aligned}$$

(b) Prime factorization in fields k of degree 11 with group $11.6 = M_{11}$ as Galois group of the Galois-hull:

<i>occurring type</i>	<i>frequency</i>
$a = (11, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$	1
$b = (3, 4, 0, 0, 0, 0, 0, 0, 0, 0, 0)$	165
$c = (3, 0, 0, 2, 0, 0, 0, 0, 0, 0, 0)$	990
$d = (2, 0, 3, 0, 0, 0, 0, 0, 0, 0, 0)$	440
$e = (1, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0)$	1980
$f = (1, 0, 0, 0, 2, 0, 0, 0, 0, 0, 0)$	1584
$g = (0, 1, 1, 0, 0, 1, 0, 0, 0, 0, 0)$	1320
$h = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1)$	1440
	$\frac{1440}{7920}$

According to (8) this means: asymptotically we have to find

$\frac{1}{7920}$	of all prime ideals being	fully decomposed
$\frac{165}{7920} = \frac{1}{48}$	of all prime ideals having	3 divisors of degree 1
	and	4 divisors of degree 2
$\frac{990}{7920} = \frac{1}{8}$	of all prime ideals having	3 divisors of degree 1
	and	2 divisors of degree 4
$\frac{440}{7920} = \frac{1}{18}$	of all prime ideals having	2 divisors of degree 1
	and	3 divisors of degree 3
$\frac{1980}{7920} = \frac{1}{4}$	of all prime ideals having	1 divisor of degree 1
		1 divisor of degree 2
	and	1 divisor of degree 8
$\frac{1584}{7920} = \frac{1}{5}$	of all prime ideals having	1 divisor of degree 1
	and	2 divisors of degree 5

$\frac{1320}{7920} = \frac{1}{6}$	of all prime ideals having	1 divisor of degree 2
		1 divisor of degree 3
	and	1 divisor of degree 6
$\frac{1440}{7920} = \frac{2}{11}$	of all prime ideals staying prime.	

TABLE 1

Occurring permutation types of primitive permutation groups up to degree 20

Degree   Permutation types

3	$a = (3, 0, 0)$ $b = (1, 1, 0)$ $c = (0, 0, 1)$
4	$a = (4, 0, 0, 0)$ $b = (2, 1, 0, 0)$ $c = (1, 0, 1, 0)$ $d = (0, 2, 0, 0)$ $e = (0, 0, 0, 1)$
5	$a = (5, 0, 0, 0, 0)$ $b = (3, 1, 0, 0, 0)$ $c = (2, 0, 1, 0, 0)$ $d = (1, 2, 0, 0, 0)$ $e = (1, 0, 0, 1, 0)$ $f = (0, 1, 1, 0, 0)$ $g = (0, 0, 0, 0, 1)$
6	$a = (6, 0, 0, 0, 0, 0)$ $b = (4, 1, 0, 0, 0, 0)$ $c = (3, 0, 1, 0, 0, 0)$ $d = (2, 2, 0, 0, 0, 0)$ $e = (2, 0, 0, 1, 0, 0)$ $f = (1, 1, 1, 0, 0, 0)$ $g = (1, 0, 0, 0, 1, 0)$ $h = (0, 3, 0, 0, 0, 0)$ $i = (0, 1, 0, 1, 0, 0)$ $k = (0, 0, 2, 0, 0, 0)$ $l = (0, 0, 0, 0, 0, 1)$
7	$a = (7, 0, 0, 0, 0, 0, 0)$ $b = (5, 1, 0, 0, 0, 0, 0)$ $c = (4, 0, 1, 0, 0, 0, 0)$ $d = (3, 2, 0, 0, 0, 0, 0)$ $e = (3, 0, 0, 1, 0, 0, 0)$ $f = (2, 1, 1, 0, 0, 0, 0)$ $g = (2, 0, 0, 0, 1, 0, 0)$

<i>Degree</i>	<i>Permutation types</i>
	$h = (1, 3, 0, 0, 0, 0, 0)$ $i = (1, 1, 0, 1, 0, 0, 0)$ $k = (1, 0, 2, 0, 0, 0, 0)$ $l = (1, 0, 0, 0, 0, 1, 0)$ $m = (0, 2, 1, 0, 0, 0, 0)$ $n = (0, 1, 0, 0, 1, 0, 0)$ $o = (0, 0, 1, 1, 0, 0, 0)$ $p = (0, 0, 0, 0, 0, 0, 1)$
8	$a = (8, 0, 0, 0, 0, 0, 0, 0)$ $b = (5, 0, 1, 0, 0, 0, 0, 0)$ $c = (4, 2, 0, 0, 0, 0, 0, 0)$ $d = (3, 0, 0, 0, 1, 0, 0, 0)$ $e = (2, 3, 0, 0, 0, 0, 0, 0)$ $f = (2, 1, 0, 1, 0, 0, 0, 0)$ $g = (2, 0, 2, 0, 0, 0, 0, 0)$ $h = (2, 0, 0, 0, 0, 1, 0, 0)$ $i = (1, 2, 1, 0, 0, 0, 0, 0)$ $k = (1, 0, 0, 0, 0, 0, 1, 0)$ $l = (0, 4, 0, 0, 0, 0, 0, 0)$ $m = (0, 1, 0, 0, 0, 1, 0, 0)$ $n = (0, 0, 1, 0, 1, 0, 0, 0)$ $o = (0, 0, 0, 2, 0, 0, 0, 0)$ $p = (0, 0, 0, 0, 0, 0, 0, 1)$
9	$a = (9, 0, 0, 0, 0, 0, 0, 0, 0)$ $b = (3, 3, 0, 0, 0, 0, 0, 0, 0)$ $c = (3, 0, 2, 0, 0, 0, 0, 0, 0)$ $d = (2, 0, 0, 0, 0, 0, 1, 0, 0)$ $e = (1, 4, 0, 0, 0, 0, 0, 0, 0)$ $f = (1, 1, 0, 0, 0, 1, 0, 0, 0)$ $g = (1, 0, 0, 2, 0, 0, 0, 0, 0)$ $h = (1, 0, 0, 0, 0, 0, 0, 1, 0)$ $i = (0, 0, 3, 0, 0, 0, 0, 0, 0)$ $k = (0, 0, 1, 0, 0, 1, 0, 0, 0)$ $l = (0, 0, 0, 0, 0, 0, 0, 0, 1)$
10	$a = (10, 0, 0, 0, 0, 0, 0, 0, 0, 0)$ $b = (4, 3, 0, 0, 0, 0, 0, 0, 0, 0)$ $c = (2, 4, 0, 0, 0, 0, 0, 0, 0, 0)$ $d = (2, 0, 0, 2, 0, 0, 0, 0, 0, 0)$ $e = (2, 0, 0, 0, 0, 0, 0, 1, 0, 0)$ $f = (1, 0, 3, 0, 0, 0, 0, 0, 0, 0)$ $g = (1, 0, 1, 0, 0, 1, 0, 0, 0, 0)$ $h = (0, 5, 0, 0, 0, 0, 0, 0, 0, 0)$ $i = (0, 1, 0, 2, 0, 0, 0, 0, 0, 0)$ $k = (0, 1, 0, 0, 0, 0, 0, 1, 0, 0)$

<i>Degree</i>	<i>Permutation types</i>
	$l = (0, 0, 0, 0, 2, 0, 0, 0, 0, 0)$ $m = (0, 0, 0, 0, 0, 0, 0, 0, 0, 1)$
11	$a = (11, 0, 0, 0, 0, 0, 0, 0, 0, 0)$ $b = (3, 4, 0, 0, 0, 0, 0, 0, 0, 0)$ $c = (3, 0, 0, 2, 0, 0, 0, 0, 0, 0)$ $d = (2, 0, 3, 0, 0, 0, 0, 0, 0, 0)$ $e = (1, 5, 0, 0, 0, 0, 0, 0, 0, 0)$ $f = (1, 1, 0, 0, 0, 0, 0, 1, 0, 0)$ $g = (1, 0, 0, 0, 2, 0, 0, 0, 0, 0)$ $h = (1, 0, 0, 0, 0, 0, 0, 0, 1, 0)$ $i = (0, 1, 1, 0, 0, 1, 0, 0, 0, 0)$ $k = (0, 0, 0, 0, 0, 0, 0, 0, 0, 1)$
12	$a = (12, 0, 0, 0, 0, 0, 0, 0, 0, 0)$ $b = (4, 4, 0, 0, 0, 0, 0, 0, 0, 0)$ $c = (3, 0, 3, 0, 0, 0, 0, 0, 0, 0)$ $d = (2, 5, 0, 0, 0, 0, 0, 0, 0, 0)$ $e = (2, 0, 0, 0, 2, 0, 0, 0, 0, 0)$ $f = (2, 0, 0, 0, 0, 0, 0, 0, 1, 0)$ $g = (1, 1, 1, 0, 0, 1, 0, 0, 0, 0)$ $h = (1, 0, 0, 0, 0, 0, 0, 0, 0, 1)$ $i = (0, 6, 0, 0, 0, 0, 0, 0, 0, 0)$ $k = (0, 2, 0, 2, 0, 0, 0, 0, 0, 0)$ $l = (0, 0, 4, 0, 0, 0, 0, 0, 0, 0)$ $m = (0, 0, 0, 3, 0, 0, 0, 0, 0, 0)$ $n = (0, 0, 0, 1, 0, 0, 0, 1, 0, 0)$ $o = (0, 0, 0, 0, 0, 2, 0, 0, 0, 0)$ $p = (0, 0, 0, 0, 0, 0, 0, 0, 0, 1)$
13	$a = (13, 0, 0, 0, 0, 0, 0, 0, 0, 0)$ $b = (5, 4, 0, 0, 0, 0, 0, 0, 0, 0)$ $c = (4, 0, 3, 0, 0, 0, 0, 0, 0, 0)$ $d = (2, 1, 1, 0, 0, 1, 0, 0, 0, 0)$ $e = (1, 6, 0, 0, 0, 0, 0, 0, 0, 0)$ $f = (1, 2, 0, 2, 0, 0, 0, 0, 0, 0)$ $g = (1, 0, 4, 0, 0, 0, 0, 0, 0, 0)$ $h = (1, 0, 0, 3, 0, 0, 0, 0, 0, 0)$ $i = (1, 0, 0, 1, 0, 0, 0, 1, 0, 0)$ $k = (1, 0, 0, 0, 0, 2, 0, 0, 0, 0)$ $l = (1, 0, 0, 0, 0, 0, 0, 0, 1, 0)$ $m = (0, 0, 0, 0, 0, 0, 0, 0, 0, 1)$
14	$a = (14, 0, 0, 0, 0, 0, 0, 0, 0, 0)$ $b = (2, 6, 0, 0, 0, 0, 0, 0, 0, 0)$ $c = (2, 0, 4, 0, 0, 0, 0, 0, 0, 0)$ $d = (2, 0, 0, 3, 0, 0, 0, 0, 0, 0)$

<i>Degree</i>	<i>Permutation types</i>
	$e = (2, 0, 0, 0, 0, 2, 0, 0, 0, 0, 0, 0, 0, 0)$ $f = (2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0)$ $g = (1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0)$ $h = (0, 7, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$ $i = (0, 0, 0, 0, 0, 0, 2, 0, 0, 0, 0, 0, 0)$ $k = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1)$
15	$a = (15, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$ $b = (7, 4, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$ $c = (3, 6, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$ $d = (3, 2, 0, 2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$ $e = (3, 0, 4, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$ $f = (1, 1, 2, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0)$ $g = (1, 1, 0, 3, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$ $h = (1, 0, 0, 0, 0, 0, 2, 0, 0, 0, 0, 0, 0, 0)$ $i = (0, 0, 5, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$ $k = (0, 0, 1, 0, 0, 2, 0, 0, 0, 0, 0, 0, 0, 0)$ $l = (0, 0, 0, 0, 3, 0, 0, 0, 0, 0, 0, 0, 0, 0)$ $m = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1)$
16	$a = (16, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$ $b = (8, 4, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$ $c = (4, 6, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$ $d = (4, 0, 4, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$ $e = (2, 1, 2, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0)$ $f = (2, 1, 0, 3, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$ $g = (1, 0, 5, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$ $h = (1, 0, 1, 0, 0, 2, 0, 0, 0, 0, 0, 0, 0, 0, 0)$ $i = (1, 0, 0, 0, 3, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$ $k = (1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0)$ $l = (0, 8, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$ $m = (0, 2, 0, 0, 0, 2, 0, 0, 0, 0, 0, 0, 0, 0, 0)$ $n = (0, 0, 0, 4, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$ $o = (0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0)$ $p = (0, 0, 0, 0, 0, 0, 0, 2, 0, 0, 0, 0, 0, 0, 0)$
17	$a = (17, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$ $b = (5, 6, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$ $c = (3, 1, 0, 3, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$ $d = (2, 0, 5, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$ $e = (2, 0, 1, 0, 0, 2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$ $f = (2, 0, 0, 0, 3, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$ $g = (2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0)$ $h = (1, 8, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$ $i = (1, 0, 0, 4, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$ $k = (1, 0, 0, 0, 0, 0, 0, 2, 0, 0, 0, 0, 0, 0, 0, 0)$

<i>Degree</i>	<i>Permutation types</i>
	$l = (1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0)$ $m = (0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0)$ $n = (0, 1, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0)$ $o = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1)$
18	$a = (18, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$ $b = (2, 8, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$ $c = (2, 0, 0, 0, 4, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$ $d = (2, 0, 0, 0, 0, 0, 0, 2, 0, 0, 0, 0, 0, 0, 0, 0)$ $e = (2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0)$ $f = (1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1)$ $g = (0, 9, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$ $h = (0, 0, 6, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$ $i = (0, 0, 0, 0, 0, 3, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$ $k = (0, 0, 0, 0, 0, 0, 0, 2, 0, 0, 0, 0, 0, 0, 0, 0)$ $l = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1)$
19	$a = (19, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$ $b = (1, 9, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$ $c = (1, 0, 6, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$ $d = (1, 0, 0, 0, 0, 3, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$ $e = (1, 0, 0, 0, 0, 0, 0, 2, 0, 0, 0, 0, 0, 0, 0, 0)$ $f = (1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1)$ $g = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1)$
20	$a = (20, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$ $b = (2, 9, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$ $c = (2, 0, 6, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$ $d = (2, 0, 0, 0, 0, 3, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$ $e = (2, 0, 0, 0, 0, 0, 0, 2, 0, 0, 0, 0, 0, 0, 0, 0)$ $f = (2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0)$ $g = (1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1)$ $h = (0, 10, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$ $i = (0, 0, 0, 5, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$ $k = (0, 0, 0, 0, 4, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$ $l = (0, 0, 0, 0, 0, 0, 0, 2, 0, 0, 0, 0, 0, 0, 0, 0)$ $m = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1)$

TABLE 2
Frequencies of permutation types in primitive
permutation groups up to degree 20

<i>Degree</i>	<i>Group</i>	<i>Order</i>	<i>Occurring frequencies</i>
3	3.2	6	1a, 3b, 2c
4	4.1	12	1a, 8c, 3d
	4.2	24	a, 6b, 8c, 3d, 6e

<i>Degree</i>	<i>Group</i>	<i>Order</i>	<i>Occurring frequencies</i>
5	5.1	5	1a, 4g
	5.2	10	1a, 5d, 4g
	5.3	20	1a, 5d, 10e, 4g
	5.4	60	1a, 20c, 15d, 24g
	5.5	120	1a, 10b, 20c, 15d, 30e, 20f, 24g
6	6.1	60	1a, 15d, 24g, 20k
	6.2	120	1a, 15d, 30e, 24g, 10h, 20k, 20l
	6.3	360	1a, 40c, 45d, 144g, 90i, 40k
	6.4	720	1a, 15b, 40c, 45d, 90e, 120f, 144g, 15h, 90i, 40k, 120l
7	7.1	7	1a, 6p
	7.2	14	1a, 7h, 6p
	7.3	21	1a, 14k, 6p
	7.4	42	1a, 7h, 14k, 14l, 6p
	7.5	168	1a, 21d, 56k, 42i, 48p
	7.6	2520	1a, 70c, 105d, 504g, 630i, 280k, 210m, 720p
	7.7	5040	1a, 21b, 70c, 105d, 210e, 420f, 504g, 105h, 630i, 280k, 840l, 210m, 504n, 420o, 720p
8	8.1	56	1a, 48k, 7l
	8.2	168	1a, 56g, 48k, 7l, 56m
	8.3	168	1a, 56g, 48k, 21l, 42o
	8.4	336	1a, 28e, 56g, 56h, 48k, 21l, 42o, 84p
	8.5	1344	1a, 42c, 168f, 224g, 384k, 49l, 224m, 252o
	8.6	20160	1a, 112b, 210c, 1344d, 2520f, 1120g, 1680i, 5760k, 105l, 3360m, 2688n, 1260o
9	9.1	36	1a, 9e, 18g, 8i
	9.2	72	1a, 12b, 9e, 18g, 8i, 24k
	9.3	72	1a, 9e, 18g, 36h, 8i
	9.4	72	1a, 9e, 54g, 8i
	9.5	144	1a, 12b, 9e, 54g, 36h, 8i, 24k
	9.6	216	1a, 24c, 9e, 72f, 54g, 56i
	9.7	432	1a, 36b, 24c, 9e, 72f, 54g, 108h, 56i, 72k
	9.8	504	1a, 216d, 63e, 56i, 168l
	9.9	1512	1a, 168c, 216d, 63e, 504f, 56i, 504l
10	10.1	60	1a, 15c, 20f, 24l
	10.2	120	1a, 10b, 15c, 20f, 20g, 30i, 24l
	10.3	360	1a, 45c, 90d, 80f, 144l
	10.4	720	1a, 30b, 45c, 90d, 80f, 240g, 90i, 144l
	10.5	720	1a, 45c, 90d, 180e, 80f, 36h, 144l, 144m
	10.6	720	1a, 45c, 270d, 80f, 180k, 144l
	10.7	1440	1a, 30b, 45c, 270d, 180e, 80f, 240g, 36h, 90i, 180k, 144l, 144m

<i>Degree</i>	<i>Group</i>	<i>Order</i>	<i>Occurring frequencies</i>
11	11.1	11	1a, 10k
	11.2	22	1a, 11e, 10k
	11.3	55	1a, 44g, 10k
	11.4	110	1a, 11e, 44g, 44h, 10k
	11.5	660	1a, 55b, 110d, 264g, 110i, 120k
	11.6	7920	1a, 165b, 990c, 440d, 1980f, 1584g, 1320i, 1440k
12	12.1	660	1a, 264e, 120h, 55i, 110l, 110o
	12.2	1320	1a, 66d, 264e, 264f, 120h, 55i, 110l, 110m, 110o, 220p
	12.3	7920	1a, 165b, 440c, 1584e, 1320g, 1440h, 990k, 1980n
13	13.1	13	1a, 12m
	13.2	26	1a, 13e, 12m
	13.3	39	1a, 26g, 12m
	13.4	52	1a, 13e, 26h, 12m
	13.5	78	1a, 13e, 26g, 26k, 12m
	13.6	156	1a, 13e, 26g, 26h, 26k, 52l, 12m
	13.7	5616	1a, 117b, 104c, 936d, 702f, 624g, 1404i, 1728m
14	14.1	1092	1a, 91b, 182c, 182e, 168g, 468i
	14.2	2184	1a, 91b, 182c, 182d, 182e, 364f, 168g, 78h, 468i, 468k
15	15.1	360	1a, 45c, 40e, 90g, 40i, 144l
	15.2	720	1a, 15b, 60c, 40e, 120f, 180g, 40i, 120k, 144l
	15.3	2520	1a, 105c, 280e, 630g, 720h, 70i, 210k, 504l
	15.4	20160	1a, 105b, 210c, 1260d, 1120e, 3360f, 2520g, 5760h, 112i, 1680k, 1344l, 2688m
16	16.1	80	1a, 64i, 15l
	16.2	160	1a, 20c, 64i, 15l, 60n
	16.3	240	1a, 32g, 64i, 128k, 15l
	16.4	288	1a, 12c, 16d, 64g, 96h, 15l, 48m, 36n
	16.5	320	1a, 20c, 80f, 64i, 15l, 60n, 80p
	16.6	480	1a, 20c, 32g, 160h, 64i, 128k, 15l, 60n
	16.7	576	1a, 36c, 16d, 144f, 64g, 15l, 48m, 108n, 144p
	16.8	576	1a, 60c, 16d, 64g, 192h, 15l, 48m, 180n
	16.9	960	1a, 60c, 80d, 384i, 15l, 240m, 180n
	16.10	960	1a, 20c, 240f, 32g, 160h, 64i, 128k, 15l, 60n, 240p
	16.11	960	1a, 60c, 320g, 384i, 15l, 180n
	16.12	1152	1a, 12b, 60c, 16d, 96e, 144f, 64g, 192h, 51l, 48m, 228n, 96o, 144p
	16.13	1920	1a, 20b, 60c, 80d, 160e, 240f, 384i, 75l, 240m, 260n, 160o, 240p
	16.14	1920	1a, 100c, 240f, 320g, 320h, 384i, 15l, 300n, 240p
	16.15	2880	1a, 60c, 160d, 352g, 480h, 384i, 768k, 15l, 480m, 180n

<i>Degree</i>	<i>Group</i>	<i>Order</i>	<i>Occurring frequencies</i>
	16.16	5760	1a, 180c, 160d, 720f, 352g, 1440h, 384i, 768k, 15l, 480m, 540n, 720p
	16.17	5760	1a, 180c, 160d, 720f, 640g, 2304i, 15l, 480m, 540n, 720p
	16.18	11520	1a, 30b, 240c, 160d, 960e, 1440f, 640g, 1920h, 2304i, 105l, 480m, 840n, 960o, 1440p
17	17.1	17	1a, 16o
	17.2	34	1a, 17h, 16o
	17.3	68	1a, 17h, 34i, 16o
	17.4	136	1a, 17h, 34i, 68k, 16o
	17.5	272	1a, 17h, 34i, 68k, 136l, 16o
	17.6	4080	1a, 272d, 544f, 1088g, 255h, 1920o
	17.7	8160	1a, 68b, 272d, 1360e, 544f, 1088g, 255h, 1020i, 1632n, 1920o
	17.8	16320	1a, 68b, 1360c, 272d, 1360e, 544f, 1088g, 255h, 1020i, 4080k, 2720m, 1632n, 1920o
18	18.1	2448	1a, 153b, 306c, 612d, 288f, 272h, 816k
	18.2	4896	1a, 153b, 306c, 612d, 1224e, 288f, 136g, 272h, 272i, 816k, 816l
19	19.1	19	1a, 18g
	19.2	38	1a, 19b, 18g
	19.3	57	1a, 38c, 18g
	19.4	114	1a, 19b, 38c, 38d, 18g
	19.5	171	1a, 38c, 114e, 18g
	19.6	342	1a, 19b, 38c, 38d, 114e, 114f, 18g
20	20.1	3420	1a, 380c, 1140e, 360g, 171h, 684k, 684l
	20.2	6840	1a, 190b, 380c, 380d, 1140e, 1140f, 360g, 171h, 342i, 684k, 684l, 1368m

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