Modular Functions Arising From Some Finite Groups

By Larissa Queen

Abstract. In [2] Conway and Norton have assigned a "Thompson series" of the form

$$q^{-1} + H_1q + H_2q^2 + \dots$$

to each element m of the Fischer-Griess "Monster" group M and conjectured that these functions are Hauptmoduls for certain genus-zero modular groups. We have found, for a large number of values of N , all the genus-zero groups between $\Gamma_0(N)$ and $PSL(2, R)$ that have Hauptmoduls of the above form, and this provides the necessary verification that the series assigned in [2] to particular elements of M really are such Hauptmoduls. (Atkin and Fong [1] have recently verified that $H_n(m)$ really is a character of M for all n .) We compute Thompson series for various finite groups and discuss the differences between these groups and M . We find that the resulting Thompson series are all Hauptmoduls for suitable genus-zero subgroups of $PSL(2, R)$.

1. Summary. Some remarkable connections between the Fischer-Griess "Monster" group M and modular functions have recently been reported in [2] and [5]. It has been noticed that the coefficients in the q -series for j

$$j(z) = \sum_{n=-1}^{\infty} c_n q^n = q^{-1} + 744 + 196884q + 21493760q^2 + \dots,$$

where j is the fundamental modular function on $\Gamma = PSL(2, Z)$ and $q = e^{2\pi iz}$, are linear combinations of the character degrees d_n of M . By considering $J = j - 744$, i.e., disregarding the constant term, and replacing the coefficients c_n in the q -series for J by the corresponding representations of M , one obtains a formal power series [5]

$$H_{-1}q^{-1} + 0 + H_1q + H_2q^2 + H_3q^3 + \dots,$$

where $q = e^{2\pi iz}$ and H_n is a representation of degree c_n known as a head representation. Replacing H_n by its character value $H_n(m)$ for various elements $m \in M$, we obtain other functions [5]:

$$T_m(z) = q^{-1} + H_1(m)q + H_2(m)q^2 + H_3(m)q^3 + \dots,$$

which are called the Thompson series in [2]. The $H_i(m)$ are called head characters. Conway and Norton [2] have computed the functions $T_m(z)$ for all elements m of M , and they conjectured that, for every $m \in M$, $T_m(z)$ is a Hauptmodul for a group $F(m)$ between $\Gamma_0(N)$ for some N and its normalizer in $PSL(2, R)$, i.e., $T_m(z)$ generates a genus-zero function field invariant under $F(m)$. In this paper we describe certain related calculations. The detailed working can be found in [4]. We have explicitly verified that the modular functions assigned to various m by

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Conway and Norton [2] are actually Hauptmoduls for the groups they mention. Atkin and Fong [1] have recently verified that $H_n(m)$ really is a character of M for all n .

We have also considered other finite groups, usually derived from centralizers of elements of M , and computed their head character tables. In particular, we consider

.0 = 2.(.1)	.1 is the Conway simple group
E	Thompson's simple group
3.2.Suz	Suz is Suzuki's sporadic simple group
2.HJ	HJ is the Hall-Janko simple group
F	Harada-Norton simple group
2. A_7	Schur double cover of A_7
H	Held's simple group
M_{12}	Mathieu simple group

for which we have

elements of M	centralizer in M	
$2B$	$2^{1+24} \cdot G$	$G = .1$
$3B$	$3^{1+12} \cdot G$	$G = 2.\text{Suz}$
$3C$	$3 \times E$	
$5A$	$5 \times F$	
$5B$	$5^{1+6} \cdot G$	$G = 2.\text{HJ}$
$7A$	$7 \times H$	
$7B$	$7^{1+4} \cdot G$	$G = 2.A_7$
$11A$	$11 \times M_{12}$	

2. Notation and Terminology. As usual, for a positive integer N , we define

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : a \equiv d \equiv 1 \pmod{N}, b \equiv c \equiv 0 \pmod{N} \right\}$$

and

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : c \equiv 0 \pmod{N} \right\}.$$

We also define

$$\Gamma_0(n; h) = \left\{ \begin{pmatrix} a & b/h \\ cnh & d \end{pmatrix} : ad - bcn = 1 \right\},$$

which is a subgroup of $PSL(2, R)$ and is conjugate to $\Gamma_0(n)$. In [2] this group is denoted by $\Gamma_0(nh|h)$; however, we prefer to reserve this name to denote a subgroup of $\Gamma_0(n; h)$ of index h which has Hauptmodul $T_{nh|h}$ (see below). Thus, adapting the rest of the notation from [2], we always have the same name for the genus-zero subgroup of $PSL(2, R)$ and the corresponding Hauptmodul in the canonical form (i.e. beginning $q^{-1} + 0 + aq + \dots$). Also, this notation provides a natural way of enumerating all discrete subgroups of $PSL(2, R)$ containing $\Gamma_0(N)$ for a given N ; see [4].

Thus, we write

$$\begin{aligned} \Gamma_0(n: h) + e, f, \dots & \quad \text{for } \langle \Gamma_0(n: h), w_e, w_f, \dots \rangle, \\ \Gamma_0(n: h) + & \quad \text{when all } w_e \text{ for } \Gamma_0(n: h) \text{ are present,} \\ \Gamma_0(n: h) - \text{ or } \Gamma_0(n: h) & \quad \text{when no } w_e \neq 1 \text{ is present,} \end{aligned}$$

where

$$w_e = \left\{ \begin{pmatrix} ae & b/h \\ chn & de \end{pmatrix} : e \text{ divides } n \text{ exactly, and } ade^2 - bcn = e > 0 \right\}$$

is a single coset of $\Gamma_0(n: h)$. The w_e are called the Atkin-Lehner involutions for $\Gamma_0(n: h)$ [2]. The corresponding Hauptmoduls (when they exist) are denoted by $t_{n+ef, \dots}(hz)$, $t_{n+}(hz)$ and $t_{n-}(hz)$ or $t_n(hz)$, respectively. $\Gamma_0(n)$ and its Hauptmodul $t_n(z)$ are a particular case when $h = 1$.

$$T_{n+ef, \dots}(hz) = t_{n+ef, \dots}(hz) - \text{constant term}$$

is the canonical Hauptmodul for $\Gamma_0(n: h) + e, f, \dots$.

If F is a subgroup of $\Gamma_0(n: h) + e, f, \dots$ of index h , with Hauptmodul T and $(T(z))^h = T_n(hz) + K$, where K is a constant, we denote F by $\Gamma_0(nh | h) + e, f, \dots$ and T by $T_{nh|h+ef, \dots}$.

In this work we also define

$$\begin{aligned} \Gamma_0\left(n \frac{f}{g}\right) &= \begin{pmatrix} g & f \\ 0 & g \end{pmatrix} \Gamma_0(n) \begin{pmatrix} g & -f \\ 0 & g \end{pmatrix}, \\ \Gamma_0\left(n \frac{f}{g}\right) + &= \begin{pmatrix} g & f \\ 0 & g \end{pmatrix} \Gamma_0(n) + \begin{pmatrix} g & -f \\ 0 & g \end{pmatrix}, \\ \Gamma_0\left(n \frac{f}{g} h: h\right) + &= \begin{pmatrix} 1 & 0 \\ 0 & h \end{pmatrix} \Gamma_0\left(n \frac{f}{g}\right) + \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix}, \end{aligned}$$

where g is such that $g \mid 24$ and $(g, f) = 1$. The corresponding functions are

$$t_{n(f/g)}(z) = e^{2\pi i f / g} t_n(z + f/g), \quad t_{n(f/g)+}(z) = e^{2\pi i f / g} t_{n+}(z + f/g),$$

and $t_{n(f/g)+}(hz)$, respectively. These are used to label a wide class of groups and functions arising from various finite groups G considered in this paper.

From [2] we quote certain identities which are called there replication formulae:

$$\begin{aligned} \frac{1}{2} \{ T^2 - T_{(2)}(2z) \} &= \{ H_2 q + H_4 q^2 + \dots \} + H_1 \quad (\text{duplication}), \\ \frac{1}{3} \{ T^3 - T_{(3)}(3z) \} &= \{ H_3 q + H_6 q^2 + \dots \} + H_1 T + H_2 \quad (\text{triplication}), \\ \frac{1}{5} \{ T^5 - T_{(5)}(5z) \} &= \{ H_5 q + H_{10} q^2 + \dots \} + H_1 T^3 + H_2 T^2 \\ &\quad + (H_3 - H_1^2) T + (H_4 - H_2 H_1), \end{aligned}$$

where $T = T_m(z)$, $T_{(n)} = T_{m^n}$ and $H_r = H_r(m)$, for $m \in M$. Many identities are obtained by comparing powers of q .

3. Discussion of Calculations and Results. When computing their Thompson series, the groups mentioned above fall naturally into two classes. The first one consists of $G_2 = .0$, $G_3 = 3.2.\text{Suz}$, $G_5 = 2.\text{HJ}$ and $G_7 = 2.A_7$. Each of these groups has a central element -1 and two algebraically conjugate representations $[d_+]$ and $[d_-]$ of degree $d = 24/(p-1)$, where p is the order of the element of M whose

centralizer involves G_p for $p = 2, 3, 5, 7$. If G is one of these groups, then for every $g \in G$ the Thompson series t_g is given by the formula

$$\begin{aligned} t_g(z) &= q^{-1} \prod_{p \nmid n} (1 - \varepsilon_1 q^n)(1 - \varepsilon_2 q^n) \cdots (1 - \varepsilon_d q^n) \\ &= q^{-1} + H_0(g) + H_1(g)q + H_2(g)q^2 + \dots \end{aligned}$$

($q = e^{2\pi iz}$), where the ε 's are the eigenvalues of g in the representation $[d_{(n/p)}]$, and (n/p) is the Legendre symbol. The $H_i(g)$ is a generalized character of G , while it can be shown that $H_i(-g)$ is a proper character; see [2]. However, our calculations show that the replication order of t_g is always less than or equal to the replication order of t_{-g} , and so they cannot be interchanged. We also observe that, as was not the case for M , the constant term is significant. In fact, $H_0(-g)$ is the character of G corresponding to the representation $[d_+]$.

The second class includes E, F, H and M_{12} . To compute the Thompson series for $g \in G$, where G is one of the groups above, we have to find, by trial and error, linear combinations of irreducible characters of G that work, i.e., such that the resulting Thompson series can be identified as modular functions for certain discrete subgroups of $PSL(2, R)$. Of course, such linear combinations do not have to exist, and in fact it is quite amazing that they do. For these groups, $H_i(g)$ is a proper character of G and the constant term is immaterial.

Tables I–VIII give values of head characters for these groups, together with the decomposition of the $H_i(g)$ into the irreducible characters of G . We found that to every element $g \in G$, where G is one of the groups above, there corresponds a function

$$t_g(z) = q^{-1} + H_0(g) + H_1(g)q + H_2(g)q^2 + \dots,$$

which can be identified as a Hauptmodul for a discrete subgroup $F = F(g)$ of $PSL(2, R)$ containing $\Gamma_0(N)$ for some N and such that $F_\infty = \langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \rangle$, where F_∞ is the stabilizer of the cusp at $z = i\infty$. However, $F(g)$ is not necessarily contained in the normalizer of $\Gamma_0(N)$ in $PSL(2, R)$, as was the case for M . Tables I–VIII also include the corresponding $\Gamma_0(N)$ and the type t (i.e., the name of the fixing group and the corresponding Hauptmodul) of $t_g(z)$ for every $g \in G$.

Some observations resulting from this work, in particular some necessary alterations in the replication formulae and the more general form of fixing group $F(g)$, have already been reported upon in [2].

Let G be one of the groups considered above. Let p be the order of the element of M from whose centralizer G was derived. If $(n/p) = -1$, $H_n(g)$ are rational for all $g \in G$, and the n -tuplication formulae are used with algebraic conjugation.

If $g \in G$ of order s such that $(s, p) = 1$, then its Thompson series T_g is the same as T_m for some $m \in M$, i.e., we can obtain new functions only from the elements of G whose order is divisible by p .

Consider $T = T_g$, where $g \in M$ or $g \in G$ for one of the groups G discussed above. The replication formulae [2] define a function $T_{(n)}$, called the n -tuplicate of T . If $T = T_m$ for $m \in M$,

$$T_{(n)} = T_{m^n}.$$

For an arbitrary Thompson series T , we say that T has replication order n if

$$T_{(n)} = J.$$

We note that J is a self-replicating function, and it assumes the role of the identity.

Let G and p be as above and let $T = T_g$ for $g \in G$. If $(n, p) = 1$,

$$T_{(n)} = T_{g^n}.$$

Hence, if $g \in G$ and $o(g) = s$ such that $(s, p) = 1$,

$$T_{(s)} = T_{1A(G)},$$

where $1A(G)$ is the identity element of G , and T has replication order ps . On the other hand, if $(n, p) \neq 1$,

$$T_{(n)} = T_m,$$

for some $m \in M$. For example, we found in F that T_{5A} has replication order 5, T_{10B} has replication order 10, T_{15A} has replication order 15, T_{20C} has replication order 20 and T_{25A} , T_{25B} have replication order 25, where $5A$, $10B$, $15A$, $20C$, $25A$, and $25B$ are elements of order 5, 10, 15, 20, and 25, respectively.

We would like to note that since $G = 3.2.\text{Suz}$ contains a central element ω of order 3, for every $t_g(z)$ given in Table II we also have

$$t_{\omega g}(z) = \omega t_g(z + 1/3)$$

and

$$t_{\bar{\omega}g}(z) = \bar{\omega} t_g(z + 2/3) = t_{\omega g}^*(z),$$

where $*$ denotes algebraic conjugation.

In our calculations we used the character tables from [3].

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TABLE I
Values of head characters for .0

		-1A +1A	-2A +2A	+2B	+2C	-3A +3A	-3B +3B	-3C +3C
H_{-1}	1	1	1	1	1	1	1	1
H_0	24	+24	+8	0	0	+12	+6	+3
H_1	276	276	20	12	-12	78	15	6
H_2	24+2024	+2048	0	0	0	+364	+32	+4
H_3	1+276+299+1771+8855	11202	-62	66	66	1365	87	3
H_4	24 ² +2024 ² +4576+40480	+49152	0	0	0	+4380	+192	+12
H_5	1+276 ⁴ +299+1771 ² +8855+37674 ² +94875	184024	216	232	-232	12520	343	-8
H_6		+614400	0	0	0	+32772	+672	+12
H_7		1881471	-641	639	639	80094	1290	30
H_8		+5373952	0	0	0	+185276	+2176	+20
H_9		14478180	1636	1596	-1596	409578	3705	30
H_{10}		+37122048	0	0	0	+871272	+6336	+72
$\Gamma_o(N)$		4 2	4 4	16	8	12 6	12 6	12 6
t		4+ 2-	4- 4-	8 2+	4 2-	6 ¹ +6 6+6	12+ 6+3	12+4 6-

TABLE I (continued)

	-3D +3D	-4A +4A	+4B -4B	-4C +4C	+4D -4D	+4E -4E	+4F -4F	-5A +5A	-5B +5B	-5C +5C	-6A +6A	+6B -6B	-6C +6C	-6D +6D
H_{-1}	1	1	1	1	1	1	1	1	1	1	1	1	1	1
H_0	0	+8	0	4	0	0	0	+6	+4	+1	+4	0	+4	+5
H_1	0	36	4	4	-4	0	0	21	6	1	14	-6	5	14
H_2	+8	+128	0	0	0	0	0	+62	+8	+2	+36	0	0	+36
H_3	0	386	2	2	2	6	-6	162	17	2	85	21	-5	85
H_4	0	+1024	0	0	0	0	0	+378	+32	+2	+180	0	0	+180
H_5	28	2488	-8	-8	8	0	0	819	54	1	360	-56	9	360
H_6	0	+5632	0	0	0	0	0	+1680	+80	0	+684	0	0	+684
H_7	0	12031	-1	-1	-1	15	15	3276	116	4	1246	126	-14	1246
H_8	+64	+24576	0	0	0	0	0	+6138	+192	+2	+2196	0	0	+2196
H_9	0	48308	20	20	20	0	0	11145	290	5	3754	-258	19	3754
H_{10}	0	+91904	0	0	0	0	0	+19662	+408	+2	+6264	0	0	+6264
$\Gamma_o(N)$	36 18	8 8	8 8	8 8	16 16	64 64	32 32	20 10	20 10	20 10	12 12	48 48	12 12	12 12
t	12 3+ 6 3	8 $\frac{1}{2}$ + 8+	8- 8-	8- 8-	8 2- 8 2-	16 4+ 16 4+	8 4- 8 4-	10 $\frac{1}{2}$ +10 10+10	20+ 10+5	20+4 10-	12+12 12 $\frac{1}{2}$ +12	12 $\frac{1}{2}$ 2+6 12 $\frac{1}{2}$ 2+6	12 2+3 12 2+3	12 $\frac{1}{2}$ +12 12+12

TABLE I (continued)

	-6E +6E	-6F +6F	+6G	+6H	+6I	-7A +7A	-7B +7B	+8A	-8C +8C	+8D	-8E +8E	+8F	-9A +9A	-9B +9B
H_{-1}	1	1	1	1	1	1	1	1	1	1	1	1	1	1
H_0	-2	+1	0	0	0	+4	+3	0	+4	0	+2	0	+3	0
H_1	-1	2	-3	0	0	10	3	4	8	0	0	0	6	0
H_2	0	0	0	0	0	+24	+4	0	+16	0	0	0	+13	+1
H_3	7	1	3	0	0	51	9	10	34	2	2	-2	24	0
H_4	0	0	0	0	0	+100	+12	0	+64	0	0	0	+42	0
H_5	-9	0	-7	4	-4	190	15	24	112	0	0	0	73	1
H_6	0	0	0	0	0	+340	+24	0	+192	0	0	0	+120	0
H_7	10	-2	18	0	0	585	39	47	319	-1	-1	-1	192	0
H_8	0	0	0	0	0	+984	+52	0	+512	0	0	0	+299	+1
H_9	-23	-2	-21	0	0	1606	66	84	808	0	0	0	456	0
H_{10}	0	0	0	0	0	+2564	+96	0	+1248	0	0	0	+684	0
$\Gamma_o(N)$	12 12	12 12	24	114	72	28 14	28 14	32	16 16	16	16 16	64	36 18	36 18
t	12+3 12+3	12- 12-	12 2+3	24 6+	12 6	14 $\frac{1}{2}$ +14 14+14	28+ 14+7	16 2+ 16 $\frac{1}{2}$ 2+	16 $\frac{1}{2}$ + 16+	16- 16-	16- 16-	16 $\frac{1}{8}$ -	18 $\frac{1}{2}$ +18 18+18	36+4 18-

TABLE I (continued)

	-9C +9C	-10A +10A	$\pm 10B$	$\pm 10C$	-10D +10D	-10E +10E	$\pm 10F$	-11A +11A	-12A +12A	$\pm 12B$	$\pm 12C$	-12D +12D	-12E +12E
H_{-1}	1	1	1	1	1	1	1	1	1	1	1	1	1
H_0	± 3	± 2	0	0	± 2	± 3	0	± 2	± 4	0	0	± 1	± 2
H_1	3	5	-3	2	0	5	-2	1	6	-2	2	0	3
H_2	± 2	± 10	0	0	0	± 10	0	± 2	± 4	0	0	± 2	± 8
H_3	3	18	6	1	3	18	1	4	5	5	5	-1	11
H_4	± 6	± 30	0	0	0	± 30	0	± 4	± 20	0	0	± 2	± 16
H_5	10	51	-13	2	-4	51	-2	5	40	-8	8	4	31
H_6	± 12	± 80	0	0	0	± 80	0	± 6	± 44	0	0	± 2	± 40
H_7	15	124	24	4	4	124	4	9	46	14	14	-2	58
H_8	± 22	± 190	0	0	0	± 190	0	± 12	± 84	0	0	± 6	± 96
H_9	30	281	-39	6	-4	281	-6	13	146	-22	22	-4	125
H_{10}	± 36	± 410	0	0	0	± 410	0	± 18	± 184	0	0	± 4	± 176
$\Gamma_o(N)$	36 18	20 20	80	80	40 40	20 20	40	44 22	24 24	24	48	24 24	24 24
t	36+ 18+9	20+20 $20\frac{1}{2}+20$	$20\frac{1}{2}+2+10$	40 2+	20 2+5 20 2+5	$20\frac{1}{2}+20$ 20+20	20 2+5	44+ 22+11		$24\frac{1}{2} 2$ +12	24 2+12	$24\frac{1}{2}+8$ 24+8	$24\frac{1}{2}+$ 24+

TABLE I (continued)

	$\pm 12F$	$\pm 12G$	$-12H$ $+12H$	$-12I$ $+12I$	$\pm 12J$	$-12K$ $+12K$	$\pm 12L$	$\pm 12M$	$-13A$ $+13A$	$\pm 14A$	$-14B$ $+14B$	$-15A$ $+15A$	$-15B$ $+15B$
H_{-1}	1	1	1	1	1	1	1	1	1	1	1	1	1
H_0	0	0	∓ 1	± 2	0	∓ 3	0	0	2	0	∓ 1	± 3	∓ 2
H_1	0	1	-2	1	-1	4	0	0	3	-2	-1	3	3
H_2	0	0	0	0	0	∓ 6	0	0	6	0	0	± 1	∓ 4
H_3	-3	-1	5	-1	-1	11	0	0	9	3	1	0	5
H_4	0	0	0	0	0	∓ 18	0	0	14	0	0	0	∓ 10
H_5	0	1	-8	1	-1	28	0	0	22	-6	-1	0	15
H_6	0	0	0	0	0	∓ 42	0	0	32	0	0	± 3	∓ 22
H_7	6	2	14	2	2	62	0	0	46	9	3	9	29
H_8	0	0	0	0	0	∓ 90	0	0	66	0	0	± 9	∓ 36
H_9	0	-1	-22	-1	1	128	0	0	93	-14	-2	3	53
H_{10}	0	0	0	0	0	∓ 180	0	0	128	0	0	± 3	∓ 72
$\Gamma_o(N)$	192	24	24 24	24 24	48	24 24	576	288	52 26	112	28 28	60 30	60 30
t	$24\frac{1}{2} 4+6$	$24 2+3$	$24\frac{1}{2} 2+12$ $24\frac{1}{2} 2+12$	$24 2+3$ $24 2+3$	$24 2+3$	$24\frac{1}{2}+24$ $24+24$	$48 12+$ $24 12-$	$26\frac{1}{2}+26$ $26+26$	$28\frac{1}{2} 2+14$	$28+7$ $28+7$	$30\frac{1}{2}+6,$ $10,15$	$30\frac{1}{2}+5,$ $6,30$	$30+5,$ $10,15$

TABLE I (continued)

	-15C +15C	-15D +15D	-15E +15E	$\pm 16A$	-16B +16B	-18A +18A	-18B +18B	-18C +18C	-20A +20A	$\pm 20B$	-20C +20C	-21A +21A	-21B +21B
H_{-1}	1	1	1	1	1	1	1	1	1	1	1	1	1
H_0	0	± 1	± 2	0	± 2	± 1	± 2	± 1	± 2	0	± 1	± 2	± 1
H_1	0	0	1	-2	2	2	2	1	1	0	-1	1	1
H_2	± 2	± 2	± 1	0	± 4	± 3	± 3	0	± 2	0	0	0	± 3
H_3	0	2	2	2	6	4	4	1	6	-1	2	0	3
H_4	0	± 2	± 2	0	± 8	± 6	± 6	0	± 6	0	0	± 2	± 4
H_5	3	3	2	-4	12	9	9	0	3	0	-3	4	7
H_6	0	± 2	± 3	0	± 16	± 12	± 12	0	± 8	0	0	± 2	± 7
H_7	0	5	5	7	23	16	16	1	16	4	4	0	9
H_8	± 6	± 6	± 5	0	± 32	± 21	± 21	0	± 14	0	0	0	± 15
H_9	0	5	5	-10	42	28	28	-2	13	0	-5	1	16
H_{10}	0	± 6	± 7	0	± 56	± 36	± 36	0	± 26	0	0	± 4	± 20
$\Gamma_o(N)$	180 90	60 30	60 30	64	32 32	36 36	36 36	36 36	40 40	160	40 40	84 42	84 42
t	$30\frac{1}{2} 3$	$30\frac{1}{2}+3,$	$30\frac{1}{2}+15$	$32\frac{1}{2} 2+$	$32\frac{1}{2}+$	$36+36$	$36\frac{1}{2}+36$	$36 2+9$		$40 4+5$	$40\frac{1}{2} 2+20$	$42\frac{1}{2}+6,$	$42\frac{1}{2}+3,$
	$+10$	$5,15$										$14,21$	$14,42$
	$30 3$	$30+3,$	$30+15$		$32+$	$36\frac{1}{2}+36$	$36+36$	$36 2+9$			$40\frac{1}{2} 2+20$	$42+6,$	$42+3,$
	$+10$	$5,15$										$14,21$	$14,42$

TABLE I (continued)

	-21C +21C	* ±22A	-23A +23A	±24A	-24B +24B	±24C	±24D	±24E	-24F +24F	-28A +28A	* ±28B	-30A +30A	±30B
H ⁻¹	1	1	1	1	1	1	1	1	1	1	1	1	1
H ₀	0	0	±1	0	±2	0	0	0	±1	±1	0	±1	0
H ₁	0	-1	0	-2	2	1	0	-1	0	1	0	1	-1
H ₂	±1	0	±1	0	±2	0	0	0	0	±2	0	±1	0
H ₃	0	0	1	1	1	1	1	1	-1	1	-1	0	1
H ₄	0	0	±1	0	±2	0	0	0	0	±2	0	0	0
H ₅	0	-1	1	0	4	3	0	-3	0	3	0	0	-1
H ₆	0	0	±1	0	±6	0	0	0	0	±4	0	±1	0
H ₇	0	1	2	2	10	2	2	2	2	5	1	1	1
H ₈	±1	0	±2	0	±10	0	0	0	0	±6	0	±1	0
H ₉	0	-1	2	-6	10	3	0	-3	0	8	0	1	-3
H ₁₀	0	0	±2	0	±12	0	0	0	0	±8	0	±1	0
Γ ₀ (N)	252 126	88	92 46	96	48 48	96	192	96	48 48	56 56	448	60 60	240
t	42 ¹ / ₂ 3+7	44 2+11	46 ¹ / ₂ + 23	(48 2)		48 2+	48 4+12	48 ¹ / ₂ 2+	48 ¹ / ₂ 4 +12	56 ¹ / ₂ +	56 ¹ / ₂ 4 +14	60+12, 15, 20	60 ¹ / ₂ 2+5, 6, 30
	42 3+7		46+23						48 ¹ / ₂ 4 +12	56+		60 ¹ / ₂ + 12 15, 20	

TABLE II
Values of head characters for 3.2.Suz ($\theta = \sqrt{-3}$)

		1A		2A		± 2B		3A	
		+	-	+	-	+	-	+	-
H_{-1}	1	1	1	1	1	1	1	1	1
H_0	12*	-12	12	4	-4	0	0	6	-6
H_1	12	54	78	6	-2	6	6	27	15
H_2	220	-76	364	4	28	0	0	86	-14
H_3	$(12^*)^2 + 780^*$	-243	1365	-3	-27	21	21	243	-21
H_4	$66^2 + 78 + 429 + 2145$	1188	4380	-12	-52	0	0	594	78
H_5	$1^4 + 143^4 + 780^2 + 3432$	-1384	12520	-8	136	56	56	1370	-62
H_6	$220^4 + 572a + 572b + 4928$	-2916	32772	12	-108	0	0	2916	-132
H_7		11934	80094	30	-162	126	126	5967	399
H_8		-11580	185276	20	620	0	0	11586	-322
H_9		-21870	409578	-30	-486	258	258	21870	-426
H_{10}		79704	871272	-72	-760	0	0	39852	1332
$\Gamma_0(N)$		3	6	6	6	24	24	9	18
t		3	6+6	6	6+2	12 2+6	12 2+6	9+	

TABLE II (continued)

	3B		3C		4A		± 4B	± 4C	± 4D	5A		5B	
	+	-	+	-	+	-				+	-	+	-
H ₋₁	1	1	1	1	1	1	1	1	1	1	1	1	1
H ₀	-3	3	0	0	4	-4	0	0	0	3	-3	-2	2
H ₁	0	6	0	0	14	6	2	-2	0	9	3	-1	3
H ₂	5	13	-4	4	36	-4	0	0	0	19	-1	4	4
H ₃	0	24	0	0	85	-3	1	-3	3	42	0	-3	5
H ₄	0	42	0	0	180	12	0	0	0	78	0	-2	10
H ₅	-7	73	2	10	360	-8	0	8	0	146	0	11	15
H ₆	0	120	0	0	684	-12	0	0	0	249	-3	-6	22
H ₇	0	192	0	0	1246	30	-2	-2	6	429	9	-11	29
H ₈	3	299	12	20	2196	-20	0	0	0	695	-9	20	36
H ₉	0	456	0	0	3754	-30	-2	-6	0	1125	3	-15	53
H ₁₀	0	684	0	0	6264	72	0	0	0	1749	-3	-16	72
Γ ₀ (N)	9	18	27	54	12	12	12	24	96	15	30	15	30
t	9	18+18	9 3	18 3 +6	12+12	12+4	12	24 2+2	24 4+6	15+15	30+6; 10,15	15+5	30+5,6,30

TABLE II (continued)

	6A		6C 6B		6D		± 6E		7A		± 8A		8B		± 8C	
	+	-	+	-	+	-	+	-	+	-	+	-	+	-	+	-
H_{-1}	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
H_0	-2	2	-1+2 ω	1-2 ω	1	-1	0	0	2	-2	0	0	-2	2	0	0
H_1	3	7	6 $\bar{\omega}$	-2 ω	0	-2	0	0	5	1	2	2	0	4	0	0
H_2	-2	10	13	1	1	1	0	0	8	0	0	0	2	6	0	0
H_3	3	27	24 ω	0	0	0	0	0	16	0	5	5	-1	11	-1	-1
H_4	-6	38	42 $\bar{\omega}$	2 ω	0	2	0	0	26	-2	0	0	-2	18	0	0
H_5	10	82	73	1	1	1	2	2	44	4	8	8	4	28	0	0
H_6	-12	108	120 ω	0	0	0	0	0	66	-2	0	0	-2	42	0	0
H_7	15	207	192 $\bar{\omega}$	0	0	0	0	0	104	0	14	14	-2	62	-2	-2
H_8	-22	278	299	-1	-1	-1	0	0	152	0	0	0	6	90	0	0
H_9	30	486	456 ω	0	0	0	0	0	229	1	22	22	-4	128	0	0
H_{10}	-36	644	684 $\bar{\omega}$	-4 ω	0	-4	0	0	324	-4	0	0	-4	180	0	0
$\Gamma_o(N)$	18	18	18	18	18	18	216	216	21	42	48	48	24	24	96	96
t	18+9	18+	18 $\frac{1}{3}$ +18	18 $\frac{2}{3}$ +2	18	18+2	36 6 +6	36 6 +6	21+21	42+6; 14,21	24 2+12	24+8	24	24	24 4+2	24 4+2

TABLE II (continued)

	9B 9A	10A	± 10B	11A	12A	12B	± 12C
	+	-	+	-	+	-	+
H ₋₁	1	1	1	1	1	1	1
H ₀	-0	0	-1	1	-2	1	0
H ₁	3ω	-3/2+0/2	1	1	-1	2	1
H ₂	5	1	0	1	6	3	0
H ₃	9ω	-3/2+0/2	1	1	-5	4	3
H ₄	12ω	-20	0	2	-6	6	0
H ₅	20	4	1	2	18	9	2
H ₆	27ω	-3/2+50/2	0	3	-12	12	0
H ₇	42ω	-3-0	1	3	-17	16	7
H ₈	57	5	0	3	42	21	0
H ₉	81ω	9-100	3	4	-22	28	6
H ₁₀	108ω	-6-60	0	6	-36	36	0
Γ ₀ (N)	27	54	120	66	36	36	72
t	27 ² / ₃ +		60 2+5, 6, 30	66+6, 11, 66	36+	36+36	36 2+

TABLE II (continued)

	$\pm$ 12D	12E		$\begin{smallmatrix} 13B \\ 13A \end{smallmatrix}$		$\pm$ 14A	$\begin{smallmatrix} 15B \\ 15A \end{smallmatrix}$		15C		$\begin{smallmatrix} 18B \\ 18A \end{smallmatrix}$	
		+	-	+	-		+	-	+	-	+	-
H_{-1}	1	1	1	1	1	1	1	1	1	1	1	1
H_0	0	0	0	1	-1	0	0	0	1	-1	1	-1
H_1	0	$2\bar{\omega}$	2ω	2	0	-1	0	0	2	0	$3/2+\theta/2$	ω
H_2	0	3	3	2	0	0	1	-1	1	1	1	1
H_3	0	$4\bar{\omega}$	$4\bar{\omega}$	4	0	0	0	0	3	-1	$3/2-\theta/2$	$3\bar{\omega}$
H_4	0	$6\bar{\omega}$	6ω	5	-1	0	0	0	4	-2	$3-\theta$	2ω
H_5	0	9	9	7	1	0	2	0	5	3	4	4
H_6	0	12ω	$12\bar{\omega}$	9	-1	0	0	0	6	-2	$5/2+\theta/2$	$3\bar{\omega}$
H_7	0	$16\bar{\omega}$	16ω	13	1	0	0	0	7	-1	$1-\theta$	6ω
H_8	0	21	21	16	0	0	2	0	11	3	5	5
H_9	0	28ω	$28\bar{\omega}$	22	0	-1	0	0	15	-1	7	$9\bar{\omega}$
H_{10}	0	$36\bar{\omega}$	36ω	27	-1	0	0	0	17	-3	12	8ω
$\Gamma_0(N)$	864	36	36	39	78	168	135	270	45	90	54	54
t	72 12	$36\frac{1}{3}+36$	$36\frac{2}{3}+36$	39+39	$78+6, 26, 39$	$84 2+6, 14, 21$	$45 3, +15$	$90 3+6, 10, 15$	45+			$54\frac{2}{3}+$

TABLE II (continued)

	20A		21B 21A		24A	
	+	-	+	-	+	-
H_{-1}	1	1	1	1	1	1
H_0	-1	1	-1	1	0	0
H_1	-1	1	-1	1	-1	-1
H_2	1	1	2	0	0	0
H_3	0	2	-2	0	-1	-1
H_4	0	2	-1	1	0	0
H_5	0	2	5	1	2	2
H_6	-1	3	-3	1	0	0
H_7	1	5	-4	0	-1	-1
H_8	1	5	8	0	0	0
H_9	-1	5	-5	1	-1	-1
H_{10}	-1	7	-6	2	0	0
$\Gamma_o(N)$	60	60	63	126	144	144
t	60+12, 15, 20		60+4, 15, 60		72 2	

TABLE III
Values of head characters for E

	1A	2A	3A	3B	3C	4A	4B	5A	6A	6B
H_{-1}	1	1	1	1	1	1	1	1	1	1
H_2	248	-8	14	5	-4	8	0	-2	4	-2
H_5	4124	28	65	-7	2	28	-4	-1	10	1
H_8	1+4123	-64	156	3	12	64	0	2	20	-4
H_{11}	1+4123+30628	134	456	15	-21	134	6	1	35	8
H_{14}	1+248 ² +4123+61256+147250	-288	1066	-32	4	288	0	4	60	-6
$\Gamma_o(N)$	9	18	27	9	27	36	72	45	54	54
t	3 3	6 3	9 3+	9-	9 3	12 3+	12 6	15 3	18 3 +6	18 3+3

TABLE III (continued)

	6C	7A	8A	8B	9A	9B	9C	10A	12B 12A	12C	12D	13A	14A	15B 15A
H ₋₁	1	1	1	1	1	1	1	1	1	1	1	1	1	1
H ₂	1	3	0	0	5	-4	2	2	2	-1	0	1	-1	1
H ₅	1	8	4	0	-7	2	5	3	1	1	2	3	0	2
H ₈	-1	11	0	0	3	12	6	6	4	1	0	3	-1	2
H ₁₁	-1	25	6	-2	15	-21	12	9	8	-1	3	4	1	4
H ₁₄	0	35	0	0	-32	4	16	12	6	0	0	6	-1	4
Γ _o (N)	18	63	144	288	9	27	81	90	108	36	216	117	126	135
t	18-	21 3+	24 6+	24 12-	9-	9 3	27 3+	30 3+10	36 3+	36+4	36 6+6	39 3+	42 3+7	45 3+15

TABLE III (continued)

	18A	18B	19A	20A	21A	24B 24A	24D 24C	27A	27C 27B	28A	30B 30A	31B 31A	36A
H_{-1}	1	1	1	1	1	1	1	1	1	1	1	1	1
H_2	1	-2	1	0	0	0	0	2	-1	1	-1	0	-1
H_5	1	1	1	1	2	1	0	-1	2	0	0	1	1
H_8	-1	2	1	0	2	0	0	0	3	1	0	1	1
H_{11}	-1	-4	3	1	1	0	1	3	0	1	0	1	-1
H_{14}	0	0	2	0	2	0	0	-2	1	1	0	1	0
$\Gamma_o(N)$	18	162	171	360	189	432	864	243	243	252	270	279	36
t	18-	(54 3)	57 3+	60 6 +10	63 3+	72 6+	72 12 +6	(81 3)	(81 3)	84 3+	90 3+6 10,15	93 3+	36+4

TABLE III (continued)

	36C 36B	39B 39A
H_{-1}	1	1
H_2	-1	1
H_5	1	0
H_8	1	0
H_{11}	-1	1
H_{14}	0	0
$\Gamma_o(N)$	36	351
t	36+4	117 3+

TABLE IV
Values of head characters for 2.HJ ($\varphi = \sqrt{5}$)

			1A		2A		$\pm 2B$		3A		3B		4A	
			+	-	+	-	+	-	+	-	+	-	+	-
H_{-1}	1		1	1	1	1	1	1	1	1	1	1	1	1
H_0		6a	-6	6	2	-2	0	0	3	-3	0	0	-2	2
H_1	1+14a	6b	9	21	1	-3	3	3	9	3	0	0	1	5
H_2	36	6a+6b+14	10	62	2	6	0	0	19	-1	-2	2	-2	10
H_3	1+14a+14b+36	6a+6b+84	-30	162	2	2	6	6	42	0	0	0	2	18
H_4			6	378	-2	2	0	0	78	0	0	0	2	30
H_5			-25	819	-1	-5	13	13	146	0	-1	3	-1	51
H_6			96	1680	0	-16	0	0	249	-3	0	0	0	80
H_7			60	3276	-4	12	24	24	429	9	0	0	-4	124
H_8			-250	6138	-2	2	0	0	695	-9	2	6	2	190
H_9			45	11145	5	17	39	39	1125	3	0	0	5	281
H_{10}			-150	19662	2	-10	0	0	1749	-3	0	0	-2	410
$\Gamma_0(N)$			5	10	10	10	40	40	15	30	45	90	20	20
t			5	10+10	10	10+2	20 2+10	15+15	30+6, 10,15	15 3	30 3 +10	20+4	20+20	20+20

TABLE IV (continued)

	5B 5A		5D 5C		6A		± 6B	7A		8A	
	+	-	+	-	+	-		+	-	+	-
H_{-1}	1	1	1	1	1	1	1	1	1	1	1
H_0	-1+ φ	1- φ	3/2+ φ /2	-3/2- φ /2	-1	1	0	1	-1	0	0
H_1	3/2-5 φ /2	7/2- φ /2	4	1+ φ	1	3	0	2	0	1	1
H_2	-10	2	5	-3	-1	3	0	3	-1	0	0
H_3	5	-3	10	2	2	8	0	5	1	2	2
H_4	21+5 φ	3+3 φ	16	3- φ	-2	8	0	6	0	0	0
H_5	-25/2+25 φ /2	3/2-3 φ /2	25	-1-2 φ	2	16	1	10	0	3	3
H_6	6-30 φ	10-2 φ	36	4 φ	-3	17	0	12	0	0	0
H_7	-85	11	55	-9	5	33	0	18	0	4	4
H_8	50	-2	75	3	-5	35	0	23	-1	0	0
H_9	285/2+65 φ /2	25/2+13 φ /2	110	15-3 φ	5	59	0	31	1	5	5
H_{10}	-75+75 φ	7-7 φ	150	-8-6 φ	-7	65	0	39	-1	0	0
$\Gamma_0(N)$	25	50	25	50	30	30	360	35	70	80	80
t			25+		30+15	30+2, 15,30	60 6+10	35+35	70+10, 14,35	40 2+20	

TABLE IV (continued)

	$\pm$ 10B $\pm$ 10A	10D + 10C	12A + -	15B 15A + -	
H_{-1}	1	1	1	1	
H_0	0	$-1/2+\phi/2$	-1	$1/2-\phi/2$	
H_1	$1/2+\phi/2$	$1-\phi$	-1	$3/2+\phi/2$	
H_2	0	-3	1	-1	
H_3	1	2	0	0	
H_4	0	$3+\phi$	0	$3-\phi$	
H_5	$1/2-\phi/2$	$-1+2\phi$	0	$1-\phi$	
H_6	0	-4ϕ	-1	$3/2+3\phi/2$	
H_7	-1	-9	1	-1	
H_8	0	3	1	5	
H_9	$3/2-\phi/2$	$15+3\phi$	-1	$15/2-\phi/2$	
H_{10}	0	$-8+6\phi$	-1	$3/2-3\phi/2$	
$\Gamma_o(N)$	100	50	60	75	150
t		50+	60+4, 15,60	60+12, 15,20	

TABLE V
Values of characters for $F(\varphi = \sqrt{5})$

	1A	2A	2B	3A	3B	4A	4B	4C	5A	5B
H_{-1}	1	1	1	1	1	1	1	1	1	1
H_0	*	*	*	*	*	*	*	*	*	*
H_1	134	22	6	8	-1	6	2	-2	-6	9
H_2	760	56	-8	22	4	8	0	0	20	10
H_3	1+3344	177	17	42	-3	17	9	1	15	-30
H_4	1+133a+3344+8778a	352	-32	70	-2	32	0	0	36	6
H_5	1+133a+760+3344+35112a	870	54	155	11	54	10	-2	0	-25
H_6		1584	-80	246	-6	80	0	0	-84	96
H_7		3412	116	421	-11	116	28	4	195	60
H_8		5952	-192	722	20	192	0	0	100	-250
H_9		11442	290	1101	-15	290	30	-6	240	45
H_{10}		19240	-408	1730	-16	408	0	0	0	-150
$\Gamma_o(N)$	5	10	10	15	15	20	40	40	25	5
t	5+	10+	10+5	15+	15+5	20+	20 2+	20 2+5		5

TABLE V (continued)

	5D 5C	5E	6A	6B	6C	7A	8A	8B	9A	10A	10B	10C	10E 10D
H_{-1}	1	1	1	1	1	1	1	1	1	1	1	1	1
H_0	*	*	*	*	*	*	*	*	*	*	*	*	*
H_1	$3/2-5\phi/2$	1	4	0	3	1	0	2	2	-3	2	1	$7/2-\phi/2$
H_2	-10	4	2	-2	4	4	0	0	1	6	-4	2	2
H_3	5	5	6	2	5	6	1	1	3	2	7	2	-3
H_4	$21+5\phi$	10	10	-2	10	6	0	0	4	2	12	-2	$3+3\phi$
H_5	$-25/2+25\phi/2$	16	15	3	15	10	0	2	5	-5	0	-1	$3/2-3\phi/2$
H_6	$6-30\phi$	25	18	-2	22	10	0	0	6	-16	4	0	$10-2\phi$
H_7	-85	36	37	5	29	19	4	4	7	12	-13	-4	11
H_8	50	55	30	-6	36	22	0	0	11	2	12	-2	-2
H_9	$285/2+65\phi/2$	75	57	5	53	32	0	6	15	17	32	5	$25/2+13\phi/2$
H_{10}	$-75+75\phi$	150	70	-6	72	40	0	0	17	-10	0	2	$7-7\phi$
$\Gamma_o(N)$	25	25	30	30	30	35	160	80	45	10	50	10	50
t		25+	30+	$30+3, 5, 15$	$30+5, 6, 30$	35+	$40 4+$	$40 2+$	45+	$10+2$		10	

TABLE V (continued)

	10F	10H 10G	11A	12A	12B	12C	14A	15A	15C 15B	19B 19A	20B 20A	20C
H_{-1}	1	1	1	1	1	1	1	1	1	1	1	1
H_0	*	*	*	*	*	*	*	*	*	*	*	*
H_1	2	1- φ	2	2	0	1	1	3	3/2+ φ /2	1	1	2
H_2	1	-3	1	0	2	0	0	2	-1	0	-2	0
H_3	2	2	1	0	2	1	2	-3	2	1	2	-1
H_4	2	3+ φ	2	0	2	0	2	0	3- φ	1	2	0
H_5	5	-1+2 φ	3	1	3	1	2	0	1- φ	1	-1	0
H_6	4	-4 φ	4	0	2	0	2	6	3/2+3 φ /2	1	0	0
H_7	7	-9	6	1	5	1	3	6	-1	1	-4	3
H_8	7	3	5	0	6	0	2	-8	5	2	2	0
H_9	12	15+3 φ	8	3	5	3	4	6	15/2- φ /2	2	5	0
H_{10}	10	-8+6 φ	9	0	6	0	4	0	3/2-3 φ /2	2	-2	0
$\Gamma_o(N)$	50	50	55	120	60	120	70	75	75	95	20	100
t	50+		55+	60 2+	60+	60 2+5, 6,30	70+			95+	20+4	

TABLE V (continued)

	20E 20D	21A	22A	25B 25A	30A	30C 30B	35B 35A	40B 40A
H_{-1}	1	1	1	1	1	1	1	1
H_0	*	*	*	*	*	*	*	*
H_1	$1/2+\varphi/2$	1	0	$3/2+\varphi/2$	-1	$1/2-\varphi/2$	1	0
H_2	0	1	1	0	2	-1	-1	0
H_3	1	0	1	0	1	0	1	1
H_4	0	0	0	$1-\varphi$	0	0	1	0
H_5	$1/2-\varphi/2$	1	1	0	0	0	0	0
H_6	0	1	0	$1+\varphi$	-2	$-1/2-\varphi/2$	0	0
H_7	-1	1	2	0	2	-1	-1	4
H_8	0	1	1	0	0	1	2	0
H_9	$3/2-\varphi/2$	2	2	$5/2-3\varphi/2$	2	$1/2+\varphi/2$	2	0
H_{10}	0	1	1	0	0	$-1/2+\varphi/2$	0	0
$\Gamma_o(N)$	100	105	110	125	150	150	175	160
t		105+	110+					40 4+

TABLE VI
Values of characters for $2.A_7 (\theta = \sqrt{-7})$

	1A		3A		3B		4A		5A		6A		7B 7A	
	+	-	+	-	+	-	+	-	+	-	+	-	+	-
H_{-1}	1	1	1	1	1	1	1	1	1	1	1	1	1	1
H_0	-4	4	2	-2	-1	1	0	-1	1	-1	0	-1/2+ θ /2	1/2- θ /2	
H_1	2	10	5	1	-1	1	0	2	2	0	-1	-3/2+ θ /2	-1/2- θ /2	
H_2	8	24	8	0	-1	3	0	3	3	-1	0	-5/2-3 θ /2	-1/2+ θ /2	
H_3	-5	51	16	0	1	3	1	5	5	1	0	2	2	
H_4	-4	100	26	-2	2	4	0	6	6	0	0	3- θ	2	
H_5	-10	190	44	4	-1	7	0	10	10	0	0	-3	1	
H_6	12	340	66	-2	3	7	0	12	12	0	0	12	4	
H_7	-7	585	104	0	-1	9	1	18	18	0	0	-7/2+7 θ /2	0	
H_8	8	984	152	0	-1	15	0	23	-1	-1	0	-19/2+11/2	1/2- θ /2	
H_9	46	1606	229	1	-2	16	0	31	1	1	-1	-27/2-19 θ /2	-1/2- θ /2	
H_{10}	-36	2564	324	-4	0	20	0	39	-1	-1	0	6	2	
$\Gamma_o(N)$	7	14	21	42	21	42	224	35	70	168	49	98		
t	7	14+14	21+21	42+6, 14, 21	21+3	42+3, 14, 42	56+4, 14	35+35	70+10, 14, 35	84+2+6, 14, 21				

TABLE VII
Values of head characters for $H(\theta = \sqrt{-7})$

	1A	2A	2B	3A	3B	4A	4B	4C	5A	6A	6B
H_{-1}	1	1	1	1	1	1	1	1	1	1	1
H_0	*	*	*	*	*	*	*	*	*	*	*
H_1	51	11	3	6	0	3	3	-1	1	2	0
H_2	204	20	-4	6	3	0	4	0	4	2	-1
H_3	681	57	9	15	0	1	9	1	6	3	0
H_4	1956	92	-12	30	0	0	12	0	6	2	0
H_5	5135	207	15	41	8	7	15	-1	10	9	0
H_6	12360	312	-24	66	0	0	24	0	10	6	0
H_7	28119	623	39	111	0	7	39	3	19	11	0
H_8	60572	932	-52	146	11	0	52	0	22	14	-1
H_9	125682	1674	66	222	0	18	66	-2	32	18	0
H_{10}	251040	2464	-96	336	0	0	96	0	40	16	0
$\Gamma_o(N)$	7	14	14	21	63	56	28	28	35	42	126
t	7+	14+	14+7	21+	21 3+	28 2+	28+	28+7	35+	42+	42 3+7

TABLE VII (continued)

	7B 7A	7C	$\frac{7E}{7D}$	8A	10A	12A	12B	$\frac{14B}{14A}$	$\frac{14D}{14C}$	15A
H_{-1}	1	1	1	1	1	1	1	1	1	1
H_0	*	*	*	*	*	*	*	*	*	*
H_1	$-3/2+3\theta/2$	2	$-3/2-\theta/2$	1	1	0	0	$1/2-\theta/2$	$-1/2+\theta/2$	1
H_2	$1-\theta$	8	$-5/2+3\theta/2$	0	0	0	1	$-1+\theta$	$-1/2-\theta/2$	1
H_3	9	-5	2	2	2	1	0	1	2	0
H_4	$3-3\theta$	-4	$3+\theta$	2	2	0	0	$1-\theta$	2	0
H_5	4	-10	-3	2	2	1	0	4	1	1
H_6	12	12	12	2	2	0	0	4	4	1
H_7	0	-7	$-7/2-7\theta/2$	3	3	1	0	0	0	1
H_8	$-13+13\theta$	8	$-19/2-11\theta/2$	2	2	0	1	$1-\theta$	$1/2+\theta/2$	1
H_9	$15/2-15\theta/2$	46	$-27/2+19\theta/2$	4	4	0	0	$-5/2+5\theta/2$	$-1/2+\theta/2$	2
H_{10}	48	-36	6	8	4	0	0	0	2	1
$\Gamma_o(N)$	49	7	49	56	70	168	252	98	98	105
t		7		56+	70+	84 2+	84 3+			105+

TABLE VII (continued)

	17B 17A	21B 21A	21D 21C	28B 28A	
H_{-1}	1	1	1	1	
H_0	*	*	*	*	
H_1	0	-1	0	-1/2+ θ /2	
H_2	0	-1	-1/2+ θ /2	0	
H_3	1	1	0	1	
H_4	1	2	0	0	
H_5	1	-1	1	0	
H_6	1	3	0	0	
H_7	1	-1	0	0	
H_8	1	-1	1/2- θ /2	0	
H_9	1	-2	0	1/2- θ /2	
H_{10}	1	0	0	0	
$\Gamma_0(N)$	119	21	441	196	
t	119+	21+3			

TABLE VIII
Values of head characters for M_{12}
 $(b = -1/2 + \sqrt{-11}/2, b^* = -1/2 - \sqrt{-11}/2)$

	1A	2A	2B	3A	3B	4A	4B	5A
H_{-1}	1	1	1	1	1	1	1	1
H_0	*	*	*	*	*	*	*	*
H_1	17	5	1	-1	2	1	1	2
H_2	46	6	-2	1	4	2	2	1
H_3	116	16	4	-1	5	4	4	1
H_4	252	20	-4	0	6	4	4	2
H_5	533	41	5	2	14	5	5	3
H_6	1034	50	-6	-1	14	6	6	4
H_7	1961	97	9	-1	20	9	9	6
H_8	3540	116	-12	3	30	12	12	5
H_9	6253	197	13	-2	37	13	13	8
H_{10}	10654	246	-18	-2	46	18	18	9
$\Gamma_o(N)$	11	22	22	33	33	44	44	55
t	11+	22+	22+11	33+11	33+	44+	44+	55+