Evaluation of Integrals of Howland Type Involving a Bessel Function

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Abstract. This paper presents a method of evaluation of four integrals of Howland type, which involve a Bessel function in the integrands. With the aid of tabulated values, they are evaluated to 10D. Two of the four Howland integrals needed in the evaluation are evaluated anew to 20D in order to provide adequate accuracy.

In a recent investigation of certain problems in elasticity concerning elliptic boundaries, four integrals of Howland type involving an additional Bessel function in the integrands were encountered. We believe that they deserve special consideration. The integrals are as follows:

$$(1) \quad \begin{aligned} \frac{F_{n,k}}{F_{n,k}^*}(a) &= \frac{2^k}{k!} \int_0^\infty \frac{m^k J_n(ma)}{\sinh 2m \pm 2m} dm & (n+k \geq 1), \\ & & (n+k \geq 3), \\ \frac{E_{n,k}}{E_{n,k}^*}(a) &= \frac{2^k}{k!} \int_0^\infty \frac{m^k J_n(ma) \coth m}{\sinh 2m \pm 2m} dm & (n+k \geq 2), \\ & & (n+k \geq 4), \end{aligned}$$

where J_n is a Bessel function of the first kind of integral order n . n and k are nonnegative integers restricted as indicated above in order to render each integral convergent at the lower limit. The constant a may be real or complex.

By using the usual series expression for J_n and integrating, the first integral becomes

$$(2) \quad F_{n,k}(a) = \sum_{p=0}^{\infty} (-1)^p \binom{n+2p}{p} \binom{n+k+2p}{k} \left(\frac{a}{4}\right)^{n+2p} I_{n+k+2p},$$

where

$$(3) \quad I_k = \frac{2^k}{k!} \int_0^\infty \frac{m^k dm}{\sinh 2m + 2m} \quad (k \geq 1).$$

Since I_k tends asymptotically to unity as k tends to infinity, we write, with a view of improving convergence of the series,

$$(4) \quad \begin{aligned} F_{n,k}(a) &= K_{n,k}(a) \\ &- \sum_{p=0}^{\infty} (-1)^p \binom{n+2p}{p} \binom{n+k+2p}{k} (1 - I_{n+k+2p}) \left(\frac{a}{4}\right)^{n+2p}, \end{aligned}$$

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where [7]

$$(5) \quad \begin{aligned} K_{n,k}(a) &= \sum_{p=0}^{\infty} (-1)^p \binom{n+2p}{p} \binom{n+k+2p}{k} \left(\frac{a}{4}\right)^{n+2p} \\ &= \frac{2^{k+1}}{k!} \int_0^{\infty} m^k e^{-2m} J_n(ma) dm = \frac{(n+k)!}{k!} P_k^{-n}(t) t^{k+1}, \end{aligned}$$

in which P_k^{-n} is an associated Legendre function of the first kind [1] and

$$(6) \quad t = 2/(a^2 + 4)^{1/2}.$$

Or, in terms of Gauss' hypergeometric function,

$$(7) \quad K_{n,k}(a) = \binom{n+k}{k} \left(\frac{1-t}{1+t}\right)^{n/2} t^{k+1} {}_2F_1\left(-k, k+1; n+1; \frac{1}{2} - \frac{t}{2}\right),$$

if t is real and $-1 < t < 1$. On the other hand, if t is complex, the factor $(1-t)^{n/2}$ is replaced by $(t-1)^{n/2}$.

Similarly, the other three integrals are

$$\begin{aligned} F_{n,k}^*(a) &= K_{n,k}(a) \\ &\quad + \sum_{p=0}^{\infty} (-1)^p \binom{n+2p}{p} \binom{n+k+2p}{k} (I_{n+k+2p}^* - 1) \left(\frac{a}{4}\right)^{n+2p}, \\ (8) \quad E_{n,k}(a) &= K_{n,k}(a) \\ &\quad - \sum_{p=0}^{\infty} (-1)^p \binom{n+2p}{p} \binom{n+k+2p}{k} (1 - IV_{n+k+2p}) \left(\frac{a}{4}\right)^{n+2p}, \\ E_{n,k}^*(a) &= K_{n,k}(a) \\ &\quad + \sum_{p=0}^{\infty} (-1)^p \binom{n+2p}{p} \binom{n+k+2p}{k} (IV_{n+k+2p}^* - 1) \left(\frac{a}{4}\right)^{n+2p}, \end{aligned}$$

where

$$(9) \quad \begin{aligned} I_k^* &= \frac{2^k}{k!} \int_0^{\infty} \frac{m^k dm}{\sinh 2m - 2m} & (k \geq 3), \\ IV_k &= \frac{2^k}{k!} \int_0^{\infty} \frac{m^k \coth m dm}{\sinh 2m} & (k \geq 2), \\ IV_k^* &= \frac{2^k}{k!} \int_0^{\infty} \frac{m^k dm}{\sinh 2m \pm 2m} & (k \geq 4). \end{aligned}$$

The evaluation of the last two integrals in (9) has been considered before by the first author [4]. It was found that

$$(10) \quad \begin{aligned} IV_k &= I_k - (I_{k-1} + II_{k-1} - S_k)/k, \\ IV_k^* &= I_k^* + (I_{k-1}^* + II_{k-1}^* - S_k)/k, \end{aligned}$$

where

$$(11) \quad \begin{aligned} II_k &= \frac{2^k}{k!} \int_0^{\infty} \frac{m^k e^{-2m} dm}{\sinh 2m \pm 2m} & (k \geq 1), \\ II_k^* &= \frac{2^k}{k!} \int_0^{\infty} \frac{m^k e^{-2m} dm}{\sinh 2m \pm 2m} & (k \geq 3), \\ S_k &= \sum_{n=1}^{\infty} \frac{1}{n^k} & (k \geq 2). \end{aligned}$$

The integrals I_k , I_k^* , II_k and II_k^* are the four ordinary Howland integrals. The first two were tabulated by the first author and Lin to 25D [6], [5] and the remaining two by Nelson to 9D [8]. The Riemann zeta function S_k was tabulated by Glaisher for integral k to 32D [2]. Among these tabulated values, it appears that Nelson's 9D values of II_k and II_k^* are inadequate for the present purpose. Consequently, the values are computed anew. The evaluation of these two integrals is described in the Appendix. Table 1 shows the values to 20D.

The relations in (4) and (8) are suitable for numerical computation. With the tabulated values, the four integrals can therefore be evaluated when a is given. In case of slow convergence of the series, the Euler transformation for alternating series [3] may be applied. The following recurrence relation for the first integral is mentioned:

$$(12) \quad F_{n-1,k}(a) + F_{n+1,k}(a) = \frac{4n}{ka} F_{n,k-1}(a),$$

which is derived by virtue of the relation connecting three Bessel functions of consecutive integral orders. Similar relations can be found for the other three integrals as well as for $K_{n,k}(a)$. These recurrence relations may be used for checking purposes. They can also be used for computing an unknown integral from two known integrals.

Values of the four integrals for the particular value $a = 1$ are computed. The expression in (7) is used for computing $K_{n,k}$. The results for $n = 0(1)3$ and $k = 0(1)10$ are shown in Table 2 to 10D.

Appendix. *Evaluation of II_k and II_k^* .* The two Howland integrals in (11) may be written as

$$(13) \quad \begin{aligned} II_k &= \frac{1}{2(k!)} \int_0^\infty \frac{w^k e^{-w} dw}{\sinh w \pm w} & (k \geq 1), \\ II_k^* & & (k \geq 3). \end{aligned}$$

Expansion of the integrand yields

$$(14) \quad \frac{w^k e^{-w}}{\sinh w \pm w} = \frac{2w^k e^{-2w}}{1 \pm 2we^{-w} - e^{-2w}} = 2w^k e^{-2w} \sum_{n=0}^{\infty} (\mp 1)^n p_n(w) e^{-nw},$$

where $p_n(w)$ is a polynomial in w of degree n related to the Gegenbauer polynomial of order unity. Its form is different according as n is even or odd. For $n \geq 0$,

$$(15) \quad p_{2n}(w) = \sum_{t=0}^n \binom{n+t}{2t} (2w)^{2t}, \quad p_{2n+1}(w) = \sum_{t=0}^n \binom{n+t+1}{2t+1} (2w)^{2t+1}.$$

With the aid of the integral

$$(16) \quad \int_0^\infty w^t e^{-cw} dw = \frac{t!}{c^{t+1}} \quad (c > 0)$$

we get

$$(17) \quad \begin{aligned} II_k &= \sum_{n=1}^{\infty} (\mp 1)^{n+1} \frac{q_n(k)}{(n+1)^{k+1}}, \\ II_k^* & \end{aligned}$$

where, for $n \geq 0$,

$$(18) \quad \begin{aligned} q_{2n+1}(k) &= \sum_{t=0}^n \binom{k+2t}{k} \frac{(n+t)!}{(n-t)!} \frac{1}{(n+1)^{2t}}, \\ q_{2n+2}(k) &= \sum_{t=0}^n \binom{k+2t+1}{k} \frac{(n+t+1)!}{(n-t)!} \left(\frac{2}{2n+3}\right)^{2t+1}. \end{aligned}$$

The series in (17) is rapidly convergent when k is large but slowly convergent when k is small. For instance, an accuracy of 20D can be reached with the first six terms when $k \geq 30$, with the first twenty terms when $k \geq 22$, and with the first eighty terms when $k \geq 18$. To reach an accuracy of 25D, the corresponding values of k are not less than 35, 26, and 21, respectively.

It is noted that these two integrals, unlike I_k and I_k^* , cannot be evaluated by Plana's method. In terms of I_k and I_k^* , the following relations are obtained by expanding e^{-w} into series of w and then applying the Kummer transformation [3]:

$$(19) \quad \begin{aligned} \frac{1}{2^{k+1}} - II_k &= \sum_{n=0}^{\infty} (-1)^n \binom{n+k}{n} (1 - I_{n+k}), \\ II_k^* - \frac{1}{2^{k+1}} &= \sum_{n=0}^{\infty} (-1)^n \binom{n+k}{n} (I_{n+k}^* - 1). \end{aligned}$$

They are suitable for computation only for the first few values of k . When k increases, accuracy is rapidly lost due to the binomial coefficient involved in the series. However, it appears that the loss can be reduced if the computation is carried out in several steps through some intermediate integrals. Suppose that s steps are taken and in each step a factor $e^{-w/s}$ is expanded into series instead of e^{-w} . Denote the intermediate integrals in the r th step by

$$(20) \quad \begin{aligned} I_k^{(r)} &= \frac{1}{2(k!)} \int_0^{\infty} \frac{w^k e^{-rw/s} dw}{\sinh w \pm w} \quad (k \geq 1), \\ I_k^{*(r)} &= \frac{1}{2(k!)} \int_0^{\infty} \frac{w^k e^{-rw/s} dw}{\sinh w \pm w} \quad (k \geq 3), \end{aligned}$$

where $r = 0, 1, 2, \dots, s$. These expressions give I_k and I_k^* when $r = 0$, and II_k and II_k^* when $r = s$. In a similar manner, it is found that

$$(21) \quad \begin{aligned} \left(\frac{s}{s+r}\right)^{k+1} - I_k^{(r)} &= \sum_{n=0}^{\infty} \frac{(-1)^n}{s^n} \binom{n+k}{n} \left\{ \left(\frac{s}{s+r-1}\right)^{n+k+1} - I_{n+k}^{(r-1)} \right\}, \\ I_k^{*(r)} - \left(\frac{s}{s+r}\right)^{k+1} &= \sum_{n=0}^{\infty} \frac{(-1)^n}{s^n} \binom{n+k}{n} \left\{ I_{n+k}^{*(r-1)} - \left(\frac{s}{s+r-1}\right)^{n+k+1} \right\}. \end{aligned}$$

It is seen that the binomial coefficient involved in the series is now divided by a factor s^n . Hence, the loss of accuracy is considerably reduced, especially when s is large. If the computation is carried out recurrently s times, the integrals II_k and II_k^* are obtained. Also, it turns out that the total loss of accuracy in s steps together is smaller than that by (19) in a single step. For instance, when $k = 20$, the total loss is 10S if $s = 2$, 7S if $s = 4$, and 5S if $s = 7$. The loss is less when k is smaller. On the other hand, the computing time required for s steps is increased s times. Furthermore, analogous to (17), the expansions of the intermediate integrals in the r th step are

$$(22) \quad \begin{aligned} I_k^{(r)} &= \sum_{n=1}^{\infty} (-1)^{n+1} q_n^{(r)}(k) \left(\frac{s}{ns+r}\right)^{k+1}, \\ I_k^{*(r)} &= \sum_{n=1}^{\infty} (-1)^{n+1} q_n^{(r)}(k) \left(\frac{s}{ns+r}\right)^{k+1}, \end{aligned}$$

where, for $n \geq 0$,

$$(23) \quad \begin{aligned} q_{2n+1}^{(r)}(k) &= \sum_{t=0}^n \binom{k+2t}{k} \frac{(n+t)!}{(n-t)!} \left\{ \frac{2s}{(2n+1)s+r} \right\}^{2t}, \\ q_{2n+2}^{(r)}(k) &= \sum_{t=0}^n \binom{k+2t+1}{k} \frac{(n+t+1)!}{(n-t)!} \left\{ \frac{2s}{(2n+2)s+r} \right\}^{2t+1}. \end{aligned}$$

Note that when $r = s$, these expressions become those for II_k and II_k^* as given in (17) and (18). Suppose that in the present computation, an accuracy not less than 20D is prescribed to the values of II_k and II_k^* . With the 25D values of I_k and I_k^* , the loss of accuracy must therefore be kept within 5S. In each step, the relations in (21) with $s = 7$ are used for k up to 20. For larger k , the relations in (22) are used instead. The values are also computed by both methods for some overlapping values of k to serve as a check. Further checks on the final results are provided by the following relations:

$$(24) \quad \begin{aligned} \sum_{k=0}^{\infty} II_{2k+1} &= \frac{1}{2} - II_1, & \sum_{k=1}^{\infty} k II_{2k} &= \frac{1}{4} - II_2, \\ \sum_{k=1}^{\infty} II_{2k+1}^* &= \frac{1}{2}, & \sum_{k=2}^{\infty} k II_{2k}^* &= \frac{1}{4}. \end{aligned}$$

The values rounded to 20D are shown in Table 1. No discrepancy is noticed in Nelson's 9D values when they are compared with the present values.

Corrigenda. The factor $(-1)^{m+n}$ in (3) and (7) of the paper [6] and also in (3) of the paper [5] should be deleted. The expressions (6) in the paper [6] should be revised accordingly. These corrections, however, do not effect the numerical results of the Howland integrals I_k and I_k^* in these two papers.

TABLE I
Values of II_k and II_k^*

k	II_k				II_k^*			
1	0.22011	95814	42489	13267	-	-	-	-
2	0.08792	72351	54623	64461	-	-	-	-
3	0.04334	78620	32253	03164	0.46071	37190	35659	66356
4	0.02258	30042	91563	34696	0.09931	55321	31928	69508
5	0.01192	34729	94286	16469	0.03241	26902	77966	60244
6	0.00628	79721	18696	33642	0.01261	69084	78437	55392
7	0.00329	50133	54822	42863	0.00539	11167	37422	13901
8	0.00171	32982	31483	05028	0.00243	29999	83832	20264
9	0.00088	41479	97290	09130	0.00113	59968	30348	45397
10	0.00045	32145	74349	76403	0.00054	22017	96217	86263
11	0.00023	10097	74078	29735	0.00026	25896	85716	89211
12	0.00011	72094	40904	63806	0.00012	84309	57923	71136
13	0.00005	92535	02802	31223	0.00006	32389	09903	71072
14	0.00002	98701	48236	51008	0.00003	12834	43520	29328
15	0.00001	50252	72536	49084	0.00001	55254	16440	55194
16	0.00000	75457	13757	46890	0.00000	77222	95797	22959
17	0.00000	37848	78742	55501	0.00000	38470	73132	06282
18	0.00000	18967	75182	64532	0.00000	19186	28537	52976
19	0.00000	09499	41039	13811	0.00000	09576	01987	42877
20	0.00000	04755	23929	87590	0.00000	04782	03676	61263

TABLE I (continued)

k	II_k				II_k^*			
21	0.00000	02379	58146	32115	0.00000	02388	93568	18747
22	0.00000	01190	48321	96864	0.00000	01193	74219	70259
23	0.00000	00595	48513	89530	0.00000	00596	61850	89582
24	0.00000	00297	82793	38128	0.00000	00298	22142	27687
25	0.00000	00148	94380	42812	0.00000	00149	08020	42756
26	0.00000	00074	48230	48966	0.00000	00074	52951	81493
27	0.00000	00037	24477	12334	0.00000	00037	26109	13887
28	0.00000	00018	62364	19592	0.00000	00018	62927	61929
29	0.00000	00009	31225	63641	0.00000	00009	31419	91799
30	0.00000	00004	65627	88197	0.00000	00004	65694	80066
31	0.00000	00002	32819	14519	0.00000	00002	32842	17086
32	0.00000	00001	16411	36812	0.00000	00001	16419	28317
33	0.00000	00000	58206	30278	0.00000	00000	58209	02107
34	0.00000	00000	29103	36435	0.00000	00000	29104	29710
35	0.00000	00000	14551	75540	0.00000	00000	14552	07520
36	0.00000	00000	07275	90285	0.00000	00000	07276	01241
37	0.00000	00000	03637	96006	0.00000	00000	03637	99757
38	0.00000	00000	01818	98299	0.00000	00000	01818	99582
39	0.00000	00000	00909	49251	0.00000	00000	00909	49690
40	0.00000	00000	00454	74660	0.00000	00000	00454	74810
41	0.00000	00000	00227	37342	0.00000	00000	00227	37393
42	0.00000	00000	00113	68675	0.00000	00000	00113	68693
43	0.00000	00000	00056	84339	0.00000	00000	00056	84345
44	0.00000	00000	00028	42170	0.00000	00000	00028	42172
45	0.00000	00000	00014	21085	0.00000	00000	00014	21086
46	0.00000	00000	00007	10543	0.00000	00000	00007	10543
47	0.00000	00000	00003	55271	0.00000	00000	00003	55271
48	0.00000	00000	00001	77636	0.00000	00000	00001	77636
49	0.00000	00000	00000	88818	0.00000	00000	00000	88818
50	0.00000	00000	00000	44409	0.00000	00000	00000	44409
51	0.00000	00000	00000	22204	0.00000	00000	00000	22204
52	0.00000	00000	00000	11102	0.00000	00000	00000	11102
53	0.00000	00000	00000	05551	0.00000	00000	00000	05551
54	0.00000	00000	00000	02776	0.00000	00000	00000	02776
55	0.00000	00000	00000	01388	0.00000	00000	00000	01388
56	0.00000	00000	00000	00694	0.00000	00000	00000	00694
57	0.00000	00000	00000	00347	0.00000	00000	00000	00347
58	0.00000	00000	00000	00173	0.00000	00000	00000	00173
59	0.00000	00000	00000	00087	0.00000	00000	00000	00087
60	0.00000	00000	00000	00043	0.00000	00000	00000	00043
61	0.00000	00000	00000	00022	0.00000	00000	00000	00022
62	0.00000	00000	00000	00011	0.00000	00000	00000	00011
63	0.00000	00000	00000	00005	0.00000	00000	00000	00005
64	0.00000	00000	00000	00003	0.00000	00000	00000	00003
65	0.00000	00000	00000	00001	0.00000	00000	00000	00001
66	0.00000	00000	00000	00001	0.00000	00000	00000	00001

TABLE 2
Values of four integrals for $a = 1$

n	k	$F_{n,k}(1)$	$F_{n,k}^*(1)$	$E_{n,k}(1)$	$E_{n,k}^*(1)$
0	1	0.54084 46591	-	-	-
	2	0.34194 08513	-	0.63469 34541	-
	3	0.18795 53763	1.15707 58719	0.27562 06093	-
	4	0.04801 62464	0.33813 39129	0.08078 15488	0.64599 26848
	5	-0.06316 63029	0.04109 33656	-0.05054 94481	0.09969 22234
	6	-0.13339 10175	-0.09670 05792	-0.12874 07789	-0.08136 34422
	7	-0.16004 74673	-0.14907 54405	-0.15851 54846	-0.14476 57105
	8	-0.14933 52743	-0.14758 67013	-0.14894 97848	-0.14645 35865
	9	-0.11299 26003	-0.11398 91334	-0.11297 69522	-0.11377 37369
	10	-0.06461 18449	-0.06600 73063	-0.06468 26817	-0.06602 70468
1	0	0.16069 18196	-	-	-
	1	0.26110 17772	-	0.41598 35324	-
	2	0.33007 03722	1.14211 61369	0.40565 04816	-
	3	0.34825 03408	0.72059 05605	0.39028 30431	1.05051 69033
	4	0.31430 77838	0.51523 39857	0.33771 73885	0.60270 22003
	5	0.24089 51857	0.35123 15466	0.25357 33417	0.38323 09369
	6	0.14764 05342	0.20700 47013	0.15426 00266	0.22007 55146
	7	0.05455 61086	0.08540 39244	0.05787 80057	0.09096 34111
	8	-0.02254 70977	-0.00718 85853	-0.02094 79499	-0.00480 47337
	9	-0.07432 31433	-0.06705 20664	-0.07358 68206	-0.06604 11886
	10	-0.09816 79533	-0.09493 09795	-0.09784 54429	-0.09451 33296
2	0	0.03700 02514	-	0.05672 84626	-
	1	0.10192 26191	0.31247 28392	0.12168 33587	-
	2	0.18026 27032	0.33106 76512	0.19727 36106	0.45768 54746
	3	0.25213 84533	0.36574 56439	0.26524 66995	0.41202 63376
	4	0.30023 40944	0.38245 66476	0.30950 14944	0.40452 42185
	5	0.31461 25299	0.37109 38229	0.32072 33589	0.38246 95368
	6	0.29398 78080	0.33085 49436	0.29778 96734	0.33685 07335
	7	0.24441 34868	0.26736 38413	0.24666 40712	0.27052 31475
	8	0.17661 33286	0.19028 86636	0.17788 87876	0.19193 52921
	9	0.10297 16679	0.11079 42066	0.10366 67523	0.11163 82998
	10	0.03488 25876	0.03918 64797	0.03524 79535	0.03961 05714
3	0	0.00940 63033	0.02727 77106	0.01109 03091	-
	1	0.03490 02343	0.06101 33066	0.03784 41688	0.08236 46842
	2	0.07762 01043	0.10777 52201	0.08108 29533	0.11967 76422
	3	0.13245 02011	0.16225 65094	0.13577 99185	0.16997 76958
	4	0.18996 91228	0.21625 73021	0.19277 60106	0.22135 04748
	5	0.23947 93654	0.26069 90895	0.24162 90493	0.26400 78127
	6	0.27184 28389	0.28778 70626	0.27337 11186	0.28988 38678
	7	0.28142 99577	0.29271 60111	0.28245 30496	0.29400 88557
	8	0.26696 05846	0.27455 24265	0.26761 20212	0.27532 78811
	9	0.23131 27687	0.23619 75451	0.23171 01874	0.23665 03371
	10	0.18054 52877	0.18356 63448	0.18077 88447	0.18382 39694

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