REVIEWS AND DESCRIPTIONS OF TABLES AND BOOKS

The numbers in brackets are assigned according to the revised indexing system printed in Volume 28, Number 128, October 1974, pages 1191–1194.

21[2.10].—H. Engels, *Numerical Quadrature and Cubature*, Academic Press, London, 1980, xiv + 441 pp. Price $74.00.

When I was asked to review this book, I was not completely unfamiliar with its contents. I had read it cursorily and did not have too favorable an opinion of it. The book appeared to be loaded with descriptions of the author's researches and did not give a realistic picture of the state-of-the-art in numerical integration, even up to June 1978, the date the book was submitted for publication as indicated in the Preface. I had also recalled the following sentence which may give some indication of the author's viewpoint: 'Somewhat exotic methods for the approximate calculation of integrals are based on statistical methods using random numbers' (p. 78). Finally, I had had the impression that the book was priced too highly. Thus I had a problem in deciding whether to accept the editor's request to review this book. I accepted despite my initial bias against it since I believe I would be able to give it a fair review. I leave it to the reader to judge whether I have succeeded.

After a thorough reading of this work, I was left with a feeling of sloppiness, tedium, irrelevance and narrowness punctuated by a few bright spots which I shall mention later. Since it is much easier to spot errors in the work of others than in one's own work, I have found not only an abundance of typographical errors but also some errors of substance. One example is the statement on p. 344 essentially repeated on p. 353: 'If $f \in C[-1, 1]$, then $f(x)$ possesses a convergent series expansion of the type $f(x) = \sum_{k=0}^{\prime\infty} a_k \omega_k(x)$. The polynomials $\omega_k(x)$ are assumed to be orthogonal with respect to a certain weight function...'. Another example is the statement on p. 371 which states essentially that if the error functional does not have an asymptotic expansion in integer powers of h, then polynomial extrapolation should not be used. There are also many errors in English, both vocabularly and usage, which would make it difficult for a reader not already familiar with the subject matter to follow the text. Thus Engels uses 'edges' in place of 'vertices', 'normed' instead of 'normalized', 'however' in place of 'moreover', 'so that' instead of 'such that', etc. In addition, there are several very problematic statements, of which we give two examples: 'The orthogonal polynomials in two or more dimensions are of comparable importance for cubature formulae to the one-dimensional orthogonal polynomials for quadrature formulae' (p. 239), and 'The requirement of analyticity is not such a serious disadvantage as it might appear, because we use only discrete values of a function and there always exist analytic functions passing

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through these discrete values' (p. 124 in the introduction to Section 3.5: Error bounds without derivatives for quadratures on analytic function spaces). One final example of sloppiness occurs in Section 2.7 on the Monte Carlo method. After a rather unsatisfactory description of this method, the author gives a FORTRAN program implementing this method. Unfortunately, the program does not implement the method given in the text but a different method which, in fact, is better than that given there.

So much for sloppiness. As for tedium, time after time I made notations that certain derivations and treatments were tedious and that certain results and tables were uninteresting. However, it would be too tedious to list examples of these. The following list of topics either unmentioned or barely touched upon is the basis for my criticism of narrowness. Numerical integration of data and of functions with singularities including Cauchy principal value integrals, nonlinear transformation of the independent variable, the epsilon algorithm, the Golub-Welsch method for generating Gaussian abscissas and weights, the fast Fourier transform for use with the Clenshaw-Curtis method, the Patterson extension of the Kronrod scheme and other topics of interest are not mentioned. Adaptive integration, methods for oscillatory integrands, integrals over an infinite range, and sampling methods are only briefly mentioned. (The adaptive scheme proposed by the author is of little merit.) On the other hand, a lot of space is devoted to subjects of little practical importance such as the Davis construction of positive cubature rules, Wilf's optimal formulas, Möller's work on cubature formulas with a minimal number of nodes, equally-weighted quadrature formulas, etc. This does not mean that these subjects should not be treated; only that this should not be done at the expense of those subjects which were omitted.

I shall now briefly list those sections of the book which were of interest inasmuch as they contained material not available in other books on the subject. The section on general Lagrange and Hermite interpolation is quite good as is part of the section on composite cubature rules. Section 3.5 on error bounds on analytic function spaces contains some worthwhile material as does Section 5.3 on implicitly-defined orthogonal polynomials. Also good are the above-mentioned sections on the work of Davis and Möller. Chapter 7 'Refined Interpolatory Quadrature' has quite a few interesting sections and is the most rewarding chapter in the book although it also suffers from the same shortcomings as the other chapters. The titles of these chapters are: 1. Introduction, 2. Construction Principles for Quadrature and Cubature Formulae, 3. Error Analysis for Quadrature and Cubature Formulae, 4. Convergence of Quadrature and Cubature Procedures, 5. Orthogonal Polynomials, 6. Interpolatory Quadrature and Cubature Formulae-Preassigned Nodes or Weights, 8. Non-Interpolatory Quadratures, 9. Auxiliary Material.

The contents of Chapter 9 deserves some further comment. In Table 9.2.1, there is a list of values of the integrals of a set of test functions for testing one-dimensional integration programs. While this list is useful, a more useful list would be of multiple integrals since not many numerical values of multiple integrals, which are not products of one-dimensional integrals, are readily available. Another useful addition would have been a set of families of integrals parametrized by one or two parameters, which have proved useful in comparison studies. There is also a list of published programs which, with some omissions, is up-to-date until 1975, except for one program by the author from 1977. In some cases, I have the impression that the author has not personally inspected the papers he lists, but classifies them, incorrectly, according to their titles. Thus, the programs by Gautschi and by Golub and Welsch should be listed under 'Computation of nodes and weights' rather than under 'Gaussian quadrature programs'. Boland's programs entitled 'Product-type formulae' are not cubature programs but quadrature programs while the two references to Welsch which appear in the list of tables, 'Abscissas and weights for Gregory/Romberg quadrature' belong in the list of programs.

I could report on other amusing and not-so-amusing flaws in the book, but I shall conclude with the following evaluation. For specialists and researchers in the field of numerical integration, this book contains some items of interest. However, the nonspecialist who is interested more in the practical aspects of numerical integration is advised to refer to the standard texts, *Methods of Numerical Integration* by Davis and Rabinowitz and *Approximate Calculation of Multiple Integrals* by Stroud.

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22[2.05].—M. J. D. POWELL, *Approximation Theory and Methods*, Cambridge Univ. Press, New York, 1981, ix + 339 pp., $23\frac{1}{2}$ cm. Price $57.50 hardcover, $19.95 paperback.

This book grew out of the material of an undergraduate course in Approximation Theory given by Professor Powell at the University of Cambridge. There are 24 chapters, from 9 to 15 pages in length, and, quoting from the preface, ..."it is possible to speak coherently on each chapter for about an hour...".

A wide range of topics, from classical to current, are covered. The selection of topics agrees with this reviewer; some are treated in more detail like minimax approximation and various topics in spline theory. Here is a list of the chapters: 1. The approximation problem and existence of best approximations. 2. The uniqueness of best approximations. 3. Approximation operators and some approximating functions. 4. Polynomial interpolation. 5. Divided differences. 6. The uniform convergence of polynomial approximations. 7. The theory of minimax approximation. 8. The exchange algorithm. 9. The convergence of the exchange algorithm. 10. Rational approximation by the exchange algorithm. 11. Least squares approximation. 12. Properties of orthogonal polynomials. 13. Approximation to periodic functions. 14. The theory of best L_1 approximation. 15. An example of L_1 approximation and the discrete case. 16. The order of convergence of polynomial approximations. 17. The uniform boundedness theorem. 18. Interpolation by piecewise polynomials. 19. *B*-splines. 20. Convergence properties of spline approximations. 21. Knot positions and the calculation of spline approximations. 22. The Peano kernel theorem. 23. Natural and perfect splines. 24. Optimal interpolation.

In spite of the topical nature, notation and style are unified throughout the book. A recurrent theme is that of Lebesgue constants. It must not be inferred from the shortness of the chapters that the treatment is fast and loose: It is not. What topics are taken up are exposed in good detail and with rigor. For a lecture course, it ought to be easy to pull out parts, substitute etc. The condensed mathematical style would probably make this book unsuitable for a first self-study book of the subject.

It is remarkable what an ideal student, with previous knowledge of computer programming, could do after reading this book and working its many exercises. Faced with an approximation problem (not involving differential equations or multi-dimensional functions on general domains), he or she could:

(i) Make a rational choice of method.

(ii) Program it, or use canned programs.

(iii) Furnish meaningful error estimates and insight in the properties and expected behavior of the method.

The second point is not stressed, but there is always given just enough algorithmic detail where it matters.

With the time constraints of an (US) undergraduate curriculum, where typically not more than two courses are devoted to Numerical Analysis, it is not likely that a whole course will be given in Approximation Theory. The demand to include Numerical Linear Algebra, numerical quadrature (to a larger extent than in this book) and numerical solution of integral equations, ordinary and partial differential equations, will preclude this. Therefore, this book can be expected to find its (US) audience among graduate students.

In conclusion, I call this a perfect no-nonsense introduction to Approximation Theory for a mathematically mature audience.

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23[2.25].—JET WIMP, *Sequence Transformations and Their Applications*, Mathematics in Science and Engineering, Vol. 154, Academic Press, New York, 1981, xix + 257 pp., $23\frac{1}{2}$ cm. Price $38.50.

It was remarked by Benjamin Disraeli that whenever he wished to learn something of a subject, he wrote a book about it. This principle is widely practiced by writers upon mathematics and computer science; their works are offered less as statements of existing knowledge than as exercises soliciting the appraisal of the informed reader. A summary judgement upon the book under review is that its contents derive more from uncritical reading of recent papers than from profound study.

The scope of the book is very briefly indicated by the following chapter synopses. 1 gives definitions concerning comparison of rates of convergence, and miscellaneous results dealing with some special sequences. 2–4 deal with Toeplitz methods, Richardson extrapolation, special methods using known properties of orthogonal polynomials and other linear methods. 5–8 consider nonlinear algorithms—Aitken's δ^2-process, the ε- and ρ-algorithms—and their connections with the algebraic theory of continued fractions. 9 deals briefly with the acceleration of sequences of vectors and with other nonlinear algorithms. 10, 11 deal with a general theory having roughly the following import: most convergence acceleration algorithms produce numbers which may be represented as components in the solutions of sets of linear algebraic equations; the point of each algorithm is that these numbers may alternatively be obtained by use of a simple algebraic recursion; the theory places emphasis upon the replacement of the simple recursion by the solution of a set of linear equations at each stage. 12 concerns acceleration methods based upon statistical considerations. 13 considers analytic transformations of two-dimensional infinite sums. The level of treatment is indicated by the author's prefatory remarks that it was not feasible to include very detailed and computational proofs, and that where he thinks that abstraction confuses rather than elucidates he has left well alone.

The author's choice of level of treatment has the immediately visible consequence that the proofs of many of the given theorems are stated as "*Obvious.*" or "*Left to the reader.*" or "*Trivial.*". A further consequence, which becomes apparent upon reading, is that the superficial is often preferred to the valuable. For example, one of the earlier theorems stated (attributed to Brezinski) is that if $\{v_j\}$ is a totally monotone sequence with $v_0 < \rho$, $c_i \geqslant 0$ $(i = 0, 1, \dots)$ and the series $\Sigma\, c_i \rho^i$ converges, then the sequence $\Sigma\, c_i v_j^i$ is totally monotone. As the author remarks, this result is an obvious consequence of the slightly more general result that if $\{v_{i,j}\}$ $(i = 0, 1, \dots)$ are all totally monotone, and with all $c_i \geqslant 0$, $\Sigma\, c_i v_{i,0}$ converges, then $\Sigma\, c_i v_{i,j}$ is totally monotone; the latter result itself being obvious. If results concerning power series and total monotonicity are to be given, perhaps the above might have been replaced by some auxiliary results given in a classic paper by Fejér [1], namely that if $\{c_j\}$ is totally monotone, then for $|\rho| < 1$ and $v_j = \Sigma_{i=j}^{\infty} c_i \rho^i$, the sequences $\{\mathrm{Re}\, v_j\}$, $\{\mathrm{Im}\, v_j\}$ and $|v_j|^2$ are also totally monotone; these results, not particularly difficult to derive, would interest the good student far more. Reading through the book it is often possible to suggest alternative results more interesting and useful than those stated.

The absence of treatment in depth results in some serious omissions, particularly with regard to acceleration algorithms connected with the analytic theory of continued fractions. This theory is equipped with *a posteriori* error estimates, deriving from the first stage in a convergence proof, and with *a priori* error bounds, deriving from comparison, made in the second stage, of these estimates with terms of a sequence tending to zero. Furthermore, the continued fractions obtained from power series whose coefficients are moments of a bounded nondecreasing function are associated with functions having a positive imaginary part in the upper half-plane, i.e., with solutions, having one sign, of Laplace's equation over a half-plane which can be used as a convenient reference domain to derive results concerning other domains. Naturally such functions have wide application in applied mathematics; in particular, much recent work in theoretical physics is the formulation of simple corollaries to the above convergence theory. As the reviewer has shown [2], classical series of numerical analysis—Newton's interpolation series, Newton's series for the derivative, the Euler-Maclaurin integration series—derived from extensive classes of functions, also generate continued fractions of the above type and may be accelerated with the security of rigorous error bounds. In the book under review, the analytic theory of continued fractions, together with its important applications, are entirely neglected.

Even within the limited frame of reference adopted, there are some significant gaps in the presentation. One of the most effective devices for the transformation of power series whose coefficients are, with alternating sign, moments of a bounded nondecreasing function, is a variant of the ε-algorithm in which a staircase sequence of numbers in the ε-array is taken as the initial sequence for the construction of a further array. For example, the sum of the first six terms of the series $\Sigma\binom{-1/2}{i}(i+1)^{-1} = 2(2^{1/2}-1) = 0.82842\ 71247\ 43\ldots$ is $0.81\ldots$; simple application of the ε-algorithm to these terms yields the estimate $0.82840\ldots$ and repeated application $0.82842\ 71247\ 49\ldots$. This mode of repetition of the ε-algorithm is not mentioned. (The author bases a number of his comparisons of numerical performance upon a recent survey by Ford and Smith in which similar omission occurs.) There are other omissions of the same kind.

Treatments of convergence behavior, where given, are largely concerned with comparisons of rates of convergence of initial and derived sequences. Thus with $\Sigma\, a_i, \Sigma\, b_i$ two convergent series, the second is said to converge faster than the first if $\Sigma_{i=n}\, b_i = o\{\Sigma_{i=n}\, a_i\}$ and so on. The given results might be of some help to the student of mathematics as elementary exercises in the use of the symbols O, o before he is ready for more substantial analysis, but they are of little use to the working numerical analyst who wishes to know, for example, the law concealed by the symbol o, and the numerical values of the dominant constants in this law.

In essence, the contents of the book reduce to a collection of algorithms, each presented with a motivation, some without error analysis, and some even without convergence proof. The numerical performance of the algorithms considered is for the most part illustrated with respect to standard series, $\Sigma(-1)^i(i+1)^{-1}$, $\Sigma(i+1)^{-2}$, $\Sigma(-1)^i i!$ and so on. Where error analyses and convergence proofs are not given, the reader is thus encouraged to judge the significance of a transformed sequence produced by a method when applied to an example for which the correct limit is unknown, from the appearance of the sequence. This is highly dubious practice. After some experience, every working numerical analyst encounters convergent sequences which initially appear to converge to the wrong limit. Indeed Gautschi [3] gives a nice example of such spurious convergence, and takes the trouble to explain how it arises. Such cases may, of course, be dismissed as examples of bad luck; but if one gambles often enough, one is sure to encounter misfortune.

The subject of the book is of prime importance. The fact that information otherwise to be obtained from billions of iterations of a recursive process can, subject to suitable preliminary theoretical investigation, be extracted with complete security from a half dozen or so iterations, has evident implications in all branches of applied mathematics and in numerical analysis in particular. But it is precisely its wide range of application that makes of convergence acceleration a difficult matter upon which to write. It is required of an author who writes with authority that he should be firmly grounded in the function theoretic bases of the algorithms considered, that he should be conversant with the branches of science in which they are applied, and that he should have sufficient practical experience to distinguish that which is useful from that which is not. In default of such an author, it is to be expected that Disraeli's principle will frequently be invoked; we may comfortably look forward to a number of books upon sequence transformations and their applications.

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1. L. FEJÉR, "Potenzreihen mit mehrfach monotoner Koeffizientenfolge und ihre Legendre-Polynome," *Proc. Cambridge Philos. Soc.*, v. 31, 1935, pp. 307–316.
2. P. WYNN, "Accélération de la convergence de séries d'opérateurs en analyse numérique," *C. R. Acad. Sci. Paris Ser. A-B*, v. 276A, 1973, pp. 803–806.
3. W. GAUTSCHI, "Anomalous convergence of a continued fraction for ratios of Kummer functions," *Math. Comp.*, v. 31, 1977, pp. 994–999.

24[4.05.2, 4.10.3, 4.15.3].—E. P. DOOLAN, J. J. H. MILLER & W. H. A. SCHILDERS, *Uniform Numerical Methods for Problems with Initial and Boundary Layers*, Boole Press, Dublin, 1980, xvi + 324 pp., 24 cm. Price \$60.00.

This monograph systematically addresses a relatively new class of numerical methods for singularly perturbed initial and boundary value problems, typical examples of which are

$$\text{(IVP)} \qquad \varepsilon u_x(x) + a(x)u(x) = f(x) \quad \text{for } x > 0,\ u(0) = A,$$

and

$$\text{(BVP)} \qquad \varepsilon u_{xx}(x) + a(x)u_x(x) - b(x)u(x) = f(x) \quad \text{for } 0 < x < 1,$$
$$u(0) = A \text{ and } u(1) = B.$$

In these problems ε is a positive constant in $(0, 1]$ which may be very small, $a(x) > 0$, $b(x) \geqslant 0$, and A and B are given constants. When ε is small, near $x = 0$ the solution $u(x)$ of (IVP) and (BVP) displays a boundary layer, i.e., a large gradient.

The presentation is expository while centering around the authors' research on finite difference methods for problems of the type (IVP) and (BVP) whose convergence is *uniform* for ε in $(0, 1]$ in the sense described below. Many of the results are new and have appeared previously in at most an abbreviated form.

Denoting the approximate solution obtained using a given finite difference scheme on an equally spaced mesh of size h by u^h (having value u_i^h at the ith mesh point), the scheme is said to be *uniformly convergent with order p* if the difference between u^h and the exact solution u at all the grid points is bounded by Ch^p where C and p are independent of *h and* ε. Uniformly convergent methods can be expected to be reliable for all values of ε even on coarse meshes. Such methods may thus also provide a sound starting point for various mesh refinement algorithms.

When ε is small relative to the mesh size, use of classical "centered" difference methods is quickly seen to lead to instability; e.g., defining $\rho = h/\varepsilon$ and approximating the solution of (IVP) when $a \equiv 1$ and $f \equiv 0$ with

$$\text{(C1)} \qquad \varepsilon(u_{i+1} - u_i)/h + (u_{i+1} + u_i)/2 = 0, \quad u_0 = A,$$

leads to

$$\text{(C2)} \qquad u_{i+1} = (1 - \rho/2)u_i/(1 + \rho/2)$$

which oscillates when $\rho > 2$. This type of instability can be suppressed by the use of "upwinding", e.g.,

$$\text{(W1)} \qquad \varepsilon(u_{i+1} - u_i)/h + u_{i+1} = 0, \quad u_0 = A,$$

however this still does not achieve uniform (in ε) convergence, since when $\rho = 1$ the error at $x = h$ remains a fixed nonzero quantity as $h \to 0$.

In the text necessary conditions are given for a finite difference scheme to be uniformly convergent for (IVP) or (BVP) (and for related problems). The general idea is that the scheme should be exact for the constant coefficient homogeneous problem, or equivalently, that the fundamental (exponential) solution behavior should be built into the coefficients of the difference scheme. Such schemes are called *exponentially fitted*. A uniformly accurate scheme for (IVP) is

$$\text{(U1)} \qquad \varepsilon\sigma_i(\rho)(u_{i+1} - u_i)/h + a(x_i)u_i = f(x_i), \quad u_0 = A,$$

where $\rho \equiv h/\varepsilon$ and the *exponential fitting factor* σ_i is defined by

$$\text{(U2)} \qquad \sigma_i(\rho) = \rho a(x_i)/[1 - \exp(-\rho a(x_i))].$$

For (BVP), the original uniform scheme, which was formulated by Allen and Southwell [1], is

$$\text{(U3)} \qquad \begin{aligned} \varepsilon\sigma_i(\rho)(u_{i-1} - 2u_i + u_{i+1})/h^2 + a(x_i)(u_{i+1} - u_{i-1})/(2h) - b(x_i)u_i \\ = f(x_i), \qquad i = 1,\dots,N-1, \\ \rho \equiv h/\varepsilon, N \equiv 1/h, u_0 = A, u_N = B, \sigma_i(\rho) = \tfrac{1}{2}\rho a(x_i)\coth\left(\tfrac{1}{2}\rho a(x_i)\right). \end{aligned}$$

Both these schemes are uniformly convergent with order 1.

The error analysis for these (and many other) finite difference methods is carried out through the use of, and in a manner designed to illustrate, three general approaches. All utilize a priori analysis of the behavior of the solution of the original problem, and the fact that in each case the differential equation and its difference approximation satisfy a maximum principle. The *two mesh method*, used first by Il'in [2] to prove uniform first order convergence for (U3), and posed as a systematic approach by Miller [4], states that a scheme is uniformly convergent with order p if and only if the scheme is convergent (for each fixed ε) and the difference in grid values for a successive mesh halving is uniformly of order p, i.e.,

$$|u_i^h - u_{2i}^{h/2}| \leqslant C_2 h^p$$

with C_2 and p independent of h, i, and ε.

The second approach, which the authors attribute to Emelyanov, Shishkin, and Titov, is to use a classical error bound based on the local truncation error for $\varepsilon \geqslant h^r$, for some appropriate choice of r, and then to use an asymptotic expansion of the solution to obtain an error bound for $\varepsilon \leqslant h^r$; the combination of the two estimates yielding the desired result.

The third approach hinges on the choice of comparison (barrier) functions derived specifically from the difference scheme being analyzed. This, together with certain a priori knowledge of the behavior of the solution, can be used to produce error estimates, as typified by the work of Kellogg and Tsan [3].

The text is divided into three parts, the first treating the initial value problem (cf. (IVP)). Basic properties and asymptotic expansions of the solution of the continuous problem are developed, and the behavior and limitations of classical difference schemes are described. Necessary conditions for a scheme to be uniformly convergent are given, and some specific exponentially fitted schemes are proven to be uniformly convergent. Other topics considered are extrapolation, uniformly accurate higher order schemes, systems, nonlinear problems, and open questions. In the second part of the text, boundary value problems (cf. (BVP)) are treated along an analogous program. In addition to (BVP), the selfadjoint problem

$$\text{(SA)} \qquad -\varepsilon u_{xx}(x) + b(x)u(x) = f(x) \quad \text{for } 0 < x < 1,\ u(0) = A \text{ and } u(1) = B,$$

where $b(x) > 0$, is considered, as well as the conservation form equations corresponding to (BVP) and (SA). Mixed boundary conditions are also treated. The last section contains a wide range of numerical results illustrating the behavior of the finite difference methods discussed in the first two parts, along with a representative Fortran program listing. Very helpful lists of notation and terminology are included, as is an extensive bibliography.

Altogether, this monograph presents a very lucid account of the use and analysis of exponential fitting to obtain uniformly convergent finite difference schemes for singular perturbation problems. Many of the results are new and anyone working in this field will want to have ready access to this text. It also provides a concise and accessible introduction to this area of study. In particular, the first section dealing with initial value problems provides a superb introduction to the fundamental concepts while the algebra involved is quite tractable (in contrast to the convergence proofs for boundary value problems where the algebra is rather formidable, regardless of the approach taken to attain the result).

While no errors affecting the validity of the results were noted, the following comments might perhaps save the reader some effort in following a few parts of the exposition. On page 24, u_1^h should be $e^{-\rho}$ etc. The equality on page 28 for Q_i can be verified by comparing terms involving $\sigma(\rho)$ and by using the identity $\coth(z) = (e^z + e^{-z})/(e^z - e^{-z})$. On page 42 the second term inside the braces expressing $V_{1,i}$ should read $-\exp(-\rho a(x_i))$. Also Theorem 1 in Appendix B is not correct as stated (e.g. take $\rho = 1$, $a \equiv 1$, $\alpha_1 = 2$, $\alpha_2 = -1$, $\alpha_3 = -1$, $\beta_1 = 3$, $\beta_2 = 1$, $\beta_3 = 5$; then (b) fails); however, wherever it is invoked the approach of writing $e^x - e^y = (x - y)e^{\xi} = (x - y)e^y + .5(x - y)^2 e^{\eta}$ (for some ξ and η between x and y) and recalling the fact that $x^r \exp(-cx)$ is bounded for $x \geqslant 0$ (for r and c fixed positive constants) can be used to obtain the desired bound. The equality used in the proof of Lemma 10.1 on page 60 did not seem to be obvious; it can be verified by multiplying through by Qe^s, comparing coefficients of s^pQ^q for $p, q \geqslant 0$, and then using induction on p to establish the necessary combinatorial identity. The inequality on the top of page 107 in the brief sketch of the proof that the scheme (U3) is uniformly accurate is not right. The direction of the inequality should be reversed, and then the result is still only valid for $h \geqslant \varepsilon$ (e.g., it is clearly not correct for $\varepsilon = 1$). The (lengthy) complete proof can be found in Miller [4] (two mesh method) and Kellogg and Tsan [3] (comparison functions). Also the second term on the right side of (7.5) is bounded by a constant times the first and so can be omitted; the error estimate for (U3) is thus

$$|u(x_i) - u_i^h| \leqslant Ch^2/(h + \varepsilon) \quad \text{for each } i.$$

In the discussion below (7.6) on page 109 there is no contradiction since the result quoted also requires that the Q weights be nonnegative and evaluations of f occur only at x_{i-1}, x_i and x_{i+1} (this discussion is later correctly continued on pages 181–182).

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1. D. N. DE G. ALLEN & R. V. SOUTHWELL, "Relaxation methods applied to determine the motion, in two dimensions, of a viscous fluid past a fixed cylinder," *Quart J. Mech. Appl. Math.*, v. 8, 1955, pp. 129–145.

2. A. M. IL'IN, "Differencing scheme for a differential equation with a small parameter affecting the highest derivative," *Mat. Zametki*, v. 6, 1969, pp. 237–248 = *Math. Notes*, v. 6, 1969, pp. 596–602.

3. R. B. KELLOGG & A. TSAN, "Analysis of some difference approximations for a singular perturbation problem without turning points," *Math. Comp.*, v. 32, 1978, pp. 1025–1039.

4. J. J. H. MILLER, "Sufficient conditions for the convergence, uniformly in epsilon, of a three point difference scheme for a singular perturbation problem," *Numerical Treatment of Differential Equations in Applications* (R. Ansorge and W. Tornig, Eds.), Lecture Notes in Math., vol. 679, Springer-Verlag, Berlin and New York, 1978, pp. 85–91.

25[5.00, 6.30].—R. GLOWINSKI, J. L. LIONS & R. TREMOLIERS, *Numerical Analysis of Variational Inequalities*, North-Holland, Amsterdam, 1981, xxx + 776 pp., 23 cm. Price $109.75, Dfl. 225.-.

This book is really a compilation of three volumes. Chapters 1–3 and Chapters 4–6 are the respective English translations of volumes I and II of the French edition which appeared in 1976. Following these chapters there are six appendices covering material on variational inequalities developed since the publication of the French edition.

Since a review of the French edition appeared in *Math. Comp.*, v. 32, 1978, pp. 313–314, we give only a brief synopsis of the first six chapters and concentrate on the additional material contained in the appendices.

Chapter 1 deals with the general theory of stationary variational inequalities, Chapter 2 with solving the finite dimensional optimization problems which result from the approximation schemes, and Chapter 3 with the specific model problem of elasto-plastic torsion of a cylindrical bar. The problem of a nondifferentiable cost functional is considered in Chapters 4 and 5, with examples such as the steady flow of a Bingham fluid in a cylindrical duct. Chapter 6 contains a discussion of some general approximation schemes for time dependent variational inequalities.

It is the goal of the appendices to treat what the authors consider to be the most important contributions to the subject since the publication of the original French edition. That substantial progress has been made is evidenced by the fact that the appendices comprise about one third of this book.

For example, one important development has been the estimation of approximation errors in connection with the use of finite element approximation schemes. This material is now heavily represented with results for the obstacle problem in Appendix 1, the elasto-plastic torsion problem in Appendix 2, and the steady flow of a Bingham fluid in Appendix 4.

Besides further discussion of topics presented in the earlier edition such as optimization algorithms, the appendices also contain new applications of the ideas of variational inequalities. These include the solution of nonlinear Dirichlet problems, a brief discussion of quasi-variational inequalities, and the numerical simulation of the transonic potential flow of ideal compressible fluids.

With the additional material now included in the present volume, this book is certainly an essential reference for anyone interested in the numerical solution of problems that can be formulated as variational inequalities.

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26[2.05.3].—HERBERT E. SALZER, NORMAN LEVINE & SAUL SERBEN, *Tables for Lagrangian Interpolation Using Chebyshev Points*, manuscript of 54 pages typewritten text + 267 pages of tables, xeroxed and slightly reduced from computer print-out sheets, deposited in the UMT file.

For n-point Lagrangian interpolation for $f(x)$ given at the Chebyshev points $x_{n,i} = -\cos[(2i - 1)\pi/2n]$, $i = 1(1)n$, there are two tables. The first, which is an auxiliary table of $x_{n,i}$ for every n, and $s_{n,i} = \sin[(2i - 1)\pi/2n]$ for the old values of n, for $n = 2(1)25(5)50(10)100$, to 25 significant figures, is intended primarily for storage in a computer program for calculating the interpolation coefficients in barycentric form. The second, which is the main table, giving the interpolation coefficients themselves, just for $n = 20$, but for $x = -1(0.001)1$, to 20 significant figures, is convenient also for desk calculation with small computers.

The following topics are included in the introductory text: Relation of tables, use of tables for interpolation and quadrature, possible application to equally spaced arguments, advantages in Chebyshev-point interpolation (minimal remainder term, with convergence and stability of coefficients for increasing n), use of tables for Chebyshev economization as an alternative to the methods of C. Lanczos and C. W. Clenshaw, further development of computational methods using interpolation at Chebyshev nodes (especially in numerical integration), description of computation and checking of the tables, and 44 references.

These are some of the more important points in the text which have not been sufficiently noted or emphasized elsewhere in the literature: For practical applications, the advantage in the much smaller upper bound for the classical remainder term is not nearly so important as the *convergence of the interpolation polynomial* as $n \to \infty$ for the wide class of continuous functions satisfying the Dini-Lipschitz condition in the real interval $[-1, 1]$ (this includes functions with a bounded first derivative which in turn includes analytic functions) *in conjunction with the much smaller interpolation coefficients* (e.g., for $n = 100$ the largest barely exceeds 1, whereas for equal spacing some coefficients exceed 10^{25}; furthermore, since the sum of the absolute values of the coefficients $\leqslant 1 + (2/\pi)\ln n$, the factor for total round-off error is < 4). On the basis of the preceding remarks, instead of the global methods of Lanczos which employ the properties of the Chebyshev polynomials to produce coefficients of an economized polynomial, or of Clenshaw which operate with the coefficients in Chebyshev series expansions, here functional values $f(x_{n,i})$ replace the polynomial or Chebyshev coefficients, for use over the entire range in x (normalized to $[-1, 1]$). After all operations and calculations pertaining to any of a wide class of problems have been completed, we end up with a skeletal set of final answers $f(x_{n,i})$, $i = 1(1)n$, from which $f(x)$ is found immediately by using these tables which are capable of *global interpolation*.

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27[9.00].—M. I. KNOPP, (Editor), *Analytic Number Theory* (Proc. Conf. held at Temple University, Philadelphia, May 12–15, 1980), Lecture Notes in Math., vol. 899, Springer-Verlag, Berlin and New York, 1981, x + 478 pp., 22 cm. Price $24.50.

The conference mentioned in the title was held on the occasion of the proforma retirement of Emil Grosswald.

The volume contains detailed versions of most of the lectures given at the conference and covers a wide range of subjects in analytic number theory. Of particular interest from the standpoint of computation are the following six articles, for which we include capsule reviews:

(1) Ronald Alter, "Computations and generalizations of a remark of Ramanujan." This paper presents extensive tables of $r(m, n, s)$, the smallest positive integer that can be expressed as a sum of m positive nth powers in s different ways.

(2) Robert J. Anderson and Harold Stark, "Oscillation theorems." The authors give an illuminating discussion of oscillation theorems for the sum-functions of some familiar arithmetic functions; specifically, they discuss various methods for obtaining numerical estimates for the lim sup and lim inf of $x^{-1/2}M(x)$, where $M(x) = \sum_{n \leq x} \mu(n)$ and μ denotes the Möbius function.

(3) Harold G. Diamond and Kevin S. McCurley, "Constructive elementary estimates for $M(x)$." The paper shows how arguments akin to those of Chebyshev can be combined with a finite amount of computation to produce elementary numerical upper estimates of very small size for lim sup $x^{-1}|M(x)|$; needless to say, the prime number theorem implies that this lim sup is actually zero.

(4) Steven M. Gonek, "The zeros of Hurwitz's zeta function on $\sigma = \frac{1}{2}$". The author shows that for certain rational values of x the proportion of zeros of $\zeta(s, \alpha)$, which have real part $\frac{1}{2}$, is definitely less than one.

(5) Peter Hagis, Jr., "On the second largest prime divisor of an odd perfect number." On the basis of extensive computer calculations and searches the paper proves that the prime mentioned in the title must be greater than 1000, under the assumption that odd perfect numbers exist.

(6) Julia Mueller, "Gaps between consecutive zeta zeros." Assuming the Riemann Hypothesis, the author proves that, if the zeros of the Riemann zeta function in the

upper half-plane are $\frac{1}{2} + \gamma_1, \frac{1}{2} + \gamma_2, \ldots$, where $\gamma_1 \leqslant \gamma_2 \leqslant \cdots$, and if

$$\lambda_n = (2\pi)^{-1}(\gamma_{n+1} - \gamma_n)\log \gamma_n,$$

then lim sup $\lambda_n > 1.9$; it is well known that the average value of λ_n is 1.

Other authors represented in this valuable collection are G. Andrews, P. Bateman, B. Berndt, D. Bressoud, H. Cohn, T. Cusick, P. Erdös, L. Goldstein, B. Gordon, E. Grosswald, J. Hafner, K. Hughes, M. Knopp, J. Lagarias, D. Lehmer, E. Lehmer, J. Lehner, T. Metzger, M. Nathanson, D. Newman, M. Newman, A. Parson, C. Pomerance, M. Sheingorn, E. Straus, A. Terras, and L. Washington.

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