On the Convergence Behavior of Continued Fractions with Real Elements*

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Abstract. We define the notion of transient (geometric) convergence rate for infinite series and continued fractions. For a class of continued fractions with real elements we prove a monotonicity property for such convergence rates which helps explain the effectiveness of certain continued fractions known to converge "only" sublinearly. This is illustrated in the case of Legendre's continued fraction for the incomplete gamma function.

1. Introduction. Continued fractions, as is well known, can be viewed in terms of infinite series. To describe the convergence behavior of a series it is useful to consider the notion of *transient (geometric) convergence rate*. Given a convergent series $\sum_{n=0}^{\infty} t_n$, the n th transient convergence rate is the quantity $|\rho_n|$, $n = 1, 2, \dots$, where $t_n = \rho_n t_{n-1}$ (assuming $t_{n-1} \neq 0$). If $\lim_{n \rightarrow \infty} |\rho_n| = r$, $0 \leq r \leq 1$, convergence is *linear* (geometric) with convergence rate r , if $0 < r < 1$, *superlinear*, if $r = 0$, and *sublinear* if $r = 1$. It is important to note, however, that these concepts are asymptotic in nature, hence not necessarily relevant for numerical (finite!) computation. Thus, a series need not be dismissed as useless, simply because it converges only sublinearly. The approach of $|\rho_n|$ to the limit 1 indeed may be so slow that the series has "converged to machine precision" long before $|\rho_n|$ reaches the neighborhood of 1. For this reason, convergence of a series ought to be judged on the basis of the complete sequence $\{\rho_n\}$ of convergence rates, and not just on the basis of asymptotic properties of ρ_n . In this connection, properties of monotone behavior significantly add to the understanding of the quality of convergence.

The purpose of this note is to prove a criterion for the sequence $\{|\rho_n|\}$ to be (ultimately) monotonically increasing, in the case where the partial sums of the series are convergents of a continued fraction with real elements. We illustrate the result with Legendre's continued fraction for the incomplete gamma function, which, though sublinearly convergent, provides an effective tool of numerical computation.

2. Continued Fractions and Infinite Series. We consider continued fractions of the form

$$(2.1) \quad c = \frac{1}{1 + \frac{a_1}{1 + \frac{a_2}{1 + \dots}}},$$

where, for some integer $k_0 \geq 1$,

$$(2.2) \quad \begin{aligned} a_k &> 0 && \text{for } 1 \leq k \leq k_0 - 1, \\ a_k &< 0 && \text{and } |a_k| \leq \frac{1}{4} \text{ for } k \geq k_0. \end{aligned}$$

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It can be seen from Worpitzky's theorem (Henrici [3, p. 506]) that the tail of the continued fraction (2.1) beginning with the element a_{k_0} , hence also the complete continued fraction, converges. The infinite series

$$(2.3) \quad s = \sum_{k=0}^{\infty} t_k \quad (t_0 = 1)$$

is equivalent to the continued fraction (2.1) if its n th partial sum

$$(2.4) \quad s_n = 1 + \sum_{k=1}^{n-1} t_k$$

is equal to the n th convergent of c , for each $n = 1, 2, 3, \dots$. According to Euler,

$$(2.5) \quad s_1 = 1, \quad s_{k+1} = s_k + t_k, \quad k = 1, 2, 3, \dots,$$

where

$$(2.6) \quad \left. \begin{aligned} \rho_0 &= 0, \quad t_0 = 1, \\ \rho_k &= \frac{-a_k(1 + \rho_{k-1})}{1 + a_k(1 + \rho_{k-1})} \\ t_k &= \rho_k t_{k-1} \end{aligned} \right\} \quad k = 1, 2, 3, \dots$$

This represents a convenient algorithm for evaluating the continued fraction c , and is also useful for analyzing qualitative properties of convergence. Note indeed that the quantities ρ_n in (2.6) yield the transient convergence rates $|\rho_n|$ of the series (2.3).

Slightly more convenient for analytical purposes are the quantities $\sigma_k = 1 + \rho_k$, which satisfy

$$(2.7) \quad \sigma_0 = 1, \quad \sigma_k = \frac{1}{1 + a_k \sigma_{k-1}}, \quad k = 1, 2, 3, \dots$$

3. Convergence Behavior. Some first insights into the convergence behavior of the continued fraction (2.1) can be gained from the following lemma.

LEMMA 3.1. *If the partial numerators a_k in (2.1) satisfy (2.2), then the quantities σ_k in (2.7) satisfy*

$$(3.1) \quad 0 < \sigma_k < 1 \quad \text{for } 1 \leq k \leq k_0 - 1,$$

and

$$(3.2) \quad 1 < \sigma_k \leq \frac{2(k - k_0 + 2)}{k - k_0 + 3} \quad \text{for } k \geq k_0.$$

Proof. The inequalities (3.1) follow immediately from the positivity of a_k and (2.7). To prove (3.2), we use induction. Since $-\frac{1}{4} \leq a_{k_0} < 0$ and $0 < \sigma_{k_0-1} \leq 1$, we have $1 < \sigma_{k_0} \leq 4/3$, so that (3.2) is true for $k = k_0$. Assuming its truth for some $k \geq k_0$, we obtain

$$1 < \sigma_{k+1} = \frac{1}{1 + a_{k+1} \sigma_k} \leq \frac{1}{1 - \frac{1}{4} \frac{2(k - k_0 + 2)}{k - k_0 + 3}} = \frac{2(k - k_0 + 3)}{k - k_0 + 4},$$

which is (3.2) with k replaced by $k + 1$. $\square$

Lemma 3.1, in particular, implies $0 < \sigma_k < 2$, hence $-1 < \rho_k < 1$, for all $k \geq 1$. The series (2.3), therefore, has terms that are strictly decreasing in absolute value. Furthermore, by (3.1) and (3.2),

$$(3.3) \quad -1 < \rho_k < 0 \quad \text{for } 1 \leq k \leq k_0 - 1, \quad \text{and} \quad 0 < \rho_k < 1 \quad \text{for } k \geq k_0,$$

so that the series initially (if $k_0 > 1$) behaves like an alternating series and subsequently turns into a monotone series.

A more detailed description of convergence is provided by the following theorem.

THEOREM 3.1. *If the partial numerators a_k in (2.1) satisfy (2.2), and in addition $-\frac{1}{4} \leq a_{k+1} \leq a_k < 0$ for $k \geq k_0$, then*

$$(3.4) \quad -1 < \rho_k < 0 \quad \text{for } 1 \leq k \leq k_0 - 1 \quad \text{and} \quad \rho_{k+1} > \rho_k > 0 \quad \text{for } k \geq k_0.$$

In particular,

$$(3.5) \quad \lim_{k \rightarrow \infty} \rho_k = \rho, \quad \rho = \frac{1 - \sqrt{1 + 4a}}{1 + \sqrt{1 + 4a}},$$

where $a = \lim_{k \rightarrow \infty} a_k$; the continued fraction (2.1) converges linearly, with convergence rate ρ , if $a > -\frac{1}{4}$, and sublinearly if $a = -\frac{1}{4}$.

Proof. The first inequalities in (3.4) have already been noted in (3.3). The others are equivalent to $\sigma_{k+1} > \sigma_k > 1$ for $k \geq k_0$. Since $\sigma_k > 1$, by (3.2), it suffices to prove

$$(3.6) \quad \sigma_{k+1} > \sigma_k \quad \text{for } k \geq k_0.$$

We first show

$$(3.7) \quad \sigma_{k-1} < \frac{2}{1 + \sqrt{1 - 4|a_k|}} \quad \text{for } k \geq k_0.$$

This is true for $k = k_0$, since by (3.1) (and (2.7), if $k_0 = 1$) $\sigma_{k_0-1} \leq 1$, while the expression on the right of (3.7) is greater than 1. Using induction, assume that (3.7) holds for some $k \geq k_0$. Then

$$(3.8) \quad \begin{aligned} \sigma_k &= \frac{1}{1 + a_k \sigma_{k-1}} < \frac{1}{1 - |a_k| \frac{2}{1 + \sqrt{1 - 4|a_k|}}} \\ &\leq \frac{1}{1 - \frac{2|a_{k+1}|}{1 + \sqrt{1 - 4|a_{k+1}|}}}, \end{aligned}$$

where in the last inequality we have used $|a_{k+1}| \geq |a_k|$. Now observe that, for any $\alpha \leq \frac{1}{4}$,

$$\begin{aligned} \frac{1}{1 - \frac{2\alpha}{1 + \sqrt{1 - 4\alpha}}} &= \frac{1 + \sqrt{1 - 4\alpha}}{1 + \sqrt{1 - 4\alpha} - 2\alpha} \\ &= \frac{1 - (1 - 4\alpha)}{1 - (1 - 4\alpha) - 2\alpha(1 - \sqrt{1 - 4\alpha})} = \frac{2}{1 + \sqrt{1 - 4\alpha}}. \end{aligned}$$

Using this in (3.8), with $\alpha = |a_{k+1}|$, yields (3.7) with k replaced by $k+1$, and thus establishes (3.7) for all $k \geq k_0$.

Now (3.6), in view of (2.7), is equivalent to

$$\frac{1}{1 + a_{k+1}\sigma_k} > \sigma_k \quad \text{for } k \geq k_0,$$

which in turn, since $1 + a_{k+1}\sigma_k > 0$ and $a_{k+1} < 0$ for $k \geq k_0$, is equivalent to

$$|a_{k+1}|\sigma_k^2 - \sigma_k + 1 > 0.$$

The quadratic function $|a_{k+1}|t^2 - t + 1$ is convex and has two real zeros $t_{1,k+1} < t_{2,k+1}$, the smaller of which is

$$t_{1,k+1} = \frac{2}{1 + \sqrt{1 - 4|a_{k+1}|}}.$$

By (3.7), $\sigma_k < t_{1,k+1}$, hence $|a_{k+1}|\sigma_k^2 - \sigma_k + 1 > 0$, which implies $\sigma_{k+1} > \sigma_k$. This proves (3.6).

Since the sequence $\{a_k\}$ is monotonically decreasing for $k \geq k_0$, and bounded below by $-\frac{1}{4}$, the limit $\lim_{k \rightarrow \infty} a_k = a$ exists, and $-\frac{1}{4} \leq a < 0$, since $a_{k_0} < 0$. Similarly, $\lim_{k \rightarrow \infty} \rho_k = \rho$, $0 < \rho \leq 1$, and $\lim_{k \rightarrow \infty} \sigma_k = \sigma$ with $\sigma = 1 + \rho$. Going to the limit $k \rightarrow \infty$ in (2.7) then gives

$$\sigma = \frac{1}{1 + a\sigma}, \quad \sigma = \frac{2}{1 \pm \sqrt{1 + 4a}}.$$

Since $\sigma \leq 2$ and $-\frac{1}{4} \leq a < 0$, the minus sign in the last equation for σ cannot hold (unless $a = -\frac{1}{4}$), and we conclude that

$$\sigma = \frac{2}{1 + \sqrt{1 + 4a}}, \quad \rho = \sigma - 1 = \frac{1 - \sqrt{1 + 4a}}{1 + \sqrt{1 + 4a}},$$

which is (3.5). The last statement of the theorem is an immediate consequence of (3.5). This completes the proof of Theorem 3.1.

4. Truncation. In practice, the continued fraction (2.1) is evaluated by carrying out (2.5) and (2.6) for $k = 1, 2, \dots, n$ and taking s_{n+1} to approximate the value of s (or c) of the continued fraction. It is important, then, to be able to choose n in such a way that s_{n+1} approximates s to any prescribed accuracy.

Assuming first $k_0 = 1$, hence $0 < \rho_k < 1$ by (3.3) and $0 < t_k < 1$, it follows from a result of Merkes [4, Eq. (12)] that

$$(4.1) \quad |s - s_{n+1}| \leq \frac{1 + \rho_n}{1 - \rho_n} t_n.$$

This suggests the following *stopping rule*: Given a prescribed (relative) accuracy ε , stop the recursion (2.6) at the first integer $k = n$ for which

$$(4.2) \quad (1 + \rho_n)t_n \leq (1 - \rho_n)s_{n+1}\varepsilon.$$

By (4.1), this implies $|s - s_{n+1}| \leq s_{n+1}\varepsilon$, hence

$$\frac{|s - s_{n+1}|}{s + |s_{n+1} - s|} \leq \frac{|s - s_{n+1}|}{s_{n+1}} \leq \varepsilon,$$

from which $|s - s_{n+1}| \leq s\varepsilon + |s_{n+1} - s|\varepsilon$, that is,

$$(4.3) \quad \left| \frac{s - s_{n+1}}{s} \right| \leq \frac{\varepsilon}{1 - \varepsilon}.$$

Our stopping rule therefore achieves the desired accuracy, at least asymptotically for $\varepsilon \rightarrow 0$.

To avail oneself of this simple stopping rule, when $k_0 > 1$, one ought to first evaluate the “tail”

$$(4.4) \quad c_{k_0} = \frac{1}{1 +} \frac{a_{k_0}}{1 +} \frac{a_{k_0+1}}{1 +} \dots$$

of the continued fraction (2.1), to which Merkes’ result applies, and then compute

$$(4.5) \quad c_k = \frac{1}{1 + a_k c_{k+1}} \quad \text{for } k = k_0 - 1, k_0 - 2, \dots, 1,$$

to get the complete continued fraction $c = c_1$. Since $c_{k_0} > 0$ and $a_k > 0$ for $k < k_0$, the computation in (4.5) involves the addition of positive numbers and division, hence only numerically stable operations.

5. An Example. Theorem 3.1 is applicable to Legendre’s continued fraction for the incomplete gamma function,

$$(5.1) \quad \begin{aligned} (x - \alpha + 1)x^{-\alpha}e^x\Gamma(\alpha, x) &= \frac{1}{1 +} \frac{a_1}{1 +} \frac{a_2}{1 +} \dots, \\ a_k &= \frac{k(\alpha - k)}{(x - \alpha + 2k - 1)(x - \alpha + 2k + 1)}, \quad k = 1, 2, 3, \dots, \end{aligned}$$

which is used in [1, p. 475], [2] to compute the incomplete gamma function in the domain D : $x \geq 1.5$, $-\infty < \alpha \leq x + \frac{1}{4}$. Assuming α not a positive integer (otherwise, the continued fraction (5.1) would terminate and our assumption (2.2) would be violated), we have for $(x, \alpha) \in D$

$$(5.2) \quad k_0 = \begin{cases} 1 & \text{if } \alpha < 1, \\ 1 + [\alpha] & \text{if } \alpha > 1. \end{cases}$$

If $k \geq k_0$, the condition $|a_k| \leq \frac{1}{4}$ is equivalent to $(x - \alpha)^2 + 4kx \geq 1$, hence satisfied if $x \geq \frac{1}{4}$ (since $k \geq 1$). An elementary calculation furthermore shows that $|a_{k+1}| \geq |a_k|$ for $k \geq k_0$ whenever $x \geq \frac{1}{2}$. It follows, in particular, that all assumptions of Theorem 3.1 are satisfied when $(x, \alpha) \in D$. Since clearly $a = \lim_{k \rightarrow \infty} a_k = -\frac{1}{4}$, we are in a case of sublinear convergence. (This is also noted by Henrici [3, p. 629] by way of a different analysis.) Nevertheless, the continued fraction is known to be quite useful as a computational tool, at least in a domain such as D . The reason for this is readily understood on the basis of Theorem 3.1: Although the transient convergence rates ρ_k eventually increase monotonically to 1, the limit is approached quite slowly. We can see this from Table 5.1 which, in the case of the continued fraction (4.4), and for selected x and α , displays the values of

$$n_\nu = \max\left(k: |\rho_k| \leq \frac{\nu}{4}\right) \quad \text{and} \quad \varepsilon_\nu = \frac{4 + \nu}{4 - \nu} t_{n_\nu}, \quad \nu = 1, 2, 3.$$

TABLE 5.1
Convergence behavior of the continued fraction (5.1)
(Numbers in parentheses indicate decimal exponents.)

x	α	n_1	ε_1	n_2	ε_2	n_3	ε_3
1.5	1.75	3	1.9(-3)	13	2.9(-7)	75	6.7(-18)
	.875	3	9.2(-4)	13	1.2(-7)	75	2.7(-18)
	0.0	3	6.8(-3)	13	1.4(-6)	75	3.7(-17)
	-3.5	3	7.3(-3)	13	1.4(-6)	75	3.5(-17)
	-7.0	4	8.9(-4)	15	5.8(-8)	79	5.5(-19)
5.0	5.25	9	1.9(-8)	41	5.4(-21)	243	2.9(-56)
	2.625	10	1.0(-9)	41	7.3(-22)	243	3.0(-57)
	0.0	10	9.8(-10)	42	2.9(-22)	244	1.6(-57)
	-10.5	9	1.9(-8)	42	1.6(-21)	244	8.5(-57)
	-21.0	12	2.7(-11)	48	1.2(-25)	254	8.8(-62)
10.0	10.25	18	1.1(-16)	81	1.2(-41)	483	3.0(-112)
	5.125	18	8.6(-17)	81	7.1(-42)	483	1.5(-112)
	0.0	20	8.0(-20)	83	8.1(-45)	485	1.9(-115)
	-20.5	19	1.7(-17)	83	6.9(-43)	485	1.6(-113)
	-41.0	24	1.5(-22)	95	5.0(-51)	504	4.2(-123)
20.0	20.25	36	3.0(-33)	161	4.8(-83)	963	2.9(-224)
	10.125	36	1.2(-33)	162	5.2(-84)	964	3.1(-225)
	0.0	40	2.9(-40)	165	3.4(-90)	967	1.6(-231)
	-40.5	38	4.3(-35)	165	8.8(-86)	968	3.4(-227)
	-81.0	48	3.1(-45)	188	1.2(-101)	1004	6.8(-246)

Note that by virtue of (4.1), and the fact that $1 < s = c_{k_0} \leq 2$ (cf. [3, Theorem 12.3c]),

$$(5.3) \quad \left| \frac{s - s_{n_\nu+1}}{s} \right| \leq |s - s_{n_\nu+1}| \leq \varepsilon_\nu, \quad \nu = 1, 2, 3.$$

Thus, for example, if $x = 5$, $\alpha = 0$, by the time the transient convergence rate has risen to $\frac{1}{2}$, the continued fraction has already converged to within a (relative) error of about 3×10^{-22} .

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