MATHEMATICS OF COMPUTATION
VOLUME 41, NUMBER 163
JULY 1983, PAGES 13-25

On the Simplified Hybrid-Combined Method*

By Zi-Cai Li and Guo-Ping Liang*****

Abstract. In order to solve the boundary value problems of elliptic equations, especially with singularities and unbounded domains, the simplified hybrid-combined method, which is equivalent to the coupling method of Zienkiewicz et al. [15], is presented. This is a combination of the Ritz-Galerkin and the finite element methods. Its optimal error estimates are proved in this paper, and the solution strategy of its algebraic equation system is discussed.

1. Introduction. It has been shown to be advantageous to use a combination of the Ritz-Galerkin and finite element methods for the boundary value problems of elliptic equations, especially with singularities and unbounded domains which are, with difficulty, solved by the finite element method. A combination with the coupling trick of the simplified hybrid method is given in this paper.

Let us consider the general elliptic equation

$$\mathfrak{L}u = -\frac{\partial}{\partial x}\left(\beta\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(\beta\frac{\partial u}{\partial y}\right) + cu = f, \qquad (x, y) \in S, \tag{1.1}$$

with the Dirichlet boundary condition

$$u = g, \qquad (x, y) \in \Gamma, \tag{1.2}$$

where S is a bounded and simply connected domain with the boundary Γ, the operator

$$\mathfrak{L} = -\frac{\partial}{\partial x}\beta\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\beta\frac{\partial}{\partial y} + c,$$

the functions β, c and f are sufficiently smooth, $c = c(x, y) \geqslant 0$, and $\beta = \beta(x, y) \geqslant \beta_0 > 0$; here β_0 is a constant.

The problem (1.1) and (1.2) can be expressed in a weak form

$$a(u, v) = f(v), \qquad v \in H_0^1(S), \tag{1.3}$$

where the true solution $u \in H_*^1(S)$; the notations are

$$a(u, v) = \int_S \left[\beta(u_x v_x + u_y v_y) + cuv\right], \tag{1.4}$$

$$f(v) = \int_S fv, \tag{1.5}$$

Received February 23, 1981; revised March 10, 1982.

1980 *Mathematics Subject Classification.* Primary 65N10, 65N30.

*This research was supported by Department of Computer Science, University of Toronto, Shanghai Institute of Computer Technology, and Institute of Mathematics, Academia Sinica.

***Present address*: Department of Computer Science, University of Toronto, Toronto, Canada M5S 1A7.

***Institute of Mathematics, Academia Sinica, Peking, China.

0025-5718/83/0000-1371/$03.75

and the spaces are defined as

$$H_0^1(S) = \{v, v_x, v_y \in L^2(S), v|_\Gamma = 0\}, \tag{1.6}$$

$$H_*^1(S) = \{v, v_x, v_y \in L^2(S), v|_\Gamma = g\}. \tag{1.7}$$

As is well known, the finite element method is a procedure based on (1.3) for the admissible functions v in the subspace consisting of piecewise low-order interpolation polynomials, but the Ritz-Galerkin method is another procedure based on (1.3) for v in the subspace consisting of analytic functions or singular functions. The admissible functions in both procedures are defined on the total solution domain S.

Let S be divided into two subdomains S_1 and S_2 with a common boundary Γ_0. A combination of the Ritz-Galerkin and finite element methods is obtained if on one of the subdomains, for example, the boundary subdomain S_1, piecewise low-order interpolation polynomials are taken as admissible functions, but on the other subdomain S_2, analytic functions or singular functions are taken as admissible functions. Here, the key is how to couple two quite different methods on their common boundary Γ_0. A direct coupling trick was given by Li and Liang [7] where both kinds of admissible functions were directly constrained to be continuous only on the element nodes on Γ_0.

As for the combination with the simplified hybrid trick in this paper, an important condition is

$$f \equiv 0 \quad \text{on } S_2. \tag{1.8}$$

Obviously, it holds for homogeneous equations $\mathfrak{L}u = 0$. Even for the nonhomogeneous equation (1.1) which is satisfied by a particular solution u^* on S_2, if such a particular solution can be found, (1.1) reduces to

$$\mathfrak{L}w = 0 \quad \text{on } S_2$$

with a new variable $w = u - u^*$. Hence, we assume that (1.8), i.e.,

$$\mathfrak{L}u = 0 \quad \text{on } S_2 \tag{1.9}$$

always holds in this paper.

Define a space

$$H = \{v \in L_2(S), v \in H^1(S_1), v \in H^1(S_2) \text{ and } \mathfrak{L}v = 0 \text{ on } S_2\}, \tag{1.10a}$$

and its subspace

$$H_0 = \{v \in H \text{ and } v|_\Gamma = 0\}. \tag{1.10b}$$

Let $V_h^0 \in H_0$ be a finite-dimensional collection of the functions such that, for $v \in V_h^0$,

(1) $v|_{S_1}$ are piecewise low-order interpolation polynomials on a regular triangulation of S_1 with the maximum boundary length h,

(2) $v|_{S_2} = \Sigma_{i=1}^N a_i \psi_i$, $\mathfrak{L}\psi_i = 0$,

where a_i are unknown coefficients, $\{\psi_i\}$ are complete basis functions of linear independence. Such basis functions can be found in Bergman [1] and Vekua [14].

Moreover, let $V_h^* \in H$ be a finite-dimensional collection of the functions satisfying (1) and (2) as well as $v|_\Gamma = g$.

Remark 1. For simplicity in analyses, here suppose that the functions $v \in V_h^*$ strictly satisfy the boundary condition (1.2); otherwise, the analyses are like Ciarlet [3] and Strang and Fix [11].

Under the condition of (1.8), the simplified hybrid-combined method is the procedure to find an approximate solution u_h^* only in V_h^* such that

$$B(u_h^*, w) = f(w), \qquad w \in V_h^0, \tag{1.11}$$

where the bilinear form is (Figure 1)

$$\begin{aligned} B(v, w) = {} & \int_{S_1} [\beta \nabla v \nabla w + cvw] + \int_{S_2} [\beta \nabla v \nabla w + cvw] \\ & + \int_{\Gamma_0} \beta \left[\frac{\partial v_2}{\partial n} w_1 - \frac{\partial w_2}{\partial n} v_1 \right], \end{aligned} \tag{1.12a}$$

i.e.,

$$B(v, w) = \int_{S_1} [\beta \nabla v \nabla w + cvw] + \int_{\Gamma_0} \beta \frac{\partial v_2}{\partial n} w_2 + \int_{\Gamma_0} \beta \left[\frac{\partial v_2}{\partial n} w_1 - \frac{\partial w_2}{\partial n} v_1 \right]; \tag{1.12b}$$

the linear functional is

$$f(w) = \int_{S_1} fw; \tag{1.13}$$

we use the notations

$$v_1 = v|_{S_1}, \qquad v_2 = v|_{S_2},$$

and n is the normal to Γ_0 shown in Figure 1.

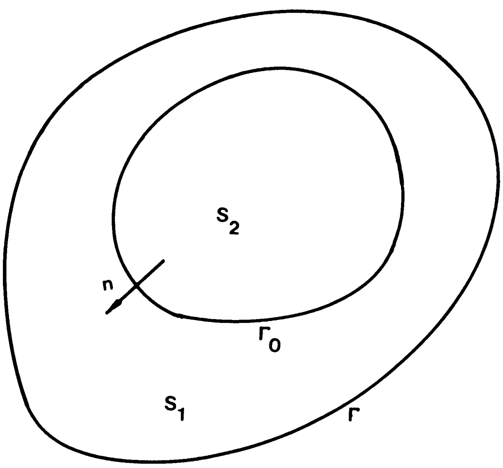

FIGURE 1
The division of the solution domain

The equivalence of (1.12a) and (1.12b) is derived from the following important equalities:

$$\int_{S_2} [\beta \nabla v \nabla w + cvw] = \int_{\Gamma_0} \beta \frac{\partial v_2}{\partial n} w_2 = \int_{\Gamma_0} \beta \frac{\partial w_2}{\partial n} v_2 \tag{1.14}$$

for $v \in H$ and $w \in H$. Eqs. (1.14) are easily proved from Green's theorem and the homogeneous equation (1.9) which is satisfied by the functions $v \in H$ and $w \in H$.

In (1.12), there is an additional integral on Γ_0:

$$\int_{\Gamma_0} \beta\left[\frac{\partial v_2}{\partial n} w_1 - \frac{\partial w_2}{\partial n} v_1\right], \tag{1.15}$$

which plays a role in coupling the Ritz-Galerkin method and the finite element method on Γ_0. Eq.(1.11) is called the simplified hybrid-combined method because the integral form (1.15) is somewhat like that in the simplified hybrid-finite element method of Fix [5], Raviart and Thoms [9] and Tong, Pian and Lasry [13].

Now, let us prove the equivalence of (1.11) and the method of Zienkiewciz et al. [15].

Define a potential energy on H for (1.1) and (1.2):

$$\Pi(v) = \frac{1}{2}\int_{S_1}\left[\beta(\nabla v)^2 + cv^2\right] + \frac{1}{2}\int_{S_2}\left[\beta(\nabla v)^2 + cv^2\right] - \int_{\Gamma_0} \lambda[v_2 - v_1] - \int_{S_1} fv, \tag{1.16}$$

with a Langrange multiplier λ which is due to the noncontinuity of v on Γ_0. It is reasonable to take the Lagrange multiplier λ as

$$\lambda = \beta\frac{\partial v_2}{\partial n} \tag{1.17}$$

because of the true value $\lambda = \beta(\partial u/\partial n)$. Hence, an approximate solution $\bar{v}$ is obtained by minimizing the potential energy [15]:

$$\Pi^*(\bar{v}) = \operatorname*{Min}_{v \in V_h^*} \Pi^*(v), \tag{1.18}$$

where we use the notation

$$\Pi^*(v) = \frac{1}{2}\int_{S_1}\left[\beta(\nabla v)^2 + cv^2\right] + \frac{1}{2}\int_{S_2}\left[\beta(\nabla v)^2 + cv^2\right] - \int_{\Gamma_0} \beta\frac{\partial v_2}{\partial n}(v_2 - v_1) - \int_{S_1} fv, \qquad v \in V_h^*. \tag{1.19}$$

Performing the variation on (1.18), we obtain

$$\int_{S_1}[\beta\nabla\bar{v}\nabla w + c\bar{v}w] + \int_{S_2}[\beta\nabla\bar{v}\nabla w + c\bar{v}w] - \left\{\int_{\Gamma_0}\beta\frac{\partial \bar{v}_2}{\partial n}(w_2 - w_1) + \int_{\Gamma_0}\beta\frac{\partial w_2}{\partial n}[\bar{v}_2 - \bar{v}_1]\right\} = \int_{S_1} fw. \tag{1.20}$$

for $\bar{v} \in V_h^*$ and $\forall w \in V_h^0$.

The functions w_2 and w_1 in (1.20) are arbitrary and independent of each other. Then, we may let them be equal to zero, respectively, so that we obtain two equalities:

$$\int_{S_1}[\beta\nabla\bar{v}\nabla w + c\bar{v}w] + \int_{\Gamma_0}\beta\frac{\partial \bar{v}_2}{\partial n} w_1 = \int_{S_1} fw \tag{1.21}$$

and

$$\int_{S_2}[\beta\nabla\bar{v}\nabla w + c\bar{v}w] - \int_{\Gamma_0}\beta\frac{\partial\bar{v}_2}{\partial n}w_2 - \int_{\Gamma_0}\beta\frac{\partial w_2}{\partial n}[\bar{v}_2 - \bar{v}_1] = 0. \tag{1.22a}$$

(1.22a) is written by applying (1.14) as

$$-\int_{\Gamma_0}\beta\frac{\partial\bar{v}_2}{\partial n}w_2 + \int_{\Gamma_0}\beta\frac{\partial w_2}{\partial n}\bar{v}_1 = 0. \tag{1.22b}$$

Therefore, the combined method (1.11) is obtained from (1.21) and (1.22b). Similarly, (1.18) can be obtained from (1.11). The equivalence of the method (1.11) and the method of Zienkiewicz et al. [15] is thus proved.

2. Error Analyses. Recently, an analysis for this method for the Poisson equation on an unbounded domain was given by Johnson and Nedelec [6]. Here, we shall give the analyses of this method for general cases.

Define a norm on H as

$$\|v\|_H = \left[\|v\|^2_{H^1(S_1)} + \|v\|^2_{H^1(S_2)}\right]^{1/2}, \tag{2.1}$$

where $\|\cdot\|_{H^1(S_1)}$ and $\|\cdot\|_{H^1(S_2)}$ are the norms in the Sobolev space. Then H is a Hilbert space.

For simplicity in analyses, suppose that S is a convex polygon, Γ_0 is a piecewise straight line (Figure 2), and the effects from the nonconforming element on Γ are not taken into account; otherwise, see [3], [11]. Then, we have

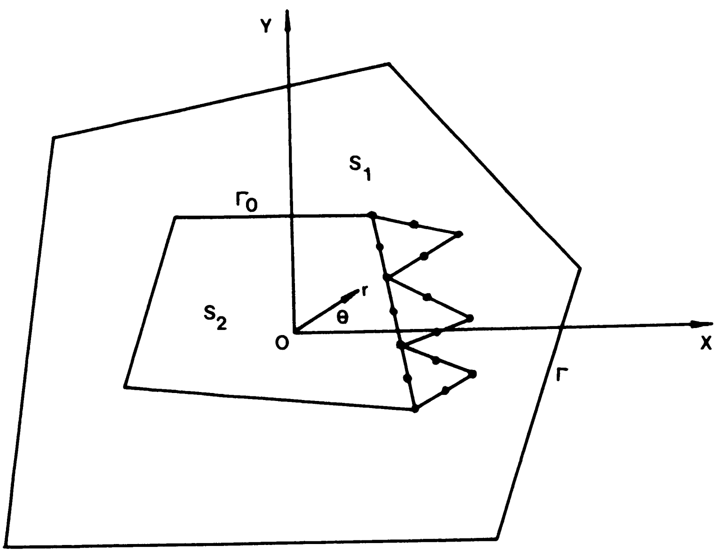

FIGURE 2
The division in the combined method

THEOREM 1. *Let* (1.8) *be given, and suppose that the bilinear form in* (1.11) *is uniformly* V_h^0*-elliptic, i.e., there exists a positive constant* α *independent of h and N such that*

$$\alpha\|v\|_H^2 \leqslant B(v,v), \qquad v \in V_h^0. \tag{2.2}$$

Then, the solution u_h^* *of* (1.11) *has the error bounds*

$$\|u - u_h^*\|_H \leqslant \inf_{\tilde{w} \in V_h^*} \|u - \tilde{w}\|_H. \tag{2.3}$$

Proof. Let u be the solution of (1.1) and (1.2) under the condition (1.8). Since u and $\partial u/\partial n$ are continuous on Γ_0, we see from Green's theorem and (1.14) that, for any $v \in H_0$,

$$\begin{aligned} B(u, v) &= \int_{S_1} fv + \int_{\Gamma_0} \left[-\beta \frac{\partial u}{\partial n} v_1 + \beta \frac{\partial u}{\partial n} v_2 \right] + \int_{\Gamma_0} \beta \left[\frac{\partial u}{\partial n} v_1 - \frac{\partial v_2}{\partial n} u \right] \\ &= \int_{S_1} fv + \int_{\Gamma_0} \beta \left[\frac{\partial u}{\partial n} v_2 - \frac{\partial v_2}{\partial n} u \right] = \int_{S_1} fv. \end{aligned} \tag{2.4}$$

Hence, the true solution u also satisfies (1.11). We have

$$B(u - u_h^*, w) = 0 \quad \forall w \in V_h^0, \tag{2.5}$$

Since v_2 satisfies the homogeneous equation (1.9) we find from the trace theorem of Lions and Magènes [8] (also see Babuška and Aziz [2, pp. 32–33]) that

$$\left\| \frac{\partial v_2}{\partial n} \right\|_{H^{-1/2}(\Gamma_0)} \leqslant K_1 \|v\|_{H^1(S_2)}, \tag{2.6}$$

with a bounded constant K_1. Also, we see from the imbedding theorem of Sobolev [10] that

$$\|v_1\|_{H^{1/2}(\Gamma_0)} \leqslant K_1 \|v\|_{H^1(S_1)}. \tag{2.7}$$

(Throughout, the notation K_1 represents a generic constant with possibly different values in different contexts.) Thus we have from (1.12a), (2.6) and (2.7) that, for $v \in H_0$ and $w \in H_0$,

$$\begin{aligned} |B(v, w)| &\leqslant \|v\|_H \|w\|_H + \operatorname{Max} \beta \int_{\Gamma_0} \left[\left| \frac{\partial v_2}{\partial n} w_1 \right| + \left| \frac{\partial w_2}{\partial n} v_1 \right| \right] \\ &\leqslant \|v\|_H \|w\|_H + \operatorname{Max} \beta \left\{ \left\| \frac{\partial v_2}{\partial n} \right\|_{H^{-1/2}(\Gamma_0)} \|w_1\|_{H^{1/2}(\Gamma_0)} \right. \\ &\qquad \left. + \left\| \frac{\partial w_2}{\partial n} \right\|_{H^{-1/2}(\Gamma_0)} \|v_1\|_{H^{1/2}(\Gamma_0)} \right\} \\ &\leqslant K_1 \|v\|_H \|w\|_H, \end{aligned} \tag{2.8}$$

So, the bilinear form $B(v, w)$ is bounded on H_0.

Let $\tilde{w} \in V_h^*$ be arbitrary. We then see from (2.2), (2.5) and (2.8) that

$$\begin{aligned} \alpha \|u_h^* - \tilde{w}\|_H^2 &\leqslant B(u_h^* - \tilde{w}, u_h^* - \tilde{w}) = B(u - \tilde{w}, u_h^* - \tilde{w}) \\ &\leqslant K_1 \|u - \tilde{w}\|_H \|u_h^* - \tilde{w}\|_H. \end{aligned}$$

Hence,

$$\|u_h^* - \tilde{w}\|_H \leqslant \frac{K_1}{\alpha} \|u - \tilde{w}\|_H. \tag{2.9}$$

Thus

$$(2.10)\qquad \|u-u_h^*\|_H \leqslant \|u-\tilde{w}\|_H + \|\tilde{w}-u_h^*\|_H \leqslant \left(1+\frac{K_1}{\alpha}\right)\|u-\tilde{w}\|_H.$$

Consequently, inequality (2.3) is obtained. Theorem 1 is thus proved. □

Theorem 1 is an optimal estimate of errors for the simplified hybrid-combined method (1.11).

THEOREM 2. *Let* (1.8) *be given, and suppose that* $u \in H^{k+1}(S_1)$, *the* k*-order Lagrange finite element method is used on* S_1 *and the uniformly* V_h^0*-elliptic inequality* (2.2) *holds. Then*

$$(2.11)\qquad \|u-u_h^*\|_H \leqslant K_1\left\{h^k|u|_{H^{k+1}(S_1)} + \left[\|R_N\|_{H^0(\Gamma_0)}\left\|\frac{\partial R_N}{\partial n}\right\|_{H^0(\Gamma_0)}\right]^{1/2}\right\},$$

where R_N *is the remainder of an approximate expansion* u_N *of* u, *which is expressed as*

$$(2.12)\qquad u_N = \sum_{i=1}^{N} \bar{a}_i\psi_i$$

with the expansion coefficients $\bar{a}_i$.

Proof. Let u_h be the piecewise k-order Lagrange interpolation polynomial of u on the triangulation of S_1. An auxiliary function $\bar{w}$ is constructed such that

$$(2.13)\qquad \bar{w} = \begin{cases} u_h, & (x,y)\in S_1,\\ u_N, & (x,y)\in S_2.\end{cases}$$

Then $\bar{w} \in V_h^*$. We obtain from Theorem 1 that

$$(2.14)\qquad \|u-u_h^*\|_H \leqslant K_1 \inf_{\tilde{w}\in V_h^*}\|u-\tilde{w}\|_H \leqslant K_1\|u-\bar{w}\|_H.$$

Moreover, we see from (2.2) that, for $\delta = u-\bar{w}$,

$$(2.15)\qquad \begin{aligned}\alpha\|u-\bar{w}\|_H^2 = \alpha\|\delta\|_H^2 &\leqslant B(\delta,\delta)\\ &= \int_{S_1}\left[\beta(\nabla\delta)^2 + c\delta^2\right] + \int_{S_2}\left[\beta(\nabla\delta)^2+c\delta^2\right]\\ &\leqslant K_1\|u-u_h\|_{H^1(S_1)}^2 + \int_{S_2}\left[\beta(\nabla R_N)^2 + cR_N^2\right].\end{aligned}$$

Then,

$$(2.16)\qquad \|u-u_h^*\|_H \leqslant K_1\left(\|u-u_h\|_{H^1(S_1)} + \left\{\int_{S_2}\left[\beta(\nabla R_N)^2 + cR_N^2\right]\right\}^{1/2}\right).$$

Note the error bounds for the finite element method [2], [3], [11],

$$(2.17)\qquad \|u-u_h\|_{H^1(S_1)} \leqslant k_1 h^k |u|_{H^{k+1}(S_1)}.$$

Next, the remainder R_N satisfies the homogeneous equation (1.9). Then we have from (1.14) that

$$(2.18)\qquad \int_{S_2}\left[\beta(\nabla R_N)^2 + cR_N^2\right] = \int_{\Gamma_0}\beta R_N\frac{\partial R_N}{\partial n} \leqslant K_1\|R_N\|_{H^0(\Gamma_0)}\left\|\frac{\partial R_N}{\partial n}\right\|_{H^0(\Gamma_0)}.$$

Inequality (2.11) is obtained by combining (2.16)–(2.18). Theorem 2 is proved. □

Now, let us examine the uniformly V_h^0-elliptic inequality (2.2). The bilinear form for $u = v \in V_h^0$ is

$$B(v, v) = \int_{S_1} \left[\beta(\nabla v)^2 + cv^2\right] + \int_{S_2} \left[\beta(\nabla v)^2 + cv^2\right]. \tag{2.19}$$

Since $v|_{\Gamma \wedge S_1} = 0$ for $v \in V_h^0$ and $\text{Meas}(\Gamma \wedge S_1) > 0$, we see from the Poincaré-Friedrichs inequality [3] that there exists a positive constant α_1 independent of h and N such that

$$\alpha_1 \|v\|_{H^1(S_1)} \leq \int_{S_1} \left[\beta(\nabla v)^2 + cv^2\right]. \tag{2.20}$$

Similarly, if $\text{Meas}(\Gamma \cap S_2) > 0$, we also have the inequality

$$\alpha_2 \|v\|_{H^1(S_2)} \leq \int_{S_2} \left[\beta(\nabla v)^2 + cv^2\right], \tag{2.21}$$

with α_2 a positive constant. Hence, the uniformly V_h^0-elliptic inequality (2.2) holds from (2.20) and (2.21).

However, in the general case S_2 might be all inside S, i.e., $\text{Meas}(\Gamma \wedge S_2) = 0$. (2.21) can also hold provided that the function c in (1.1) satisfies

$$c = c(x, y) \not\equiv 0 \quad \text{on } S_2. \tag{2.22}$$

(This proof is like that in Ciarlet [3].) Then,

LEMMA 2.1. *The uniformly V_h^0-elliptic inequality* (2.2) *holds provided that either* $\text{Meas}(\Gamma \wedge S_2) > 0$ *or* (2.22) *holds.*

Remark 2. For the case where neither of the conditions in Lemma 2.1 holds, for example, if

$$\mathfrak{L}^* u = -\frac{\partial}{\partial x}\beta\frac{\partial u}{\partial x} - \frac{\partial}{\partial x}\beta\frac{\partial u}{\partial y} = 0, \qquad (x, y) \in S_2, \tag{2.23}$$

with S_2 inside S, the uniformly V_h^0-elliptic inequality (2.2) does not hold because an arbitrary constant is permitted for the admissible functions v_h on S_2.

In this case, the spaces H and H_0 shall again satisfy a constraint condition, for example, $\int_{S_2} v = 0$. Then, we may define the spaces

$$H^* = \left\{v \in L_2(S), v \in H^1(S_1), v \in H^1(S_2), \right. \\ \left. \mathfrak{L}^* v = 0 \text{ on } S_2 \text{ and } \int_{S_2} v = 0\right\}, \tag{2.24a}$$

and

$$H_0^* = \{v \in H^* \text{ and } v|_\Gamma = 0\}. \tag{2.24b}$$

Moreover, let the subspaces $V_h^* \in H^*$ and $V_h^0 \in H_0^*$. Therefore, the corresponding uniformly elliptic inequality on $V_h^0 \in H_0^*$ for the simplified hybrid-combined method still holds so that the combined method (1.11) and Theorems 1 and 2 all are valid.

3. The Strategy for Solving the Algebraic Equation System. An algebraic equation system is obtained from (1.11)

$$Av + Ed = b_1, \tag{3.1}$$

$$-E^{T}v + Dd = b_2, \tag{3.2}$$

where $\mathbf{v}$ is the unknown vector with the elements $v_{i,j}$, $\mathbf{d}$ is the unknown vector with the coefficients a_i, $\mathbf{b}_1$ and $\mathbf{b}_2$ are known vectors, A, D, E and E^{T} are matrices, and E^{T} is the transposed matrix of E.

The matrix A is positive definite, symmetric and sparse because it comes from the integral $\int_{S_1}[\beta\nabla v\nabla w + cvw]$, and the matrix D is also positive definite and symmetric because it comes from the integral $\int_{S_2}[\beta\nabla v\nabla w + cvw]$ (see (2.21)), and the matrices E and E^{T} are from the integrals $\int_{\Gamma_0}\beta(\partial v_2/\partial n)w_1$ and $\int_{\Gamma_0}\beta(\partial w_2/\partial n)v_1$, respectively.

Since the coefficient matrix $\left(\begin{smallmatrix} A & E \\ -E^{T} & D\end{smallmatrix}\right)$ in (3.1) and (3.2) is nonsymmetric, the following strategy for solving them is recommended.

We see from (3.2) that

$$\mathbf{d} = D^{-1}\left[E^{T}\mathbf{v} + \mathbf{b}_2\right]. \tag{3.3}$$

Then we obtain, by substituting $\mathbf{d}$ into (3.1), that

$$F\mathbf{v} = \mathbf{b} \tag{3.4}$$

with the matrix

$$F = A + ED^{-1}E^{T} \tag{3.5}$$

and the known vector

$$\mathbf{b} = \mathbf{b}_1 - ED^{-1}\mathbf{b}_2. \tag{3.6}$$

Obviously, the solution $\mathbf{v}$ is easily evaluated from (3.4) because the matrix F is also positive definite, symmetric and sparse. Then the solution $\mathbf{d}$ is obtained from (3.3).

Now, let us consider the stability of (3.4), which is measured by the bounds of the following condition number of the matrix F:

$$\rho(F) = \lambda_{\max}(F)/\lambda_{\min}(F), \tag{3.7}$$

where $\lambda_{\max}(F)$ and $\lambda_{\min}(F)$ are the maximum and minimum eigenvalues of F, respectively.

THEOREM 3. *Let there be given either* (2.22) *or* Meas$(\Gamma \wedge S_2) > 0$. *Then*

$$\rho(F) \leqslant K_1 h_{\min}^{-2}\left\{1 + \lambda_{\max}(EE^{T})/\lambda_{\min}(D)\right\}, \tag{3.8}$$

where $h_{\min}$ is the minimum boundary length of triangular elements on S_1.

Proof. Since D is a positive definite and symmetric matrix, we have

$$\lambda_{\max}(ED^{-1}E^{T}) \leqslant \lambda_{\max}(EE^{T})/\lambda_{\min}(D). \tag{3.9}$$

Also $\lambda_{\min}(ED^{-1}E^{T}) \geqslant 0$. Then, we see that

$$\begin{aligned} \rho(F) &\leqslant \frac{\lambda_{\max}(A) + \lambda_{\max}(ED^{-1}E^{T})}{\lambda_{\min}(A) + \lambda_{\min}(ED^{-1}E^{T})} \\ &\leqslant \left[\lambda_{\max}(A) + \lambda_{\max}(ED^{-1}E^{T})\right]/\lambda_{\min}(A) \\ &\leqslant \left[\lambda_{\max}(A) + \lambda_{\max}(EE^{T})/\lambda_{\min}(D)\right]/\lambda_{\min}(A). \end{aligned} \tag{3.10}$$

Hence, (3.8) is obtained from the following estimates in the finite element method [11]:

$$\lambda_{\max}(A) \leq K_1, \qquad \lambda_{\min}(A) = O(h^2_{\min}).$$

Theorem 3 is proved. $\square$

It is shown in Theorem 3 that the condition number $\rho(F)$ will not be too large if the ratio of $\lambda_{\max}(EE^{\mathrm{T}})/\lambda_{\min}(D)$ is not too large.

As to (3.3), the analysis of stability is obvious.

4. Examples. In this section, we take the following model problem as an example for the application of the simplified hybrid-combined method (1.11):

$$-\Delta u + u = f, \quad (x, y) \in S_1, \tag{4.1}$$

$$-\Delta u + u = 0, \quad (x, y) \in S_2, \tag{4.2}$$

$$u = g, \quad (x, y) \in \Gamma. \tag{4.3}$$

The condition (2.22) holds because of $c \equiv 1$ on S.

The solution u on S_2 can be expanded, with the help of the method of separation of variables (see Tikhonov and Samarskii [12]), as

$$u = \bar{a}_0 I_0(r) + \sum_{n=1}^{N} I_n(r)\left[\bar{a}_n \cos n\theta + \bar{b}_n \sin n\theta\right] + R_N, \tag{4.4}$$

where $\bar{a}_n$ and $\bar{b}_n$ are expansion coefficients, R_N is the remainder, and $I_n(r)$ is the Bessel function for a purely imaginary argument, defined by

$$I_n(r) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+1)\Gamma(k+n+1)} \left(\frac{r}{2}\right)^{2k+n}. \tag{4.5}$$

Then, the admissible functions should be taken as (Figure 2):

$$v = \begin{cases} v_k, & (x, y) \in S_1, \\ a_0 I_0(r) + \sum_{n=1}^{N} I_n(r)[a_n \cos n\theta + b_n \sin n\theta], & (r, \theta) \in S_2, \end{cases} \tag{4.6}$$

where a_n and b_n are unknown coefficients, and v_k are piecewise k-order Lagrange polynomials on the triangulation of S_1.

The basis functions $I_0(r)$, $I_n(r)\cos n\theta$ and $I_n(r)\sin n\theta$ all satisfy (4.2). So, the space V_h^* consisting of (4.6) does belong to H, as defined by (1.10a); similarly $V_h^0 \in H_0$.Therefore, an approximate solution can be calculated from the combined method (1.11) and Theorems 1–3 hold true.

Next, consider a singularity problem of a crack lying on the axis x, with the following boundary condition on the crack (Figure 3).

$$u|_{\Gamma} = 0 \qquad (y = 0 \text{ and } x \geq 0). \tag{4.7}$$

There exists a singularity at the origin, which is placed on S_2 (see Figure 3). The solution on S_2 can be similarly found as

$$u = \sum_{n=1}^{2N} \bar{a}_n I_{n/2}(r) \sin\frac{n}{2}\theta + R_N, \tag{4.8}$$

with the coefficients $\bar{a}_n$ and the remainder R_N. Obviously, the derivative $\partial u/\partial r$ is unbounded when $r \to 0$, so that the numerical solutions of the single finite element method or the single finite difference method only have a poor precision [11].

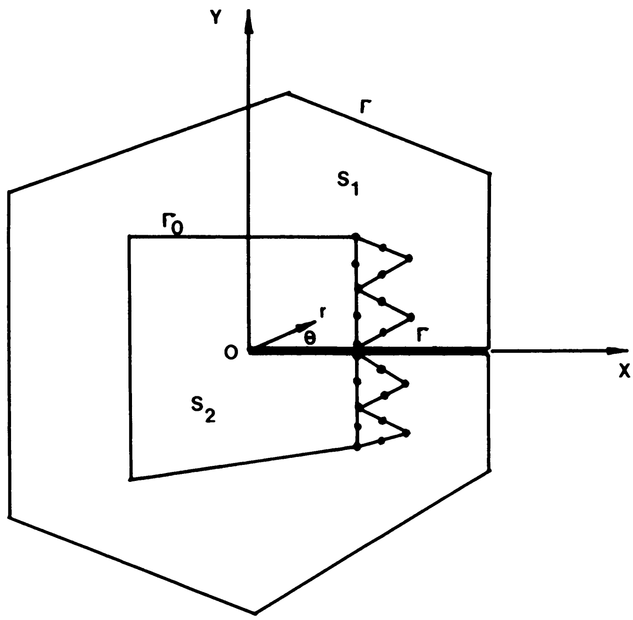

FIGURE 3
The crack problem

Here, we use the combined method for solving the crack problem and take the admissible functions:

$$v = \begin{cases} v_k, & (x, y) \in S_1, \\ \sum_{n=1}^{2N} a_n I_{n/2}(r) \sin \frac{n}{2}\theta, & (r, \theta) \in S_2, \end{cases} \tag{4.9}$$

with the unknown coefficients a_n and the basis functions $I_{n/2}(r) \sin n\theta/2$, which satisfy (4.2) and (4.7).

It is worth pointing out that even for the singularity problems, Theorems 1–3 still hold.

COROLLARY 4.1. *Suppose that the conditions in Theorem* 3 *hold, and* $u|_{\Gamma_0}$ *has bounded partial derivatives of order* μ. *Then, the solution* u_h^* *of* (1.11) *satisfies the following error bounds*:

$$\|u - u_h^*\|_H \leq K_1 \left[h^k |u|_{H^{k+1}(S_1)} + \frac{1}{N^{\mu - 1/2}} \right]. \tag{4.10}$$

Proof. For the remainders R_N in (4.4) and (4.8), we see from Eisenstat [4] that

$$\|R_N\|_{H^0(\Gamma_0)} \leq K_1 \frac{1}{N^\mu} \quad \text{and} \quad \left\| \frac{\partial R_N}{\partial n} \right\|_{H^0(\Gamma_0)} \leq K_1 \frac{1}{N^{\mu-1}}.$$

Then, (4.10) is obtained from Theorem 2. □

The corollary leads to

$$\|u - u_h^*\|_h \leqslant K_1 h^k \tag{4.11}$$

provided that we choose the optimal integer

$$N = N_{\text{opt}} = O(h^{-k/(\mu-1/2)}). \tag{4.12}$$

In this case, the total number of unknown quantities in (1.11) is

$$O(h^{-2}) + O(N_{\text{opt}}) = O(h^{-2}) + O(h^{-k/(\mu-1/2)}).$$

Generally, $\mu \geqslant k + 1$, and then the number N of the unknown coefficients a_n and b_n is less than $O(h^{-1})$, which is much less than $O(h^{-2})$. The latter is the number of element nodes in the finite element method. Hence, the calculation and storage space in the combined method (1.11) are substantially less than those in the single finite element method on S.

Obviously, the larger S_2 and μ are, the less the calculation and storage space in (1.11) are. Corollary 4.1 still holds for general elliptic equations if we take the admissible functions according to the expansions of solutions in Bergman [1] and Vekua [14].

Concluding Remarks. According to the above analyses, the combined method (1.11) with the simplified hybrid trick should be used for singularity problems and unbounded problems, instead of the single finite element method. Also, we recommend that the combined method (1.11) be used for common boundary value problems of elliptic equations if there exists a large subdomain where the solution is sufficiently smooth. Finally, we would like again to remind the reader of the necessary condition (1.8) for the combined method in this paper.

Acknowledgements. We wish to express our gratitude to the referee who read the first manuscript and gave us many valuable suggestions. Special thanks are due to Professor R. Mathon, Department of Computer Science, University of Toronto. Without his support and encouragement, this study could not have been completed.

Department of Computer Science
University of Toronto
Toronto, Ontario, Canada M5S 1A7

1. S. BERGMAN, *Integral Operations in the Theory of Linear Partial Differential Equations*, Springer-Verlag, Berlin and New York, 1969.

2. I. BABUŠKA & A. K. AZIZ, "Survey lectures on the mathematical foundations of the finite element method," in *The Mathematical Foundations of the Finite Element Method with Applications to Partial Differential Equations* (A. K. Aziz, Ed.), Academic Press, New York and London, 1972, pp. 3–359.

3. P. G. CIARLET, *The Finite Element Method for Elliptic Problems*, North-Holland, Amsterdam, 1978.

4. S. C. EISENSTAT, *On the Rate of Convergence of the Bergman-Vekua Method for the Numerical Solution of Elliptic Boundary Value Problems*, Research Report No. 72–2, Department of Computer Science, Yale University, 1972.

5. G. J. FIX, "Hybrid element method," *SIAM Rev.*, v. 18, 1976, pp. 460–484.

6. C. JOHNSON & C. NEDELEC, "On the coupling of boundary integral and finite element methods," *Math. Comp.*, v. 35, 1980, pp. 1063–1079.

7. Z. C. LI & G. P. LIANG, "On Ritz-Galerkin-F.E.M. combined method of solving the boundary value problems of elliptic equations," *Sci. Sinica*, v. 24, 1981, pp. 1497–1508.

8. J. L. LIONS & E. MAGÈNES, *Problèmes aux Limites non Homogènes et Applications*, Vol. 1, Travaux et Recherches Mathematiques, no. 17, Dunod, Paris, 1968.

9. P. A. RAVIART & J. M. THOMS, "Primal hybrid finite element method for 2nd order elliptic equations," *Math. Comp.*, v. 31, 1977, pp. 391–413.

10. S. L. SOBOLEV, *Applications of Functional Analysis in Mathematical Physics*, Transl. Math. Monos., vol. 7, Amer. Math. Soc., Providence, R.I., 1963.

11. G. STRANG & G. J. FIX, *An Analysis of the Finite Element Method*, Prentice-Hall, Englewood Cliffs, N.J., 1973.

12. A. N. TIKHONOV & A. A. SAMARSKII, *Equations of Mathematical Physics* (Transl. by A. P. N. Robson & P. Basu), Macmillan, New York, 1973.

13. P. TONG, T. H. H. PIAN & S. J. LASRY, "A hybrid element approach to crack problems in plane elasticity," *Internat. J. Numer. Methods Engrg.*, v. 7, 1973, pp. 297–308.

14. I. N. VEKUA, *New Method for Solving Elliptic Equations*, North-Holland, Amsterdam, 1967.

15. O. C. ZIENKIEWICZ, D. W. KELLY & P. BETTESS, "The coupling of the finite element method and boundary solution procedures," *Internat. J. Numer. Methods Engrg.*, v. 11, 1977, pp. 355–375.