MATHEMATICS OF COMPUTATION
VOLUME 42, NUMBER 166
APRIL 1984, PAGES 465-492

A Priori Estimates and Analysis of a Numerical Method for a Turning Point Problem

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Abstract. Bounds are obtained for the derivatives of the solution of a turning point problem. These results suggest a modification of the El-Mistikawy Werle finite difference scheme at the turning point. A uniform error estimate is obtained for the resulting method, and illustrative numerical results are given.

I. Introduction. We will examine the following two-point boundary value problem with Dirichlet data at the endpoints:

$$Ly = -\varepsilon y_{xx}(x) - p(x)y_x(x) + q(x)y(x) = f(x) \quad \text{for } -1 < x < 1, \tag{1.1a}$$

$$y(-1) = d_1, \qquad y(1) = d_2. \tag{1.1b}$$

Here ε is a constant in $(0, 1]$, p is assumed to be in $C^2[-1, 1]$, q and f are required to be in $C^1[-1, 1]$, and d_1 and d_2 are given constants. The function $p(x)$ is allowed to have a finite number of zeros located at points $z_1, \dots, z_r$ in $(-1, 1)$. The zeros of p are assumed to be simple, and $p(-1)p(1)$ must not vanish. The points z_i are called *turning points* of (1.1). Also $q(x)$ is required to be bounded below by some positive constant k_q, so we are thus excluding the so-called resonance cases, e.g. [1]. The above assumptions will be in force throughout the rest of this paper. This type of problem arises, e.g., as a linearized one-dimensional slice of a fluid flow problem having a region of recirculation. Under these conditions (1.1a) satisfies a maximum principle [16, p. 6], i.e.,

(1.2) if $y(x)$ in $C^2[-1, 1]$ is such that $Ly \geqslant 0$ on $(-1, 1)$ and $y(\pm 1) \geqslant 0$, then $y(x) \geqslant 0$ for $-1 \leqslant x \leqslant 1$.

Existence and uniqueness of the solution of (1.1) follow easily from (1.2) and existence of solutions of the initial value problem for (1.1a).

We will see below that the bounds on the behavior of $y(x)$ near a given turning point z_i depend specifically on ε and on the constant $\beta_i = q(z_i)/p_x(z_i)$. If $\beta_i < 0$, it will be shown that $y(x)$ is "smooth" near $x = z_i$; on the other hand if $\beta_i > 0$, then there is in general an "internal layer" at $x = z_i$, the nature of which depends in a fundamental way on β_i. Results in [12] will be used to show that in general y has a

Received August 20, 1982.

1980 *Mathematics Subject Classification*. Primary 34E20; Secondary 65L10.

*Research supported jointly by the Office of Naval Research (Contract No. N0001481WR10009, contract id. no. NR044-547) and by the Naval Surface Weapons Center Independent Research Fund.

**Research supported in part by National Institutes of Health grant R01-AM20373.

boundary layer at $x = -1$ $[x = +1]$ if and only if $p(-1) > 0$ $[p(1) < 0]$. These results will be stated precisely in Section 2, and their proofs will be given in Section 4.

The a priori estimates given in Section 2 are direct explicit bounds on the derivatives of $y(x)$ which are obtained by local examination of $y(x)$ near each turning point. When $\beta_i > 0$ this entails employing appropriate parabolic cylinder functions and the Green's function for a local approximation of the operator L. This is a somewhat different approach from the asymptotic expansions obtained by, e.g. [7], [8], [9]. The results obtained here remain valid as β varies though positive integer values.

In Section 3 we describe the modification for use with turning point problems of the El-Mistikawy Werle exponential finite difference scheme [6] which is suggested by the results in Section 2. A uniform error estimate is also proven in this section by using comparison functions and the a priori estimates, and some illustrative numerical results are displayed in Section 5. We note that Farrell [8], [9], [21] has obtained a set of general sufficient conditions for a scheme to be uniformly accurate for turning point problems. Other results for numerical methods for turning point problems have been obtained in [2], [13], [14], [15].

II. Statement and Discussion of the A Priori Estimates. We first use the maximum principle to show that the solution of (1.1) is bounded. Then we make some further preliminary observations concerning (1.1) which will effectively reduce the situation to considering the case of one turning point located at $x = 0$ for which $\beta > 0$. The a priori estimates will then be stated.

For any given function $g(x)$ in $C^k[-1, 1]$ (k a nonnegative integer) let $\|g\|_k$ denote $\sum_{i=0}^{k} \max_{-1 \leqslant x \leqslant 1} |D_x^i g(x)|$, where $D_x^i g(x)$ denotes the ith derivative of g (and where $D_x^0 g(x) \equiv g(x)$ and $D_x \equiv D_x^1$). Let $y(x)$ be the solution of (1.1) and set $\phi(x) \equiv \|f\|_0/k_q + \max(|d_1|, |d_2|)$.

Then, applying the maximum principle (1.2) to $\phi(x) \pm y(x)$, one finds that

$$\|y\|_0 \leqslant \|f\|_0/k_q + \max\bigl(|d_1|, |d_2|\bigr). \tag{2.1}$$

From (2.1) and (1.2) we now show that the turning points and boundary points can be treated individually for the purpose of studying the regularity of the solution. Suppose $[a, b]$ is a subinterval of $[-1, 1]$ which contains none of the turning points $\{z_1, \dots, z_r\}$. Recall (2.1) provides a bound for $y(a)$ and $y(b)$. Then Lemma 2.3 of [12] can be used to bound the derivatives of y on $[a, b]$; we restate a form of this lemma here making more precise what the constants in the estimates depend on. Suppose p, q, and f are in $C^m[a, b]$ with m a positive integer, $|p(x)| \geqslant B_1$ for $a \leqslant x \leqslant b$ (B_1 a positive constant), and let $S_1(m)$ denote the set $\{\|p\|_m, \|q\|_m, \|f\|_m, B_1, b - a, y(a), y(b), m\}$ (here the C^m norms of p, q, and f are on the interval $[a, b]$). Then

LEMMA 2.1 [12]. *There are positive constants η and C depending only on $S_1(m)$ such that if $p(x) > 0$ on $[a, b]$, then*

$$\bigl|D_x^i y(x)\bigr| \leqslant C + C\varepsilon^{-i} \exp(-2\eta(x - a)/\varepsilon) \tag{2.2a}$$

$$\textit{for } i = 1, \dots, m + 1,\ a \leqslant x \leqslant b,$$

while if $p(x) < 0$ *on* $[a, b]$, *then*

(2.2b) $$\left|D_x^i y(x)\right| \leqslant C + C\varepsilon^{-i}\exp(-2\eta(b-x)/\varepsilon) \quad \text{for } i = 1,\dots, m+1,\ a \leqslant x \leqslant b.$$

Lemma 2.1 provides bounds on the behavior of y at the endpoints $x = \pm 1$, and shows that if $p(-1) < 0$ $[p(1) > 0]$, then there is no boundary layer at $x = -1$ $[x = 1]$, since, for k and c given positive constants,

(2.3) $$s^k \exp(-cs) \quad \text{is bounded for } s \geqslant 0.$$

Another consequence of Lemma 2.1, (2.1) and (2.3) is the fact that the solution $y(x)$ of (1.1) is "smooth" away from $\{-1, 1, z_1, \dots, z_r\}$, i.e.,

Remark 2.2. Let $[a_1, b_1]$ be a subinterval of $[-1, 1]$ contained in an interval (a, b) such that $[a, b]$ contains none of the points $\{-1, 1, z_1, \dots, z_r\}$. Assume f, p and q are in $C^m[-1, 1]$ with m a positive integer and let $S_2(m)$ denote the set $\{\|p\|_m, \|q\|_m, \|f\|_m, \min_{a \leqslant x \leqslant b}|p(x)|, b-a, b-b_1, a_1-a, k_q, |d_1|, |d_2|, m\}$. Then there is a constant C depending only on $S_2(m)$ such that

(2.4) $$\left|D_x^i y(x)\right| \leqslant C \quad \text{for } i = 1,\dots, m+1,\ a_1 \leqslant x \leqslant b_1.$$

Lemma 2.1 and Remark 2.2 and (2.1) reduce the matter of a priori estimates for $y(x)$ to producing bounds for $D_x^k y(x)$ in a neighborhood N_i of each turning point z_i. Toward this end one can easily verify

Remark 2.3. There is a positive constant δ depending only on the set $S_3 = \{|p(-1)|, |p(1)|, \|p\|_2, \max(|\beta_1|, \dots, |\beta_r|), k_q\}$ such that for $i = 1,\dots, r$ the neighborhood $N_i \equiv [z_i - \delta, z_i + \delta]$ of the turning point z_i does not contain any other turning point of (1.1) or the points ± 1. Furthermore

(2.5) $$|p_x(x)| \geqslant |p_x(z_i)/2| \quad \text{for } x \text{ in } N_i.$$

The condition (2.5) will be convenient for some of the proofs. By using the transformation

(2.6) $$\tilde{x} = \delta^{-1}(x - z_i) \quad \text{for } x \text{ in } N_i$$

one may thus reduce the study of the behavior of $y(x)$ near a given turning point z_i to the case of (1.1) where $p(x)$ has precisely one zero located at $x = 0$. Note that the quantity β for a given turning point remains invariant under the change of independent variable given by (2.6). We are thus led to considering (1.1) under the following hypotheses.

(2.7a) $p(x)$ is in $C^2[-1, 1]$ and f and q are in $C^1[-1, 1]$,

(2.7b) ε is in $(0, 1]$,

(2.7c) $q(x) \geqslant k_q > 0$ on $[-1, 1]$, where k_q is a positive constant,

(2.7d) $p(x)$ has a simple zero at $x = 0$ and no other zeros on $[-1, 1]$,

(2.7e) $|p_x(x)| \geqslant |p_x(0)|/2 \quad$ for $-1 \leqslant x \leqslant 1$.

Let $\beta = q(0)/p_x(0)$, and let β_l, β_s be fixed positive constants such that $\beta_l < 1 < \beta_s$ and

(2.7f) $$\beta_l \leqslant |\beta| \leqslant \beta_s.$$

We next show that $y(x)$ is "smooth" near $x = 0$ if $\beta < 0$; cf. [1].

THEOREM 2.4. *Assume* (2.7a–f), *suppose* $\beta < 0$, *let* p, q, *and* f *be in* $C^m[-1, 1]$ *with* m *a positive integer, and define* $S_4(m) = \{\|p\|_m, \|q\|_m, \|f\|_m, \beta_s, k_q, |d_1|, |d_2|, m\}$. *Then there is a constant* C *depending only on* $S_4(m)$ *such that*

$$|D_x^k y(x)| \leqslant C \quad \text{for } k = 1, \ldots, m, \text{ and } |x| \leqslant 1/2. \tag{2.8}$$

We remark that the choice of $1/2$ in (2.8) is arbitrary, and Lemma 2.1 and Remark 2.2 can be used to describe the behavior of y for $|x| \geqslant 1/2$.

Proof. From the mean value theorem and (2.7e, f),

$$|p(x)| = |p(x) - p(0)| = |x|\,|p_x(\zeta)| \geqslant |x|\,|p_x(0)|/2 \geqslant .5|x|k_q/\beta_s. \tag{2.9}$$

Remark 2.2 implies that $|D_x^k y(\pm 1/2)| \leqslant C_1$ for $k = 1, \ldots, m$ where C_1 depends only on $S_4(m)$. For $k = 1, \ldots, m$, if (1.1a) is differentiated k times, one finds that the differential equation satisfied by $z(x) \equiv D_x^k y(x)$ is

$$-\varepsilon z_{xx} - p(x) z_x + [q(x) - kp_x(x)]z(x) = g(x), \tag{2.10}$$

where g depends on $y, \ldots, D_x^{k-1}y$ and on at most kth order derivatives of p, q, and f. Applying (2.1) with q replaced by $q - kp_x$, and using an inductive argument, we obtain (2.8).

We have thus reduced the study of the solution to the case of (2.7a–f) together with

$$\beta > 0. \tag{2.7g}$$

We now state the results for the case of (2.7a–g). The proofs will be given in Section 4.

For convenient reference define, for m any positive integer, the set $S_5(m) = \{\|p\|_2, \|q\|_1, \|f\|_1, k_q, \beta_l, \beta_s, |d_1|, |d_2|, \|p\|_m, \|q\|_m, \|f\|_m, m\}$. Then we have

THEOREM 2.5. *Assume* (2.7a–g) *and let* $y(x)$ *denote the solution of* (1.1). *Then there is a constant* C_1 *depending only on* $S_5(1)$ *such that*

$$|D_x y(x)| \leqslant C_1 (x^2 + \varepsilon)^{(\beta - 1)/2} I(x, \varepsilon, \beta) \quad \text{for } -1 \leqslant x \leqslant 1, \tag{2.11a}$$

where

$$I(x, \varepsilon, \beta) \equiv \int_{x^2+\varepsilon}^{6} s^{(-\beta - 1)/2}\, ds. \tag{2.11b}$$

The choice of 6 as the upper limit of integration in (2.11b) is a matter of convenience in the proofs (any number larger than 2 would be valid). Note also that there are constants c and C depending only on β_l and β_s such that if

$$\rho = \varepsilon^{1/2}, \tag{2.11c}$$

then

$$c(|x| + \rho)^{\beta - 1} \leqslant (x^2 + \varepsilon)^{(\beta - 1)/2} \leqslant C(|x| + \rho)^{\beta - 1}, \tag{2.12}$$

so that (2.11a) could just as well be written as

$$|D_x y(x)| \leqslant C_1 (|x| + \rho)^{\beta - 1} I(x, \varepsilon, \beta) \quad \text{for } -1 \leqslant x \leqslant 1. \tag{2.11d}$$

Also, we note that

$$(2.11e)\qquad I(x,\varepsilon,\beta)=\begin{cases}\dfrac{2}{1-\beta}\left(6^{(1-\beta)/2}-(x^2+\varepsilon)^{(1-\beta)/2}\right), & \beta\neq 1,\\ \ln\dfrac{6}{x^2+\varepsilon}, & \beta=1.\end{cases}$$

Here we are employing the convention that c, C, c_1, C_1, etc. denote generic positive constants which may depend on $S_5(m)$, but which do not depend on ε or x (or the mesh size h when the approximate problem is under discussion), and whose values may change from one usage to the next. In particular, insofar as β is concerned, these constants may depend only on β_l and β_s and the assumption that $0<\beta_l\leqslant|\beta|\leqslant\beta_s$.

To get a clearer picture of the dependence on β of the bound for $y_x(x)$ given in Theorem 2.5, one can observe the following. Suppose β is in $[\beta_l, 1-k]$ for some positive constant k. Then $I(x,\varepsilon,\beta)\leqslant C(k)$, and so (2.11d) becomes

$$|y_x(x)|\leqslant C_1(k)(|x|+\rho)^{\beta-1}\quad\text{for } -1\leqslant x\leqslant 1, \beta \text{ in } [\beta_l, 1-k].$$

If β is in $[1+k,\beta_s]$, then $I(x,\varepsilon,\beta)\leqslant C(k)(|x|+\rho)^{1-\beta}$, and so

$$|y_x(x)|\leqslant C_1(k)\quad\text{for } -1\leqslant x\leqslant 1, \beta \text{ in } [1+k,\beta_s];$$

while if $\beta=1$ evaluation of $I(x,\varepsilon,1)$ shows that

$$|y_x(x)|\leqslant C_1\ln\left[6/(x^2+\varepsilon)\right]\quad\text{for } -1\leqslant x\leqslant 1, \beta=1.$$

The following technical lemma, which one would expect to be true from (2.11), will be proven in Section 4.

LEMMA 2.6. *There is a positive constant c_2 depending only on β_l and β_s such that for ε in $(0,1]$ and $\beta_l\leqslant\beta\leqslant\beta_s$,*

$$(2.13)\qquad (x^2+\varepsilon)^{(\beta-1)/2}I(x,\varepsilon,\beta)\geqslant c_2\quad \textit{for } -1\leqslant x\leqslant 1.$$

We also have the following estimates on the higher derivatives.

THEOREM 2.7. *Assume* (2.7a–g) *and in addition assume f, p, and q are in $C^K[-1,1]$, where $K\geqslant 2$ is an integer. Then there is a constant C depending only on $S_5(K)$ such that for y the solution of* (1.1)

$$(2.14)\qquad |D_x^k y(x)|\leqslant C(|x|+\rho)^{\beta-k}I(x,\varepsilon,\beta)\quad \textit{for } -1\leqslant x\leqslant 1 \textit{ and } k=1,2,\dots,K+1.$$

When β is above 1, (2.14) is not a good estimate for the higher derivatives of y since $I(x,\varepsilon,\beta)$ increases with β. An improved result for this situation is:

THEOREM 2.8. *Suppose $\beta=m+\Delta$ where m is a positive integer and $\beta_l\leqslant\Delta\leqslant\beta_s$, and assume* (2.7a–g). (*For this result the "appropriate" choice of $[\beta_l,\beta_s]$ is $[c_0, 1+c_0]$, where c_0 is a "small" positive constant.*) *Let f, p, and q be in $C^{m+i}[-1,1]$ for $i\geqslant 2$ an integer. Then there is a constant C depending only on $S_5(m+i)$ such that for y the solution of* (1.1)

$$(2.15a)\qquad |D_x^k y(x)|\leqslant C\quad \textit{for } -1\leqslant x\leqslant 1 \textit{ and } k=1,\dots,m,$$

$$(2.15b) \quad |D_x^k y(x)| \leqslant C(|x| + \rho)^{\beta-k} I(x, \varepsilon, \Delta)$$
$$\text{for } -1 \leqslant x \leqslant 1 \text{ and } k = m+1, \ldots, m+i+1.$$

In the above situation where $i = 1$; (2.15a) is valid, and (2.15b) holds for $k = m + 1$.

We note that for $\beta > 0$ not an integer and for sufficiently smooth p, q, and f, Farrell [9] has previously shown that

$$|D_x^k y(x)| \leqslant M(k, \beta)\left[1 + (|x| + \rho)^{\beta-k}\right] \quad \text{for } -1 \leqslant x \leqslant 1, k = 1, 2, \ldots,$$

where the positive constant $M(k, \beta)$ does not depend on x or ε. This result corresponds to (2.14), (2.15) since for β not an integer one can write $\beta = m + \Delta$ with $0 < \Delta < 1$ and so $I(x, \varepsilon, \Delta) \leqslant C(\Delta)$.

The above theorems, together with Remark 2.3, may be used to derive estimates for the solution of (1.1), with turning points $x_i \neq \pm 1$, $1 \leqslant i \leqslant r$, and with possible boundary layers at $x = \pm 1$. For this we require some notation. We define the index set $I \subset \{1, \ldots, r\}$ by $I = \{i: \beta_i > 0\}$. In a similar way, we define $x_1^* = -1$, $x_2^* = +1$, and we define $I^* \subset \{1, 2\}$ by $I^* = \{j: (-1)^j p(x_j^*) < 0\}$. Either of the sets I or I^* may be empty, but it cannot happen that both sets are empty.

Let $\delta > 0$ and N_i, $1 \leqslant i \leqslant r$, be as in Remark 2.3. Let $k_p = \min\{|p(x)|: x \notin N_i, 1 \leqslant i \leqslant r\}$. Let $S_6(m) = \{\|p\|_m, \|q\|_m, \|f\|_m, |d_1|, |d_2|, k_p, k_q, \delta, \beta_l, \beta_s, m\}$. The following theorem is the generalization of Theorem 2.7 to the case of an arbitrary set of boundary layers and interior turning points. A similar generalization of Theorems 2.5 and 2.8 could also be made.

THEOREM 2.9. *Suppose f, p, q are in $C^K[-1, 1]$, where $K \geqslant 2$ is an integer. Then there are positive constants C and η, depending only on $S_6(K)$, such that if $\beta_l \leqslant |\beta_i| \leqslant \beta_s$, $1 \leqslant i \leqslant r$, then*

$$(2.16) \quad |D_x^k y(x)| \leqslant C\left\{ \sum_{i \in I} (|x - x_i| + \rho)^{\beta_i - k} I(x - x_i, \varepsilon, \beta_i) \right.$$
$$\left. + \sum_{j \in I^*} \varepsilon^{-k} \exp\left[-\eta |x - x_j^*| / \varepsilon\right] + 1 \right\},$$
$$-1 \leqslant x \leqslant 1, \ 1 \leqslant k \leqslant K + 1.$$

Finally, we note that the preceding estimates enable one to examine how the solution $y(x)$ of (1.1) approaches the solution $v(x)$ of the reduced problem (i.e., (1.1a) with $\varepsilon = 0$, and the boundary condition $v(-1) = d_1$ [$v(1) = d_2$] imposed if and only if $p(-1) < 0$ [$p(1) > 0$]). For a discussion of the reduced problem and estimates of $y - v$ see, for example, [22], [1, pp. 54, 59, 68]. In the case of a single turning point at $x = 0$, one can easily show the following result (the proof is in Section 4). A similar result could be formulated for the case of an arbitrary number of turning points.

Remark 2.10. Assume (2.7a–e), $\beta > 0$, and suppose p, q, and f are in $C^3[-1, 1]$. Then there is a constant $C(\beta)$ depending only on $S_5(3)$ and β such that

$$(2.17) \quad \|y - v\|_0 \leqslant C(\beta)\left[\varepsilon + \varepsilon^{\beta/2}(\ln(6/\varepsilon))^{\eta}\right]$$

where if $\beta = 1$, then $\eta = 1$, if $\beta = 2$, then $\eta = 2$, while $\eta = 0$ for all other $\beta > 0$. This also shows that $v(x)$ is continuous at $x = 0$.

Remark 2.11. Assume (2.7a–e), $\beta < 0$, and suppose p, q, and f are in $C^2[-1, 1]$. Then there is a constant C depending only on the set $S_4(2)$ (defined in Theorem 2.4) such that

$$|y(x) - v(x)| \leqslant C\varepsilon \quad \text{for } |x| \leqslant 1/2. \tag{2.18}$$

III. A Uniform Error Estimate for a Modification of the El-Mistikawy Werle Scheme for (1.1). In this section we will consider approximating the solution $y(x)$ of (1.1) using a modification of the exponential scheme of El-Mistikawy and Werle [6] which they constructed by using a specific choice of the general approach of Pruess [17] and Rose [18]. We will then use the bounds on $|y_x(x)|$ given in Section 2 along with appropriate comparison functions to estimate the difference in $L^\infty(-1, 1)$ between $y(x)$ and its approximation.

We now describe in detail the general approach of the El-Mistikawy Werle scheme, which is to replace (1.1) by a piecewise constant coefficient approximating differential equation. Consider (1.1) and assume (2.7a–c). Let J be a positive integer and define the uniform mesh length $h = 2/J$. Let the grid points $\{x_j\}$ be given by $x_j = -1 + jh$ for $j = 0, 1, \ldots, J$, and let Y_j denote the approximate value (to be determined) for $y_j = y(x_j)$. The solution $Y(x)$ of the problem

$$\bar{L}Y \equiv -\varepsilon Y_{xx}(x) - P(x)Y_x(x) + Q(x)Y(x) = F(x), \tag{3.1a}$$

$$Y(-1) = Y_0 = d_1, \qquad Y(1) = Y_J = d_2 \tag{3.1b}$$

is used to approximate the solution $y(x)$ of (1.1), where P, Q, and F are constants on each subinterval (x_{j-1}, x_j), $1 \leqslant j \leqslant J$ (the values of which may vary from one subinterval to the next). $Y(x)$ satisfies (3.1) in the sense that $Y(x)$ is in $C^1[-1, 1]$, (3.1b) holds, and (3.1a) is valid for x in $X' \equiv \bigcup_{j=1}^{J}(x_{j-1}, x_j)$. We will assume in what follows that $Q(x)$ is chosen such that

$$Q(x) \geqslant k_q, \qquad x \text{ in } X' \tag{3.2a}$$

and P, Q, and F satisfy

$$|P(x) - p(x)| + |Q(x) - q(x)| + |F(x) - f(x)| \leqslant Ch \quad \text{for } x \text{ in } X', \tag{3.2b}$$

where C depends only on $\|p\|_1$, $\|q\|_1$, and $\|f\|_1$. More specific choices for P, Q, and F will be made below.

The discussion in the beginning of Section 2 of [4] shows that (3.1) has a unique solution in the sense just described. (The specific choices of P and Q given there are not required in the proof, which remains valid for the case of (3.1) if (3.2a) holds.) From [6] or from Section 2 of [4] one has that at each interior grid point x_j a tridiagonal relationship of the form

$$\begin{aligned} -\varepsilon h^{-2}\left(r_j^- Y_{j-1} + r_j^c Y_j + r_j^+ Y_{j+1}\right) \\ = s_j^- f_{j-1} + s_j^c f_j + s_j^+ f_{j+1}, \quad 1 \leqslant j \leqslant J-1, \end{aligned} \tag{3.3}$$

is valid for the solution Y of (3.1) where for each j the r and s coefficients in (3.3) can be determined as follows [4]. Let P^- $[P^+]$ denote the value of $P(x)$ on (x_{j-1}, x_j)

$[(x_j, x_{j+1})]$ and similarly for Q^- and Q^+. Let $\bar{n}_1$ denote the negative root of $-\varepsilon w^2 - P^- w + Q^- = 0$, and let $\bar{k}_1$ denote the positive root. Define $n_1 = h\bar{n}_1$ and $k_1 = h\bar{k}_1$. Similarly define n_2 and k_2 using the quadratic polynomial $-\varepsilon w^2 - P^+ w + Q^+$. Define the following functions; $e(w) = \exp(w)$, $g(w) = (e(w) - 1)/w$, with $g(0) \equiv 1$, and let $2v_1 \equiv [1 - e(n_1 - k_1)]^{-1}$ and $2v_2 \equiv [1 - e(n_2 - k_2)]^{-1}$. Then (suppressing the j subscripts) the r and s coefficients in (3.3) are given by

$$
\begin{aligned}
&r^- = e(n_1)/g(n_1 - k_1), && r^+ = e(-k_2)/g(n_2 - k_2),\\
&r_1 = -n_1 - 1/g(n_1 - k_1), && r_2 = k_2 - 1/g(n_2 - k_2),\\
&r^c = r_1 + r_2, &&\\
&s^- = g(n_1)v_1 - e(n_1)g(-k_1)v_1, &&\\
&s^+ = g(-k_2)v_2 - e(-k_2)g(n_2)v_2, && s^c = s^- + s^+.
\end{aligned}
\tag{3.4}
$$

Remark 2.2 of [4] shows that the linear system (3.1b), (3.3), (3.4) has a unique solution which may be calculated using simple tridiagonal Gaussian decomposition. Thus (3.1) yields a readily implementable algorithm for obtaining an approximation to the solution of (1.1).

If $q(x) \equiv 0$ and $p(x)$ is nonzero on $[-1, 1]$, it has been shown [4] that $\|Y - y\|_0 \leqslant Ch$, with $C > 0$ independent of h and ε. If $q \equiv 0$, $P(x) = (p_{j-1} + p_j)/2$ and $F(x) = (f_{j-1} + f_j)/2$ on (x_{j-1}, x_j) for each j, then one has $\max_j |Y(x_j) - y(x_j)| \leqslant Ch^2$ [4], [10]. A similar result holds in the case that $P(x) \equiv 0$ and $q(x) > 0$ on $[-1, 1]$ [10]. We will use a numerical scheme based on (3.1) for the solution of our turning point problem. Our analysis uses a comparison function argument. For this, we require the following lemma.

LEMMA 3.1. *Consider the operator* $\bar{L}w(x) = -\varepsilon w_{xx}(x) - P(x)w_x(x) + Q(x)w(x)$, *where* $\varepsilon > 0$ *and* P *and* Q *are constant on each subinterval* (x_{j-1}, x_j), $j = 1, \dots, J$, *and where here we only need assume* $Q(x) \geqslant 0$. *Suppose* $w(x)$ *is in* $C^1[-1, 1]$, w *restricted to* $[x_{j-1}, x_j]$ *is in* $C^2[x_{j-1}, x_j]$ *for each* j, $w(-1) \geqslant 0$, $w(1) \geqslant 0$, *and* $\bar{L}w(x) \geqslant 0$ *for all* x *in* X'. *Then* $w(x) \geqslant 0$ *for* $-1 \leqslant x \leqslant 1$.

Proof. If not, then there is an x_0 in $(-1, 1)$ at which w attains its minimum and $w(x_0) < 0$. Furthermore since $w(\pm 1) \geqslant 0$, x_0 may be chosen such that x_0 is in an interval $[x_{i-1}, x_i]$ on which w is not constant. One can then use the maximum principles in [16, pp. 6–7] applied to $u = -w$ on the interval $[x_{i-1}, x_i]$ to obtain a contradiction.

The comparison function estimate for $Y(x) - y(x)$ proceeds in the following fashion. Letting $e(x) \equiv Y(x) - y(x)$, we have

$$
\begin{aligned}
\bar{L}e(x) &= F(x) - f(x) + (P(x) - p(x))y_x(x)\\
&\quad + (q(x) - Q(x))y(x) \equiv g(x) \quad \text{for } x \text{ in } X',
\end{aligned}
\tag{3.5a}
$$

$$e(-1) = e(1) = 0. \tag{3.5b}$$

Suppose we can choose a comparison function $\zeta(x)$ in $C^2[-1, 1]$ such that

$$\zeta(\pm 1) \geqslant 0 \quad \text{and} \quad \bar{L}\zeta(x) \geqslant |g(x)| \quad \text{for } x \text{ in } X'. \tag{3.6}$$

Then Lemma 3.1 applied to $w(x) = \zeta(x) \pm e(x)$ implies that

$$|e(x)| \leqslant \zeta(x) \quad \text{for } -1 \leqslant x \leqslant 1. \tag{3.7}$$

The estimates in Section 2 are used to bound $g(x)$. A suitable $\zeta(x)$ is then chosen which satisfies (3.6) thus yielding an error estimate (3.7).

We will give the error estimates for the situation when there is one turning point located at $x = 0$. The analysis of this case, together with Theorem 2.9, will make it clear how to treat (1.1) when there is more than one turning point.

THEOREM 3.2. *Assume* (2.7a–f) *and* (3.2) *and let* $\beta < 0$. *Then there is a constant* C *depending only on* $S_4(1)$ (*defined in Theorem* 2.4) *such that*

$$\|Y - y\|_0 \leqslant Ch. \tag{3.8}$$

Proof. From the maximum principle $Y(x)$ and $y(x)$ are both bounded so we may suppose h is bounded above by a fixed positive constant (described below). Recall Lemma 2.1 (on the intervals $[-1, -1/2]$ and $[1/2, 1]$ with $m = 1$) and then define the comparison functions

$$\psi_1(x) = C_1 \exp(-2\eta c_4(x+1)/\varepsilon), \tag{3.9a}$$

$$\psi_2(x) = C_2 \exp(-2\eta c_4(1-x)/\varepsilon), \tag{3.9b}$$

where C_1, C_2, c_4 are positive constants to be chosen. By direct substitution and use of (2.9) and (3.2) one can easily verify that for h near 0 and for c_4 fixed sufficiently small, there is a positive constant c_5 depending only on $S_4(1)$ such that

$$\bar{L}\psi_1(x) \geqslant c_5\varepsilon^{-1}\psi_1(x) \quad \text{for } x \text{ in } X' \text{ satisfying } -1 \leqslant x \leqslant -1/2, \tag{3.10a}$$

$$\bar{L}\psi_2(x) \geqslant c_5\varepsilon^{-1}\psi_2(x) \quad \text{for } x \text{ in } X' \text{ satisfying } 1/2 \leqslant x \leqslant 1. \tag{3.10b}$$

From (2.3) one has in addition that

$$|\bar{L}\psi_1(x)| \leqslant C \quad \text{for } x \text{ in } X' \text{ satisfying } x \geqslant -1/2, \tag{3.11a}$$

$$|\bar{L}\psi_2(x)| \leqslant C \quad \text{for } x \text{ in } X' \text{ satisfying } x \leqslant 1/2. \tag{3.11b}$$

Now let

$$\zeta(x) \equiv C_3 h + h\psi_1(x) + h\psi_2(x). \tag{3.12}$$

We conclude that (3.6) holds (with $g(x)$ given in (3.5)) if C_1 and C_2 are chosen sufficiently large and then an appropriately large C_3 is fixed: this follows from (3.2), (2.1), (2.2) (with $i = 1$), Theorem 2.4 (with $m = 1$) and (3.10), (3.11). Then (3.7) and (3.12) yield (3.8), and the proof is complete.

We now give the result for the case where $\beta > 0$ for which it will be convenient to define the following comparison function:

$$\phi(x, c) = (c^2x^2 + \varepsilon)^{(\beta-1)/2} I(cx, \varepsilon, \beta), \tag{3.13}$$

where c is a (small) positive constant to be chosen below. Note that for $c = 1$, $C_1\phi(x, c)$ is just the right side of (2.11a). We have

THEOREM 3.3. *Assume* (2.7a–f) *and* (3.2) *and let* $\beta > 0$. *Suppose* $P(x) \geqslant 0$ *for* $x \geqslant 0$ *and* $P(x) \leqslant 0$ *for* $x \leqslant 0$. *Then there are positive constants* c *and* C_1 *depending only on* $S_5(1)$ *such that for* y *the solution of* (1.1) *and* Y *the solution of* (3.1) *it is true that*

$$|Y(x) - y(x)| \leqslant C_1 h\phi(x, c) \quad \textit{for } -1 \leqslant x \leqslant 1 \tag{3.14}$$

with $\phi(x, c)$ *defined by* (3.13).

Note that Lemma 2.6 shows that for c in $(0, 1]$, $\phi(x, c)$ is bounded from below by a positive constant. In order to demonstrate (3.14) we first prove the following lemma.

LEMMA 3.4. *For any c in $(0, 1]$, $D_x\phi(x) < 0$ for $0 < x \leqslant 1$, and hence if $0 < c_1 < c_2 \leqslant 1$ and $|x| \leqslant 1$, then $\phi(x, c_1) \geqslant \phi(x, c_2)$. Assuming the hypotheses of Theorem 3.3, there are positive constants $c < 1$ and c_3 depending only on $S_5(1)$ such that*

$$\bar{L}\phi(x, c) \geqslant c_3\phi(x, c) \quad \text{for } x \text{ in } X'. \tag{3.15}$$

Proof. We first show $D_x\phi(x, c) < 0$ for $0 < x \leqslant 1$. Write out $D_x\phi(x, c)$ and consider the term containing the factor $I(cx, \varepsilon, \beta)$ and (except when $\beta = 1$) explicitly evaluate this integral. The contribution from the lower limit of integration exactly cancels the other term, and one finds that $D_x\phi(x, c)$ has the form $xd(x, \varepsilon)$ where $d(x, \varepsilon) < 0$ for $|x| \leqslant 1$ and ε in $(0, 1]$. We now prove (3.15). By (3.2a) and Lemma 2.6 one has that

$$Q(x)\phi(x, c) \geqslant k_q\phi(x, c)/2 + k_q c_2/2 \quad \text{for } c \text{ in } (0, 1] \text{ and } x \text{ in } X'.$$

Since $P(x) \geqslant 0$ for $x \geqslant 0$ and $P(x) \leqslant 0$ for $x \leqslant 0$ we have $-P(x)\phi_x(x, c) \geqslant 0$. Explicitly evaluating $-\varepsilon\phi_{xx}(x, c)$ and observing that $\varepsilon \leqslant (c^2x^2 + \varepsilon)$ and $c^2x^2 \leqslant (c^2x^2 + \varepsilon)$, one finds that, for $c > 0$ sufficiently small, $-\varepsilon\phi_{xx}(x, c) \geqslant -k_q c_2/4 - k_q\phi(x, c)/4$, and (3.15) follows.

With Lemma 3.4 in hand, the proof of Theorem 3.3 is then an immediate consequence of (3.5a), (3.2b), (2.1), and Theorem 2.5. The bound on the error given in (3.14) suffers a large growth when $|x| < h$, $\beta \leqslant 1$ and ε is small. This can be remedied with stronger conditions on the choice of $P(x)$. Numerical results given in Section 5 for the unmodified El-Mistikawy Werle scheme (i.e., for each j, $P(x) = (p_j + p_{j+1})/2$ on (x_j, x_{j+1}) and similarly for Q and F) suggest that some such stronger conditions are indeed necessary to prevent loss of accuracy when $\varepsilon \ll h$, $|x| < h$, and $\beta < 1$. We have

THEOREM 3.5. *Assume the hypotheses of Theorem 3.3 and furthermore assume $|P(x)| \leqslant C_4|x|$ for $x \in X'$. Then there is a constant $C_5 > 0$ depending only on C_4 and $S_5(1)$ such that with c the same as in Theorem 3.3*

$$|Y(x) - y(x)| \leqslant C_5 h\phi(h, c) \quad \text{for } |x| \leqslant 1. \tag{3.16}$$

The condition $|P(x)| \leqslant C_4|x|$ may be easily satisfied by slightly modifying the choice of $P(x)$ near the turning point: if there is a mesh point x_i coinciding with the turning point $x = 0$ then the condition $|P(x)| \leqslant c_4|x|$ will be satisfied if (in addition to (3.2b)) $P(x) \equiv 0$ on (x_{i-1}, x_{i+1}). If the turning point $x = 0$ is in the interior of (x_i, x_{i+1}), then the condition $|P(x)| \leqslant C_4|x|$ may be imposed by setting $P(x) \equiv p(x_i)$ on (x_{i-1}, x_i), $P(x) \equiv 0$ on (x_i, x_{i+1}), and $P(x) \equiv p(x_{i+1})$ on (x_{i+1}, x_{i+2}) (in addition to (3.2b)).

Proof of Theorem 3.5. From (3.2a), (3.5)–(3.7), and with ζ a (large) constant, it suffices to show $g(x)$ in (3.5a) is bounded by $Ch\phi(h, c)$. Since $\phi(x, c)$ is decreasing for $x > 0$ (by Lemma 3.4) it remains to prove the latter for x in X' with $0 \leqslant x \leqslant h$ (the case of $x < 0$ being symmetric). It is thus sufficient to show that

$$|P(x) - p(x)|\phi(x, c) \leqslant Ch\phi(h, c) \quad \text{for } x \text{ in } X' \text{ with } 0 \leqslant x \leqslant h,$$

and hence it suffices to demonstrate that

$$x\phi(x,c) \leqslant Ch\phi(h,c) \quad \text{for } 0 \leqslant x \leqslant h.$$

Let $b(x) = x\phi(x, c)$. Then by the mean value theorem, for any x in $[0, h)$ there is a ξ in (x, h) such that

$$x\phi(x,c) = b(x) = b(h) - (h-x)b_x(\xi)$$
$$= h\phi(h,c) - (h-x)\left[\phi(\xi,c) + \xi\phi_x(\xi,c)\right].$$

By combining $\phi(\xi, c)$ and the term in $\xi\phi_x(\xi, c)$ containing $I(c\xi, \varepsilon, \beta)$, one can verify that for ξ in $(0, h)$, $\phi(\xi, c) + \xi\phi_x(\xi, c) \geqslant -2$ and so $x\phi(x, c) \leqslant h\phi(h, c) + Ch$ which, recalling Lemma 2.6, completes the proof.

It should be noted that Theorems 3.3 and 3.5 are not sharp, e.g., when $\varepsilon = 1$ they only demonstrate $O(h)$ accuracy while for $\varepsilon = 1$ the unmodified El-Mistikawy Werle scheme is $O(h^2)$ [17]. Observe that Theorem 3.3 yields $O(h)$ accuracy away from the turning point for any β in $[\beta_l, \beta_s]$, and (3.16) implies that

(3.17a) $$\|Y - y\|_0 \leqslant C(\beta)(h^\beta + h) \quad \text{when } \beta > 0, \beta \neq 1,$$

(3.17b) $$\|Y - y\|_0 \leqslant C_5 h \ln \frac{6}{ch^2} \quad \text{when } \beta = 1.$$

Farrell [8], [9], [21] has obtained a general set of sufficient conditions on the coefficients of a large class of tridiagonal finite difference schemes for (1.1), (2.7), $\beta > 0$ not an integer, which when satisfied imply that the error at all the grid points is bounded by $C(\beta)(h^\beta + h)$.

IV. Proofs of the A Priori Estimates When $\beta > 0$. In this section we will provide the proofs of the results in Section 2 which were not proven there, starting with Theorem 2.5. Unless otherwise stated, in this section conditions (2.7a–g) will be assumed to hold for (1.1). Note that it suffices to prove the results for ε in $(0, \varepsilon_0]$ for a fixed positive $\varepsilon_0 < 1$.

We may rewrite (1.1) in the form

(4.1a) $$-\varepsilon y_{xx}(x) - p_x(0)xy_x(x) + q(0)y(x) = g_2(x) \quad \text{for } |x| < 1,$$

where

(4.1b) $$g_2(x) = f(x) + (p(x) - p_x(0)x)y_x(x) + (q(0) - q(x))y(x),$$

(4.2) $$y(-1) = d_1 \quad \text{and} \quad y(1) = d_2.$$

We first show that without loss of generality we may take $q(0) = 1$ in (4.1). Note that $q(0)$ is bounded between k_q and $\|q\|_1$. Define α by

(4.3) $$\alpha = 1/\beta = p_x(0)/q(0),$$

and then divide (4.1a) by $q(0)$, obtaining

(4.1c) $$-(\varepsilon/q(0))y_{xx}(x) - \alpha x y_x(x) + y(x) = g_2(x)/q(0).$$

It can be verified that if $\tilde{\varepsilon} = \varepsilon/q(0)$, then for k any given positive integer

$$(x^2 + \tilde{\varepsilon})^{(\beta-k)/2} \leqslant C(x^2 + \varepsilon)^{(\beta-k)/2} \quad \text{and} \quad I(x, \tilde{\varepsilon}, \beta) \leqslant CI(x, \varepsilon, \beta);$$

for this observe that in the case $c \equiv 1/q(0) < 1$

$$\int_{x^2+c\varepsilon}^{6} s^{(-\beta-1)/2}\,ds \leqslant \int_{cx^2+c\varepsilon}^{6} s^{(-\beta-1)/2}\,ds,$$

and then use the change of variable $\bar{s} = s/c$; the other parts are straightforward to check. The a priori estimates are hence not materially affected by replacing ε by $\varepsilon/q(0)$, and it will be seen below (cf. the discussion below (4.20)) that neither is the relevant behavior of $g(x)$. Thus instead of (4.1c, b), (4.2) it suffices to consider the problem

$$My \equiv -\varepsilon y_{xx}(x) - \alpha x y_x(x) + y(x) = g(x) \quad \text{for } |x| < 1, \tag{4.4a}$$

where

$$g(x) = f(x) + (p(x) - p_x(0)x)y_x(x) + (q(0) - q(x))y(x), \tag{4.4b}$$

$$y(-1) = d_1 \quad \text{and} \quad y(1) = d_2. \tag{4.4c}$$

The general approach to be used in treating (4.4) is to write $y(x)$ in the form $u_1(x) + u_2(x)$, where $Mu_1 = 0$ and $u_1(-1) = y(-1)$, $u_1(1) = y(1)$, and where $Mu_2 = g$ with $u_2(\pm 1) = 0$. A priori estimates on the behavior of the function $u_1(x)$ are obtained through the direct use of parabolic cylinder functions. These functions are also used to construct and obtain bounds on the Green's function for M which is then used to obtain the desired bound on $D_x u_2(x)$. Higher derivatives of u_2 are bounded via an inductive argument.

4.1. *Parabolic Cylinder Functions.* We recall some properties of the parabolic cylinder functions that are relevant to our analysis. Given a function $w(t)$ consider the corresponding function $\bar{w}(x)$ defined by

$$\bar{w}(x) = \phi(x) w[t(x)], \tag{4.5a}$$

where

$$t(x) \equiv \alpha^{1/2} x/\rho \quad \text{and} \quad \phi(x) \equiv \exp(-\alpha x^2/(4\varepsilon)) = \exp(-t^2/4). \tag{4.5b}$$

One can then check that $\bar{w}(x)$ satisfies $M\bar{w} = 0$ if and only if $w(t)$ satisfies

$$w_{tt} - [t^2/4 + \beta + 1/2]w = 0. \tag{4.6}$$

From [3], [20], one recalls that (4.6) determines the parabolic cylinder function with the index a of [3] given by $a = \beta + 1/2$. Following [3], there are two linearly independent solutions of (4.6), $U(a, t)$ and $V(a, t)$. These functions satisfy, for arbitrary real a:

$$U(a, t) = \exp(-t^2/4) t^{-a-1/2} \cdot (1 + \delta_1) \qquad \text{for } t \geqslant C_1(a), \tag{4.7a}$$

$$V(a, t) = (2/\pi)^{1/2} \exp(t^2/4) t^{a-1/2} \cdot (1 + \delta_2) \qquad \text{for } t \geqslant C_2(a), \tag{4.7b}$$

$$\pi V(a, t) = \Gamma(a + 1/2)\{\sin \pi a \cdot U(a, t) + U(a, -t)\}, \tag{4.7c}$$

where $C_1(a)$ and $C_2(a)$ are (large) positive constants and $|\delta_1| + |\delta_2| \leqslant 1/3$. Note that when $a = \beta + 1/2 - k$ for $k = 0$ or 1, and $\beta_l \leqslant \beta \leqslant \beta_s$, $\Gamma(a + 1/2)$ is nonzero and finite, and hence for such an a

$$U(a, -t) = (1 + \delta_3)\pi V(a, t)/\Gamma(a + 1/2) \quad \text{for } t \geqslant C_3(a), \tag{4.7d}$$

where $C_3(a)$ is a positive constant and $|\delta_3| \leqslant 1/3$. From this it follows that

$$|V(a, -t)| \leqslant C_4(a)|V(a, t)| \quad \text{for } t \geqslant C_5(a), \tag{4.7e}$$

where $C_4(a)$ and $C_5(a)$ are positive constants.

It is also true [3] that for arbitrary real a

(4.8a) $$U_t(a,t) = .5tU(a,t) - U(a-1,t),$$

(4.8b) $$V_t(a,t) = .5tV(a,t) + (a-1/2)V(a-1,t),$$

from which it follows that for arbitrary real a

(4.9a) $$\bar{U}_x(a,x) = -\alpha^{1/2}\bar{U}(a-1,x)/\rho,$$

(4.9b) $$\bar{V}_x(a,x) = (a-1/2)\alpha^{1/2}\bar{V}(a-1,x)/\rho.$$

Now consider the two functions μ^- and μ^+ which are solutions of $M\mu = 0$ and such that $\mu^-(-1) = \mu^+(1) = 0$ and $\mu^-(1) = \mu^+(-1) = 1$. We may write

(4.10) $$u_1(x) = u_1(1)\mu^-(x) + u_1(-1)\mu^+(x),$$

and thus to analyze $u_1(x)$ it suffices to analyze the behavior of μ^- and μ^+. These functions will also be used below to explicitly examine the Green's function for the operator M which will be used to obtain the desired estimates for $u_2(x)$.

4.2. *Analysis of* μ^- *and* μ^+. Let $\bar{U}(x) = \phi(x)U(t)$, $\bar{V}(x) = \phi(x)V(t)$ as in (4.5a); we suppress the dependence of these functions on a when the value of a is clear from the context. Write $\mu^+(x) = \gamma^+\bar{U}(x) + \delta^+\bar{V}(x)$, $\mu^-(x) = \gamma^-\bar{U}(x) + \delta^-\bar{V}(x)$, where $\gamma^\pm$, $\delta^\pm$ are constants whose dependence on ε we wish to determine. Then

(4.11) $$\begin{pmatrix} \mu^-(-1) \\ \mu^-(1) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = A\begin{pmatrix} \gamma^- \\ \delta^- \end{pmatrix},$$

where

$$A = \begin{bmatrix} \bar{U}(-1) & \bar{V}(-1) \\ \bar{U}(1) & \bar{V}(1) \end{bmatrix}.$$

We assume that ε is so small that for $\beta_l \leqslant \beta \leqslant \beta_s$, and $a = \beta \pm 1/2$, (4.7) holds for $t = \alpha^{1/2}/\rho$. Using (4.7), we find that the inverse of the matrix A has the form

(4.12a) $$A^{-1} = \begin{bmatrix} k_1\rho^\beta & C_1\rho^\beta \\ C_2\rho^{3\beta+1}\exp(-.5\alpha/\varepsilon) & k_2\rho^\beta \end{bmatrix},$$

where

(4.12b) $$0 < c_3 \leqslant |k_i| \leqslant C_4, \qquad i = 1,2,$$

and where c_3 and C_4 depend only on α and the upper bound on ε. We thus obtain $\gamma^- = C_1\rho^\beta$, $\delta^- = k_2\rho^\beta$. In a similar way, we may obtain γ^+, δ^+. Summarizing these calculations, we have

(4.13) $$\begin{bmatrix} \mu^-(x) & \mu_x^-(x) \\ \mu^+(x) & \mu_x^+(x) \end{bmatrix} = B\begin{bmatrix} \bar{U}(x) & \bar{U}_x(x) \\ \bar{V}(x) & \bar{V}_x(x) \end{bmatrix},$$

where

$$B = \begin{bmatrix} C_1\rho^\beta & k_2\rho^\beta \\ k_1\rho^\beta & C_2\rho^{3\beta+1}\exp(-.5\alpha/\varepsilon) \end{bmatrix}.$$

Then from (4.9) and (4.13), for $i \geqslant 1$,

(4.14a) $$D_x^i\mu^-(x) = C\rho^{\beta-i}\bar{U}(a-i,x) + C\rho^{\beta-i}\bar{V}(a-i,x),$$

(4.14b) $$D_x^i\mu^+(x) = C\rho^{\beta-i}\bar{U}(a-i,x) + C\rho^{3\beta+1-i}\exp(-.5\alpha/\varepsilon)\bar{V}(a-i,x).$$

Note that since $U(a-i,0)$ and $V(a-i,0)$ are finite, (4.7a, b) and the maximum principle for M imply that $\overline{U}(a-i,x)$ and $\overline{V}(a-i,x)$ are bounded for $0 \leqslant x \leqslant C\rho$. We now establish estimates for the derivatives of $u_1(x)$. (See also [8, Lemma 2].)

LEMMA 4.1. *Let $u_1(x)$ satisfy $Mu_1 = 0$ with $u_1(\pm 1) = y(\pm 1)$. Then for $i = 1, 2, \ldots$ there is a constant C_i such that*

(4.15a) $$|D_x^i u_1(x)| \leqslant C_i/\rho^{i-\beta} \qquad \text{for } |x| \leqslant \rho,$$

(4.15b) $$|D_x^i u_1(x)| \leqslant C_i/|x|^{i-\beta} \qquad \text{for } |x| \geqslant \rho.$$

Proof. Recalling (4.10), for $x \geqslant 0$ the result follows from (4.14) and (4.7a, b) and from observing that, for $x \geqslant \rho$, $(x/\rho)^{2i-2\beta-1}\exp(-.5\alpha x^2/\varepsilon) \leqslant C$. For the case $x < 0$, observe that

(4.16) $$\mu^+(-x) = \mu^-(x),$$

since $w(x) = \mu^+(-x)$ is a solution of $Mw = 0$, and its boundary values at $x = \pm 1$ agree with those of $\mu^-(x)$. Then, from (4.10) and (4.16), we have

(4.17) $$(-1)^i (D_x^i u_1)(-x) = u_1(1) D_x^i \mu^+(x) + u_1(-1) D_x^i \mu^-(x),$$

so the result for $x < 0$ follows from the analysis for $x \geqslant 0$.

We next turn our attention to the Green's function for M in order to obtain the desired bounds on $u_2(x)$.

4.3. *The Green's Function for M.* From, e.g., [5, p. 192] or [19] one may verify that the Green's function for M is given by

(4.18a) $$G(x,\tau) = -\mu^-(x)\mu^+(\tau)\varepsilon^{-1}\exp(.5\alpha\tau^2/\varepsilon)/W(0) \quad \text{for } x \leqslant \tau,$$

(4.18b) $$G(x,\tau) = -\mu^+(x)\mu^-(\tau)\varepsilon^{-1}\exp(.5\alpha\tau^2/\varepsilon)/W(0) \quad \text{for } x \geqslant \tau,$$

where

(4.18c) $$W(x) \equiv \mu^-(x)\mu_x^+(x) - \mu^+(x)\mu_x^-(x),$$

and the solution of $Mu_2 = g$ with $u_2(\pm 1) = 0$ is given by

(4.19) $$\begin{aligned} u_2(x) &= \int_{-1}^{1} G(x,\tau) g(\tau)\, d\tau \\ &= \int_{-1}^{x} G(x,\tau) g(\tau)\, d\tau + \int_{x}^{1} G(x,\tau) g(\tau)\, d\tau, \end{aligned}$$

and so

(4.20) $$D_x u_2(x) = \int_{-1}^{1} G_x(x,\tau) g(\tau)\, d\tau.$$

We now discuss some properties of the function g given by (4.4b). We write $g(x) = f(0) + \tilde{g}(x)$, where

(4.21) $$\tilde{g}(x) = f(x) - f(0) + [p(x) - xp_x(0)]y_x(x) + [q(0) - q(x)]y(x).$$

Using Lemma 4.4 below, we see that $|\tilde{g}(x)| \leqslant C|x|$. A solution of $Mw = f(0)$ is $w(x) \equiv f(0)$. We may then write $y = u_1 + f(0) + u_2$ with $Mu_1 = 0$, and the boundary data of u_1 adjusted so $u_1 + f(0)$ agrees with y at $x = \pm 1$, and with $Mu_2 = g(x) - f(0)$ and $u_2(\pm 1) = 0$. Hence, taking Lemma 4.4 to be true, we may without loss of generality assume g has the form

(4.22) $$g(x) = g_1(x)x, \quad \text{where } |g_1(x)| \leqslant C.$$

In order to use (4.20) to estimate $D_x u(x)$, we need some further bounds on μ^-, μ^+, and $W(0)$. From (4.13) and (4.9), for $\varepsilon \leqslant c$ one finds that

$$|W(0)| \geqslant k\rho^{2\beta-1} \tag{4.23}$$

for some constant $k > 0$. By the maximum principle, $\mu^-(x)$ and $\mu^+(x)$ are in $[0, 1]$ for $-1 \leqslant x \leqslant 1$. From (4.13), (4.9), and (4.7) we have

$$\mu^-(x) + \mu^+(x) \leqslant C\rho^\beta \qquad \text{for } 0 \leqslant x \leqslant \rho, \tag{4.24a}$$

$$\mu^-(x) \leqslant Cx^\beta \qquad \text{for } \rho \leqslant x \leqslant 1, \tag{4.24b}$$

$$\mu^+(x) \leqslant C\rho^{2\beta+1}x^{-\beta-1}\exp(-.5\alpha x^2/\varepsilon) \qquad \text{for } \rho \leqslant x \leqslant 1, \tag{4.24c}$$

and so (4.16) shows that

$$\mu^+(x) \leqslant C|x|^\beta \qquad \text{for } -1 \leqslant x \leqslant -\rho, \tag{4.24d}$$

$$\mu^-(x) \leqslant C\rho^{2\beta+1}|x|^{-\beta-1}\exp(-.5\alpha x^2/\varepsilon) \qquad \text{for } -1 \leqslant x \leqslant -\rho. \tag{4.24e}$$

Similarly we have

$$|\mu_x^-(x)| + |\mu_x^+(x)| \leqslant C\rho^{\beta-1} \qquad \text{for } |x| \leqslant \rho, \tag{4.24f}$$

$$|\mu_x^-(x)| \leqslant Cx^{\beta-1} \qquad \text{for } \rho \leqslant x \leqslant 1, \tag{4.24g}$$

$$|\mu_x^+(x)| \leqslant C\rho^{2\beta-1}\exp(-.5\alpha x^2/\varepsilon)/x^\beta \qquad \text{for } \rho \leqslant x \leqslant 1, \tag{4.24h}$$

$$|\mu_x^-(x)| \leqslant C\rho^{2\beta-1}\exp(-.5\alpha x^2/\varepsilon)/|x|^\beta \qquad \text{for } -1 \leqslant x \leqslant -\rho, \tag{4.24i}$$

$$|\mu_x^+(x)| \leqslant C|x|^{\beta-1} \qquad \text{for } -1 \leqslant x \leqslant -\rho. \tag{4.24j}$$

We can now prove

LEMMA 4.2. *Let $u_2(x)$ be given by* (4.19) *and assume $g(x)$ satisfies* (4.22). *Then there is a constant C_1 depending only on $S_5(1)$ such that*

$$|D_x u_2(x)| \leqslant C_1 + C_1\rho^{\beta-1}\int_{\max(|x|,\rho)}^1 \tau^{-\beta}\,d\tau \qquad \textit{for } |x| \leqslant \rho, \tag{4.25a}$$

$$|D_x u_2(x)| \leqslant C_1 + C_1|x|^{\beta-1}\int_{\max(|x|,\rho)}^1 \tau^{-\beta}\,d\tau \qquad \textit{for } |x| \geqslant \rho. \tag{4.25b}$$

Proof. We first show that it suffices to prove that

$$F_1(x) \equiv \int_x^1 |G_x(x,\tau)\tau|\,d\tau \tag{4.26a}$$

is bounded by the right-hand side of (4.25a) [(4.25b)] for $|x| \leqslant \rho$ [$|x| \geqslant \rho$]. Suppose $F_1(x)$ satisfies these bounds. By (4.20) and (4.22) it remains to show that

$$F_2(x) \equiv \int_{-1}^x |G_x(x,\tau)\tau|\,d\tau \tag{4.26b}$$

also satisfies these bounds. Now, by (4.18) and (4.16),

$$G_x(-x,-\tau) = -G_x(x,\tau), \tag{4.27}$$

so then

$$F_2(x) = \int_{-1}^x |G_x(-x,-\tau)\tau|\,d\tau = \int_{-x}^1 |G_x(-x,s)s|\,ds = F_1(-x),$$

so it suffices to estimate $F_1(x)$. We complete the proof by using (4.18), (4.23), and (4.24) to show that the integral of $|G_x(x,\tau)\tau|$ over the following τ intervals (and for various ranges of x) is bounded as claimed.

Case I. $-1 \leqslant x \leqslant -\rho$; $x \leqslant \tau \leqslant -\rho$, $-\rho \leqslant \tau \leqslant \rho$, $\rho \leqslant \tau \leqslant 1$.

Case II. $-\rho \leqslant x \leqslant \rho$; $x \leqslant \tau \leqslant \rho$, $\rho \leqslant \tau \leqslant 1$.

Case III. $\rho \leqslant x \leqslant 1$; $x \leqslant \tau \leqslant 1$.

We present two representative verifications of the claimed bound, the rest being similar. First consider Case I with $x \leqslant \tau \leqslant -\rho$. Then

$$|G_x(x,\tau)\tau| \leqslant -\exp(-.5\alpha(x^2-\tau^2)/\varepsilon)\tau\rho^{-2}|\tau/x|^{\beta}, \tag{4.28}$$

and, for $x \leqslant \tau \leqslant -\rho$, $|\tau/x|^{\beta} \leqslant 1$. Replacing the latter term by 1 in (4.28), we find that the integral of $|G_x(x,\tau)\tau|$ from $\tau = x$ to $\tau = -\rho$ is $\leqslant C$. For Case I with $\rho \leqslant \tau \leqslant 1$,

$$\begin{aligned} |G_x(x,\tau)\tau| &\leqslant \exp(-.5\alpha x^2/\varepsilon)\rho^{2\beta-1}|x|^{-\beta}\tau^{-\beta} \\ &= \exp(-.25\alpha x^2/\varepsilon)\left[\exp(-.25\alpha x^2/\varepsilon)(|x|/\rho)\right] \\ &\quad\cdot\left[|x|^{-1}\rho\cdot\rho^{2\beta-1}|x|^{-\beta}\cdot|x|^{1-\beta}\right]|x|^{\beta-1}\tau^{-\beta}. \end{aligned} \tag{4.29a}$$

The two bracketed terms are bounded by C and 1, respectively, and so

$$\begin{aligned} \int_{\rho}^{1}|G_x(x,\tau)\tau|\,d\tau \leqslant\ & C\int_{\rho}^{|x|}\exp(-.25\alpha x^2/\varepsilon)|x/\tau|^{\beta}|x|^{-1}\,d\tau \\ &+ C|x|^{\beta-1}\int_{|x|}^{1}\tau^{-\beta}\,d\tau. \end{aligned} \tag{4.29b}$$

Now in (4.29b) $\tau \geqslant \rho$, so $|x/\tau| \leqslant |x/\rho|$ and hence, using (2.3), the first term in the right side of (4.29b) is bounded by $(|x|-\rho)C|x|^{-1} \leqslant C$ giving the result.

4.4. *A Bound for* $y_x(x)$. We prove the following lemma.

LEMMA 4.4. *Let* $y(x)$ *be the solution of* (1.1). *Then there is a constant* C *depending only on* $S_5(1)$ *such that*

$$|xD_x y(x)| \leqslant C \quad \text{for } -1 < x < 1. \tag{4.30}$$

Proof. From the results in Section 2, $y(x)$ is "smooth" for $|x| \geqslant c$, so it is only necessary to demonstrate (4.30) in a neighborhood of $x = 0$. We will show that

$$\varepsilon|D_x^2 y(x)| \leqslant C, \tag{4.31}$$

which then implies (4.30). Let $z(x) \equiv D_x^2 y(x)$. Then since $p(0) = 0$, $|z(0)| \leqslant C/\varepsilon$. Differentiating (1.1a) once, we find that

$$Nz \equiv -\varepsilon z_x(x) - p(x)z(x) = s(x), \tag{4.32a}$$

where

$$s(x) = s_1(x) + s_2(x) \quad \text{with } s_1(x) = f_x(x) - q_x(x)y(x) \text{ and } s_2(x) = (p_x(x) - q(x))y_x(x). \tag{4.32b}$$

Let

$$P(x) \equiv -\int_0^x p(\xi)\,d\xi \quad \text{and} \quad \phi(x,\xi) \equiv \exp[(P(x)-P(\xi))/\varepsilon]. \tag{4.33a}$$

Then, as can easily be verified, the solution of (4.32a) is given by

$$z(x) = z(0)\exp(P(x)/\varepsilon) - \varepsilon^{-1}\int_0^x s(\xi)\phi(x,\xi)\,d\xi. \tag{4.33b}$$

From the conditions (2.7) we have that $p_x(x) \geqslant \gamma$, $-1 \leqslant x \leqslant 1$, for some positive constant γ depending only on $S_5(1)$. Then

$$P(x) \leqslant -\gamma x^2/2 \quad \text{for } -1 \leqslant x \leqslant 1, \tag{4.34a}$$

$$P(x) - P(\xi) = -\int_\xi^x p(\tau)\,d\tau \leqslant -.5\gamma(x^2-\xi^2) \leqslant 0 \tag{4.34b}$$

$$\text{for } 0 \leqslant \xi \leqslant x \leqslant 1 \text{ and for } -1 \leqslant x \leqslant \xi \leqslant 0,$$

and so

$$|z(0)\exp(P(x)/\varepsilon)| \leqslant C/\varepsilon, \tag{4.35a}$$

$$\left|-\varepsilon^{-1}\int_0^x s_1(\xi)\phi(x,\xi)\,d\xi\right| \leqslant C/\varepsilon. \tag{4.35b}$$

To deal with

$$I_0 \equiv -\varepsilon^{-1}\int_0^x y_x(\xi)\cdot(p_x(\xi)-q(\xi))\phi(x,\xi)\,d\xi \tag{4.36a}$$

we integrate by parts obtaining

$$\begin{aligned} I_0 = {} & \left[-\varepsilon^{-1}y(\xi)(p_x(\xi)-q(\xi))\phi(x,\xi)\right]_0^x \\ & +\varepsilon^{-1}\int_0^x y(\xi)\phi(x,\xi)D_\xi(p_x(\xi)-q(\xi))\,d\xi \\ & +\varepsilon^{-1}\int_0^x y(\xi)(p_x(x)-q(\xi))\phi_\xi(x,\xi)\,d\xi. \end{aligned} \tag{4.36b}$$

The first two terms on the right side of (4.36b) are clearly $\leqslant C/\varepsilon$, and the last term is bounded by

$$C\varepsilon^{-1}\int_0^x |\phi_\xi(x,\xi)|\,d\xi = C\varepsilon^{-1}|\phi(x,x)-\phi(x,0)| \leqslant C/\varepsilon. \tag{4.36c}$$

Equations (4.35) and (4.36) imply (4.31), and the proof of the lemma is complete.

Proof of Theorem 2.5. Using (4.15) and the inequality $I(x,\varepsilon,\beta) \geqslant c$, we find that the inequality (2.11a) is satisfied with y replaced by u_1. To show that (2.11a) is satisfied with y replaced by u_2, we use Lemma 4.2 and Lemma 2.6. We must then show that

$$\int_{\max(|x|,\rho)}^1 \tau^{-\beta}\,d\tau \leqslant CI(x,\varepsilon,\beta). \tag{4.37}$$

Using the change of variable $s = \tau^2$, one finds that

$$\int_z^1 \tau^{-\beta}\,d\tau = \int_{z^2}^1 .5s^{(-\beta-1)/2}\,ds \quad \text{for } z \text{ in } (0,1).$$

Setting

$$i(s,\beta) \equiv s^{(-\beta-1)/2}, \tag{4.38}$$

to prove (4.37) it is thus certainly sufficient to show that

$$I^* \equiv \int_{\max(x^2,\varepsilon)}^{x^2+\varepsilon} i(s,\beta)\,ds \leqslant C\int_{x^2+\varepsilon}^6 i(x,\beta)\,ds. \tag{4.39}$$

Now using the change of variable $t = s + \varepsilon$, we have

$$I^* = \int_{\max(x^2,\varepsilon)+\varepsilon}^{x^2+2\varepsilon} (t/2 + (t/2 - \varepsilon))^{(-\beta-1)/2}\, dt$$
$$\leqslant \int_{x^2+\varepsilon}^{3} (t/2)^{(-\beta-1)/2}\, dt \leqslant CI(x, \varepsilon, \beta)$$

completing the proof of (4.37).

Proof of Lemma 2.6. Using the notation (4.38), define

$$\phi(z) = z^{(\beta-1)/2}\int_z^6 i(s,\beta)\, ds. \tag{4.40}$$

An easy calculation shows that, for $z > 0$ and β in $[\beta_l, \beta_s]$,

$$D_z\phi(z) = -6^{(1-\beta)/2}z^{(\beta-3)/2} < 0. \tag{4.41}$$

Hence $\phi(x^2+\varepsilon) \geqslant \phi(2) \geqslant c_2$ for $|x| \leqslant 1$, ε in $(0, 1]$, and $\beta_l \leqslant \beta \leqslant \beta_s$, which is the desired result.

We now turn our attention to obtaining a priori bounds on the higher derivatives of $y(x)$.

4.5. *A Priori Estimates for the Higher Derivatives*. The estimates for the higher derivatives will follow from an inductive argument, using the fact that each higher derivative satisfies an equation of the form (4.32a) having a solution of the form (4.33). To begin the induction, we need to bound $D_x^2y(0)$, where y is the solution of (4.4) (and where we are continuing to assume (2.7a–g)).

LEMMA 4.5. *Let y be the solution of* (4.4). *Then*

$$|y_{xx}(0)| \leqslant C\varepsilon^{(\beta-2)/2}I(0, \varepsilon, \beta). \tag{4.42}$$

Proof. Note that Lemma 2.6 shows that the right side of (4.42) is $\geqslant C\rho^{-1}$. From the results already obtained, in particular Lemma 4.1 and (4.30), without loss of generality we may assume $y(\pm 1) = 0$ and (4.22) is valid. In this case,

$$y(x) = \int_{-1}^{x} G(x,\tau)g(\tau)\, d\tau + \int_x^1 G(x,\tau)g(\tau)\, d\tau, \tag{4.43a}$$

and by (4.22) $g(0) = 0$, hence differentiating (4.43a) twice gives

$$D_x^2y(0) = \int_{-1}^{0} G_{xx}(0,\tau)g(\tau)\, d\tau + \int_0^1 G_{xx}(0,\tau)g(\tau)\, d\tau. \tag{4.43b}$$

From (4.14) with $i = 2$ and $x = 0$,

$$|D_x^2\mu^-(0)| + |D_x^2\mu^+(0)| \leqslant C\rho^{\beta-2}.$$

Also observe that $(D_x^2G)(x,\tau) = (D_x^2G)(-x,-\tau)$, and so, using (4.22) and (4.43b), one only needs to bound the integral of $|G_{xx}(0,\tau)\tau|$ from $\tau = 0$ to ρ and from $\tau = \rho$ to 1. Using (4.18), (4.23), (4.24a, c) and (4.37), we obtain (4.42).

We can now bound the higher derivatives at $x = 0$.

LEMMA 4.6. *In addition to* (2.7a–g), *assume that f, p, and q are in $C^K[-1,1]$ where $K \geqslant 2$, and let y be the solution of* (1.1). *Then*

$$|D_x^ky(0)| \leqslant C\rho^{\beta-k}I(0,\varepsilon,\beta) \quad \textit{for } k = 1,2,\ldots, K+2. \tag{4.44a}$$

Furthermore, (4.44a) *is also valid if f is a function of x and ε satisfying*

$$\text{(4.44b)} \quad |D_x^k f(x,\varepsilon)| \leqslant C, \qquad k = 0, 1, |x| \leqslant 1,$$

$$\text{(4.44c)} \quad |D_x^k f(x,\varepsilon)| \leqslant C(|x| + \rho)^{\beta - k + 1} I(x, \varepsilon, \beta), \quad 2 \leqslant k \leqslant K, -1 \leqslant x \leqslant 1.$$

The constant in (4.44a) *depends only on $S_5(K)$ and the constants in* (4.44b, c).

Proof. We have already proven (4.44a) for $k = 1$ and $k = 2$, and we now proceed by induction, assuming the result is true for derivatives 1 through k $(2 \leqslant k \leqslant K + 1)$. Differentiating (1.1a) $k - 1$ times with respect to x and recalling that $p(0) = 0$, we find that

$$\text{(4.45)} \quad \varepsilon|D_x^{k+1} y(0)| \leqslant C|D_x^{k-1} y(0)| + C|D_x^{k-1} f(0, \varepsilon)| + \text{lower order terms.}$$

By the inductive hypothesis, $D_x^{k-1} y(0)$ is bounded by $C\rho^{\beta-(k-1)} I(0, \varepsilon, \beta)$ and so (4.45), (4.44b, c) and Lemma 2.6 yield the result.

We note that (4.44c) is overly restrictive here, but it will turn out to be appropriate for a later result. With Lemma 4.6 we can prove

LEMMA 4.7. *In addition to* (2.7a–g), *assume f, p, and q are in $C^K[-1, 1]$ where $K \geqslant 2$, and let y be the solution of* (1.1). *Then*

$$\text{(4.46)} \quad |D_x^k y(x)| \leqslant C\rho^{\beta-k} I(0, \varepsilon, \beta) \quad \textit{for } k = 1, 2, \ldots, K + 1, -1 \leqslant x \leqslant 1.$$

Furthermore (4.46) *remains valid when f is a function of x and ε satisfying* (4.44b, c). *The constant in* (4.46) *depends only on $S_5(K)$ and the constants in* (4.4b, c).

Proof. The result is true for $k = 1$ by Theorem 2.5 and (4.40), (4.41). Assume the result is true for derivatives 1 through k $(1 \leqslant k \leqslant K)$. Differentiate (1.1a) k times and let $z(x) \equiv D_x^{k+1} y(x)$. Then $z(x)$ satisfies

$$\text{(4.47)} \qquad -\varepsilon z_x(x) - p(x) z(x) = s(x) \quad \text{for } -1 \leqslant x \leqslant 1,$$

where $s(x)$ involves kth and lower order x-derivatives of y and f. Recall (4.32a), (4.33), (4.34), and Lemma 4.6, and use the inductive hypothesis to find that

$$\text{(4.48)} \quad |D_x^{k+1} y(x)| \leqslant C\rho^{\beta-(k+1)} I(0, \varepsilon, \beta)$$
$$+ C\rho^{-2} \int_0^{|x|} \rho^{\beta-k} I(0, \varepsilon, \beta) \exp[-.5\gamma(x^2 - \xi^2)/\varepsilon] \, d\xi.$$

If $|x| \leqslant \rho$, then the second term is bounded by $C\rho^{-2}|x|\rho^{\beta-k} I(0, \varepsilon, \beta)$ giving the result. For $|x| \geqslant \rho$, in the integral use the change of variable $\psi = |x| - \xi$ and then use the inequality $2|x| - \psi \geqslant |x| \geqslant \rho$ for ψ in $[0, |x|]$ to obtain the result.

We can now prove

THEOREM 4.8. *In addition to* (2.7), *assume f, p, and q are in $C^K[-1, 1]$, where $K \geqslant 2$, and let y be the solution of* (1.1). *Then*

$$\text{(4.49)} \quad |D_x^k y(x)| \leqslant C(|x| + \rho)^{\beta-k} I(x, \varepsilon, \beta)$$
$$\textit{for } k = 1, 2, \ldots, k + 1, -1 \leqslant x \leqslant 1.$$

Furthermore (4.49) *remains valid if f is a function of x and ε satisfying* (4.44b, c). *The constant in* (4.49) *depends only on $S_5(K)$ and the constant in* (4.44b, c).

Proof. Let $\pi \geqslant 1$ be a number depending only on $S_5(K)$; a specific choice of π will be made right after (4.58) below. Since it is sufficient to prove the result for ε bounded above by some fixed positive constant ε_0, we may assume $2\pi\rho \leqslant 1$. We first demonstrate that when $|x| \leqslant 2\pi\rho$, (4.49) follows from (4.46). In this case, since $(|x| + \rho)^{\beta-k}$ lies between $\rho^{\beta-k}$ and $(2\pi\rho + \rho)^{\beta-k}$, it suffices to show

$$I(0, \varepsilon, \beta) \leqslant C_\pi I(2\pi\rho, \varepsilon, \beta), \tag{4.50a}$$

and so using the notation (4.38) it suffices to show

$$\int_\varepsilon^{\varepsilon+4\pi^2\varepsilon} i(s, \beta)\, ds \leqslant C_\pi I(2\pi\rho, \varepsilon, \beta). \tag{4.50b}$$

Using the change of variable $t = s + 4\pi^2\varepsilon$, we have

$$\begin{aligned} \int_\varepsilon^{\varepsilon+4\pi^2\varepsilon} & i(s, \beta)\, ds \\ &= \int_{\varepsilon+4\pi^2\varepsilon}^{\varepsilon+8\pi^2\varepsilon} i\left[\left(4\pi^2 t/(1 + 4\pi^2) - 4\pi^2\varepsilon\right) + t/(1 + 4\pi^2), \beta\right] dt \\ &\leqslant \int_{\varepsilon+4\pi^2\varepsilon}^{\varepsilon+8\pi^2\varepsilon} \left[t/(1 + 4\pi^2)\right]^{(-\beta-1)/2} dt \leqslant C_\pi I(2\pi\rho, \varepsilon, \beta) \end{aligned} \tag{4.51}$$

as claimed. We may thus suppose $|x| \geqslant 2\pi\rho$ for the rest of the proof of (4.49).

We now proceed inductively as in the proof of Lemma 4.7, using (4.47). It will be convenient to define

$$E(x, \xi) = \exp\left[-.5\gamma(x^2 - \xi^2)/\varepsilon\right]. \tag{4.52a}$$

From (4.32), (4.33), (4.34), (4.44), (2.13) and the inductive hypothesis

$$\begin{aligned} |z(x)| &\leqslant C\rho^{\beta-k-1} I(0, \varepsilon, \beta) \exp(-\gamma x^2/(2\varepsilon)) \\ &\quad + \left| C\varepsilon^{-1} \int_0^x (|\xi| + \rho)^{\beta-k} I(\xi, \varepsilon, \beta) E(x, \xi)\, d\xi \right| \equiv T_1 + T_2. \end{aligned} \tag{4.52b}$$

Examination of (4.52) shows that we can, for convenience, suppose that $x \geqslant 0$. We first treat the term T_1. Let m be an integer larger than $2^{(\beta+1)/2}$. Then since $x \geqslant 2\pi\rho > \rho$, from (4.50a) $I(0, \varepsilon, \beta) \leqslant C_\pi I(2\pi\rho, \varepsilon, \beta) \leqslant C_\pi I(\rho, \varepsilon, \beta)$ and so using (2.3),

$$\begin{aligned} T_1 &= C\frac{\rho^{\beta-k-1}}{x^{\beta-k-1}} \cdot \frac{x^{\beta-k-1}}{(x + \rho)^{\beta-k-1}} \\ &\quad \cdot (x + \rho)^{\beta-k-1} I(x, \varepsilon, \beta) \left[\frac{I(0, \varepsilon, \beta)\rho^{2m}}{I(x, \varepsilon, \beta)x^{2m}}\right] \frac{x^{2m}}{\rho^{2m}} \exp(-\gamma x^2/(2\varepsilon)) \\ &\leqslant C(x + \rho)^{\beta-k-1} I(x, \varepsilon, \beta)\left[I(\rho, \varepsilon, \beta)\rho^{2m}/(I(x, \varepsilon, \beta)x^{2m})\right]. \end{aligned} \tag{4.53}$$

If the last term in brackets is shown to be bounded by 1, then the contribution T_1 to the bound on $D_x^{k+1}y(x)$ will be bounded as claimed. For this it suffices to show that

$$F(z) \equiv z^m \int_{z+\varepsilon}^6 i(s, \beta)\, ds \tag{4.54}$$

is increasing for z in $(0, 1]$, to which end we have

$$(4.55)\qquad D_z F(z) = mz^{m-1}\int_{z+\varepsilon}^{6} i(s, \beta)\, ds - z^m i(z+\varepsilon, \beta)$$
$$\geqslant mz^{m-1}(z+\varepsilon) i(2(z+\varepsilon), \beta) - z^m i(z+\varepsilon, \beta),$$

which is larger than zero by the choice of m.

We next turn our attention to the second term, T_2, on the right side of (4.52b). We have

$$(4.56)\qquad T_2 \leqslant C\rho^{-2}\int_0^{\pi\rho}(\xi+\rho)^{\beta-k} I(\xi, \varepsilon, \beta) E(x, \xi)\, d\xi$$
$$+ C_2\int_{\pi\rho}^{x}\left[\xi^{\beta-k-1} I(\xi, \varepsilon, \beta)\right]\left[\gamma\xi\rho^{-2}E(x, \xi)\right] d\xi \equiv T_3 + C_2 T_4.$$

Now since $x \geqslant 2\pi\rho$,

$$(4.57)\qquad T_3 \leqslant \pi\rho C\rho^{-2}\rho^{\beta-k} I(0, \varepsilon, \beta)\exp(-.25\gamma x^2/\varepsilon),$$

and so T_3 can be bounded as desired exactly as was T_1.

To bound T_4, perform integration by parts as delimited by the brackets in (4.56) and find

$$(4.58a)\qquad T_4 \leqslant \{x^{\beta-k-1} I(x, \varepsilon, \beta) - 0\}$$
$$+ \int_{\pi\rho}^{x}\left[-(\beta-k-1)\gamma^{-1}\rho^2\xi^{-2}\right]\rho^{-2}\xi^{\beta-k}\gamma I(\xi, \varepsilon, \beta) E(x, \xi)\, d\xi$$
$$+ \int_{\pi\rho}^{x} 2\xi^{-k-1} E(x, \xi)\, d\xi.$$

From Lemma 2.6, and since here $\xi \geqslant \rho$,

$$(4.58b)\qquad \int_{\pi\rho}^{x} 2\xi^{-k-1}E(x, \xi)\, d\xi \leqslant C\int_{\pi\rho}^{x}\xi^{-k-1}\xi^{\beta-1} I(\xi, \varepsilon, \beta) E(x, \xi)\, d\xi$$
$$= C_3\int_{\pi\rho}^{x}\xi^{\beta-k}\gamma(\rho^2\xi^{-2})\rho^{-2} I(\xi, \varepsilon, \beta) E(x, \xi)\, d\xi.$$

Now choose $\pi > 1$ so that $C_3\rho^2/(\pi\rho)^2 \leqslant 1/3$ and so that

$$|-(\beta-k-1)\gamma^{-1}\rho^2/(\pi\rho)^2| \leqslant 1/3.$$

Then, having selected such a π, (4.58) shows that

$$(4.59)\qquad T_4 \leqslant x^{\beta-k-1} I(x, \varepsilon, \beta) + T_4/3 + T_4/3.$$

Since here $x \geqslant \rho$, (4.59) completes by induction the establishment of (4.49) and the proof of the theorem is finished.

We now complete the establishment of the a priori estimates by proving Theorem 2.8.

Proof of Theorem 2.8. First recall that for $k = 1, \ldots, m$ the differential equation satisfied by $z(x) = D_x^k y(x)$ is given by (2.10) with the dependence of $g(x)$ on the derivatives of y and the data (p, q and f) as described below (2.10). Also note that the value of β associated with (2.10) is given by $[q(0) - kp_x(0)]/p_x(0) = m + \Delta - k \geqslant \Delta$ for $1 \leqslant k \leqslant m$, and so in particular

$$[q(0) - kp_x(0)] \geqslant \Delta p_x(0) \geqslant \Delta k_q/(m+\Delta) \geqslant \beta_l k_q/(m+\beta_s).$$

Thus $[q(x) - kp_x(x)]$ is positive in a neighborhood N of 0 and so using Remark 2.2 to bound $D_x^k y$ at the endpoints of N, one may use the maximum principle to bound $D_x^k y(x)$ for x in N (for $1 \leqslant k \leqslant m$) and thereby obtain (2.15a).

Analogous to the discussion in Remark 2.3, we can just as well assume that for $-1 \leqslant x \leqslant 1$, $[q(x) - mp_x(x)]$ is bounded above a positive constant k_q^* which depends only on $S_5(m)$. Now for $k = m$; $g(x)$ in (2.10) depends on at most mth order derivatives of p, q and f and on $y, y_x, \ldots, D_x^{m-1}y$ and so $|g(x)| + |g_x(x)| \leqslant C$, also the value of β for (2.10) with $k = m$ is Δ, and so Theorem 2.5 applies to (2.10) and yields precisely the bound (2.15b) for $D_x^{m+1}y(x)$, since $\Delta - 1 = \beta - (m+1)$. Now suppose $i \geqslant 2$. We establish the rest of (2.15b) by induction with the aid of Theorem 4.8. Suppose (2.15b) is true for derivatives $m+1$ through j where $m + 1 \leqslant j \leqslant m + i$. Then apply Theorem 4.8 to (2.10) taking $\beta = \Delta$, $K = \max(2, j - m)$ and $k = j + 1 - m$ and obtain the result.

We complete this section by proving the claims concerning the convergence as ε approaches 0 of y to the solution v of the reduced problem.

Proof of Remark 2.10. Let $e(x)$ denote $y(x) - v(x)$. Subtracting the equation satisfied by v from that satisfied by y yields

$$e_x(x) + (-q(x)/p(x))e(x) = -\varepsilon y_{xx}(x)/p(x) \quad \text{for } x \neq 0 \tag{4.60a}$$

$$e(-1) = e(1) = 0. \tag{4.60b}$$

Now the solution of an equation of the form

$$e_x(x) + a(x)e(x) = b(x), \qquad e(x_0) = 0, \tag{4.61a}$$

is given by

$$e(x) = \exp(-A(x)) \int_{x_0}^{x} \exp(A(t))b(t)\,dt \quad \text{where } A(t) = \int_{t_0}^{t} a(s)\,ds \tag{4.61b}$$

and where t_0 in $(-1, 1)$ is arbitrary.

Note that by using the change of variable $\bar{x} = -x$, the inequality (2.17) in the case $-1 \leqslant x < 0$ will follow from the case where $0 < x \leqslant 1$, so we proceed assuming $x \geqslant 0$. We may thus take $x_0 = t_0 = 1$ in (4.61b) to solve (4.60) for $x > 0$, and obtain

$$\begin{aligned} e(x) = {} & \exp\left(\int_x^1 \frac{-q(s)}{p(s)}\,ds\right) \\ & \cdot \int_x^1 \left\{ \frac{-q(t)}{p(t)} \exp\left(\int_t^1 \frac{q(s)}{p(s)}\,ds\right)\right\} \left\{ \frac{\varepsilon y_{xx}(t)}{-q(t)} \right\} dt \quad \text{for } x > 0. \end{aligned} \tag{4.62}$$

Now integrate by parts as indicated by the braces in (4.62) and obtain

$$\begin{aligned} e(x) = {} & \exp\left(\int_x^1 \frac{-q(s)}{p(s)}\,ds\right) \\ & \times \left\{ \left[\exp\left(\int_t^1 \frac{q(s)}{p(s)}\,ds\right) \cdot \frac{\varepsilon y_{xx}(t)}{-q(t)} \right]_x^1 - \int_x^1 \exp\left(\int_t^1 \frac{q(s)}{p(s)}\,ds\right) \right. \\ & \left. \cdot \left(-\varepsilon q(t) D_x^3 y(t) + \varepsilon q_x(t) y_{xx}(t)\right) q^{-2}(t)\,dt \right\}. \end{aligned} \tag{4.63}$$

Noting that $p(x) > 0$ for $x > 0$, (4.63) shows that for $x > 0$

$$(4.64) \qquad |e(x)| \leqslant |\varepsilon y_{xx}(1)/k_q| + |\varepsilon y_{xx}(x)/k_q| + \int_x^1 \varepsilon |D_x^3 y(t)| k_q^{-1}\, dt + \max_{0<t\leqslant 1} |\varepsilon q_x(t) y_{xx}(t) k_q^{-2}|.$$

The inequality (2.17) for $x > 0$ then follows from (4.64) and Theorems 2.7 and 2.8 and the argument in the first part of the proof of Theorem 2.8, while the inequality (2.17) for $x = 0$ follows from the results just mentioned and from the fact that

$$(4.65) \qquad q(0)e(0) = \varepsilon y_{xx}(0).$$

The proof of this can be organized by verifying the result for the cases $0 < \beta < 1$, $\beta = 1$, $1 < \beta < 2$, $\beta = 2$, $2 < \beta < 3$, $\beta = 3$, and $\beta > 3$.

Proof of Remark 2.11. Let $e(x) = y(x) - v(x)$. Then $e(x)$ satisfies

$$(4.66) \qquad L^0 e(x) \equiv -p(x)e_x(x) + q(x)e(x) = \varepsilon y_{xx}(x).$$

By Theorem 2.4 and (2.1) we have that for $|x| \leqslant 1/2$, y, y_x and y_{xx} are all bounded in magnitude by a constant depending only on $S_4(2)$, and so

$$|e(0)| + |\varepsilon y_{xx}(x)| \leqslant C_1\varepsilon \quad \text{for } |x| \leqslant 1/2.$$

We now use a comparison function argument to bound $e(x)$. Define the two functions ψ^+ and ψ^- by $\psi^\pm(x) = \pm e(x) + C_1\varepsilon/k_q + C_1\varepsilon$. Then

$$(4.67a) \qquad L^0\psi^\pm \geqslant \pm\varepsilon y_{xx}(x) + C_1\varepsilon + k_q C_1 \varepsilon > 0 \quad \text{for } |x| \leqslant 1/2,$$

and

$$(4.67b) \qquad \psi^\pm(0) > 0.$$

Using the fact that $q > 0$, and $p(x) < 0$ for $x > 0$ and $p(x) > 0$ for $x < 0$, one can easily check by contradiction that (4.67) implies that $\psi^\pm(x) \geqslant 0$ for $|x| \leqslant 1/2$ proving the result.

V. Numerical Results. In this section we present some numerical experiments which illustrate Theorem 3.5 (particularly (3.17a)) and which suggest that modification of the El-Mistikawy Werle scheme near the turning point to satisfy $|P(x)| \leqslant Cx$ is indeed necessary to prevent loss of accuracy when $\varepsilon \ll h$ and $|x| < h$.

Calculations were done for Eq. (1.1) on the interval $[0, 1]$ instead of $[-1, 1]$ with one turning point located at $z = 1/2$ for which $\alpha = 1/\beta$ was chosen to be either 4, 2, or 4/3. The coefficients $p(x)$ and $q(x)$ were defined by

$$(5.1) \qquad \begin{aligned} p(x) &= \alpha(x - z) + .3121\,\alpha(x - z)^2, \\ q(x) &= 1 + .2764(x - z). \end{aligned}$$

The right-hand side $f(x) \equiv Ly(x)$ and the boundary data $d_1 = y(0)$ and $d_2 = y(1)$ were determined by defining the solution $y(x)$ to be

$$(5.2) \qquad y(x) = \left[.291(x - z)^2 + \varepsilon\right]^{\beta/2} + \left[.291(x - z)^2 + \varepsilon\right]^{(\beta-1)/2}(x - z) + \exp(-.5x^2).$$

The form of the function $y(x)$ in (5.2) was chosen such that its various derivatives have behavior as bad as and no worse than the estimates in Theorem 2.7 for any given β in $(0,1)$. For a given choice of β, uniform meshes with $h = 1/J$, $J = 32, 64, \ldots, 1024$ were used. To obtain a wide variation of relationships between h and ε, the problem was solved with $\varepsilon \equiv h^s$ for various values of s, i.e., the equation solved was

$$-h^s y_{xx}(x) - p(x)y_x(x) + q(x)y(x) = f(x) \quad \text{for } 0 < x < 1, \tag{5.3a}$$

$$y(0) = d_1 \quad \text{and} \quad y(1) = d_2. \tag{5.3b}$$

The calculations presented here were done in single precision on a CDC-6500 (approximately 14 significant digits) except that the decomposition and backsolve of the linear system (3.1b), (3.3), (3.4) was done in double precision (approximately 28 significant digits). A few comparison runs using single precision throughout revealed no substantial changes in the results given here. Table 1 contains results from solving (5.3) with $\beta = 1/4$ using the El-Mistikawy Werle scheme (i.e., $P(x)$ on each interval (x_j, x_{j+1}) equals $(p_j + p_{j+1})/2$ and similarly for Q and F) but with the definition of $P(x)$ modified near the turning point as described immediately after (3.16). Results for particular values of s are given in each column. The l^∞ error defined to be the maximum over $j = 1, \ldots, J-1$ of $|Y_j - y(x_j)|$ is listed under E_∞, and the value of J is given in the first column. The numerical rate of convergence (listed under the heading rate) is determined from the E_∞ values for two successive values of J (e.g., E_∞^1 and E_∞^2 corresponding to $h = 1/J$ and $h = 1/(2J)$, respectively) by

$$\text{rate} \equiv \left(\ln E_\infty^1 - \ln E_\infty^2\right)/\ln(2). \tag{5.4}$$

TABLE 1

Numerical results for the modified El-Mistikawy Werle scheme applied to (5.1)–(5.3), $\beta = 1/4$

	$\varepsilon = 1$	$\varepsilon = h^{.5}$	$\varepsilon = h$	$\varepsilon = h^{1.5}$	$\varepsilon = h^2$	$\varepsilon = h^3$
J	E_∞ Rate	E_∞ Rate	E_∞ Rate	E_∞ Rate	E_∞ Rate	E_∞ Rate
32	6.2E-5	3.6E-4	1.9E-3	1.1E-2	3.2E-2	1.3E-1
	2.22	1.97	1.48	.60	.19	.27
64	1.3E-5	9.3E-5	6.7E-4	7.5E-3	2.8E-2	1.0E-1
	2.14	1.53	1.40	.67	.22	.32
128	3.0E-6	3.2E-5	2.5E-4	4.7E-3	2.4E-2	8.4E-2
	2.08	1.45	1.34	.73	.23	.35
256	7.2E-7	1.2E-5	1.0E-4	2.8E-3	2.0E-2	6.6E-2
	2.04	1.53	1.12	.76	.24	.36
512	1.8E-7	4.1E-6	4.6E-5	1.7E-3	1.7E-2	5.1E-2
	1.99	1.57	1.12	.79	.25	.37
1024	4.4E-8	1.4E-6	2.1E-5	9.7E-4	1.4E-2	4.0E-2

TABLE 2

Numerical results for the modified El-Mistikawy Werle scheme applied to (5.1)–(5.3), $\beta = 1/2$

	$\varepsilon = 1$	$\varepsilon = h^{.5}$	$\varepsilon = h$	$\varepsilon = h^{1.5}$	$\varepsilon = h^2$	$\varepsilon = h^3$
J	E_∞ Rate	E_∞ Rate	E_∞ Rate	E_∞ Rate	E_∞ Rate	E_∞ Rate
32	1.7E-5	1.6E-4	7.0E-4	4.5E-3	1.5E-2	3.5E-2
	2.40	1.96	1.66	.85	.43	.62
64	3.3E-6	4.1E-5	2.2E-4	2.5E-3	1.1E-2	2.3E-2
	2.27	1.96	1.47	.90	.46	.68
128	6.7E-7	1.1E-5	8.0E-5	1.4E-3	7.9E-3	1.4E-2
	2.16	1.88	1.48	.95	.48	.71
256	1.5E-7	2.9E-6	2.9E-5	7.0E-4	5.7E-3	8.6E-3
	2.08	1.51	1.47	.98	.48	.73
512	3.6E-8	1.0E-6	1.0E-5	3.6E-4	4.1E-3	5.2E-3
	1.89	1.58	1.46	1.00	.49	.74
1024	9.6E-9	3.4E-7	3.8E-6	1.8E-4	2.9E-3	3.1E-3

TABLE 3

Numerical results for the modified El-Mistikawy Werle scheme applied to (5.1)–(5.3), $\beta = 3/4$

	$\varepsilon = 1$	$\varepsilon = h^{.5}$	$\varepsilon = h$	$\varepsilon = h^{1.5}$	$\varepsilon = h^2$	$\varepsilon = h^3$
J	E_∞ Rate	E_∞ Rate	E_∞ Rate	E_∞ Rate	E_∞ Rate	E_∞ Rate
32	7.7E-6	8.7E-5	3.2E-4	1.8E-3	5.8E-3	9.4E-3
	2.47	1.94	1.84	1.05	.67	.96
64	1.4E-6	2.3E-5	8.9E-5	8.5E-4	3.6E-3	4.8E-3
	2.27	1.92	1.73	1.10	.70	1.04
128	2.9E-7	6.0E-6	2.7E-5	4.0E-4	2.2E-3	2.4E-3
	1.89	1.92	1.58	1.14	.71	1.07
256	7.8E-8	1.6E-6	8.9E-6	1.8E-4	1.4E-3	1.1E-3
	1.96	1.94	1.56	1.17	.72	1.09
512	2.0E-8	4.1E-7	3.0E-6	8.0E-5	8.3E-4	5.2E-4
	2.22	1.98	1.54	1.19	.73	1.10
1024	4.3E-9	1.1E-7	1.0E-6	3.5E-5	5.0E-4	2.4E-4

TABLE 4

Numerical results for the original El-Mistikawy Werle scheme applied to (5.1)– (5.3), $\beta = 1/4$

	$\varepsilon = 1$	$\varepsilon = h^{.5}$	$\varepsilon = h$	$\varepsilon = h^{1.5}$	$\varepsilon = h^{2}$	$\varepsilon = h^{3}$
J	E_∞ Rate	E_∞ Rate	E_∞ Rate	E_∞ Rate	E_∞ Rate	E_∞ Rate
32	4.4E-5	2.6E-4	1.4E-3	4.9E-3	1.6E-2	1.4E-2
	2.00	1.43	1.18	.66	.30	-.55
64	1.1E-5	9.5E-5	6.1E-4	3.1E-3	1.3E-2	2.1E-2
	2.00	1.46	1.21	.63	.28	-.35
128	2.7E-6	3.5E-5	2.6E-4	2.0E-3	1.1E-2	2.7E-2
	2.00	1.51	1.19	.65	.26	-.19
256	6.8E-7	1.2E-5	1.2E-4	1.3E-3	9.0E-3	3.0E-2
	2.00	1.56	1.18	.67	.25	-.08
512	1.7E-7	4.1E-6	5.1E-5	8.0E-4	7.6E-3	3.2E-2
	1.97	1.59	1.13	.66	.25	-.01
1024	4.4E-8	1.4E-6	2.3E-5	5.1E-4	6.4E-3	3.2E-2

TABLE 5

Numerical results for the original El-Mistikawy Werle scheme applied to (5.1)– (5.3), $\beta = 3/4$

	$\varepsilon = 1$	$\varepsilon = h^{.5}$	$\varepsilon = h$	$\varepsilon = h^{1.5}$	$\varepsilon = h^{2}$	$\varepsilon = h^{3}$
J	E_∞ Rate	E_∞ Rate	E_∞ Rate	E_∞ Rate	E_∞ Rate	E_∞ Rate
32	5.3E-6	5.5E-5	1.0E-4	2.3E-4	4.6E-4	2.6E-3
	2.00	1.66	1.58	1.23	.55	.57
64	1.3E-6	1.7E-5	3.5E-5	9.9E-5	3.1E-4	1.8E-3
	2.00	1.75	1.39	1.27	.55	.55
128	3.3E-7	5.1E-6	1.3E-5	4.1E-5	2.2E-4	1.2E-3
	2.00	1.83	1.46	.91	.66	.27
256	8.4E-8	1.5E-6	4.8E-6	2.2E-5	1.4E-4	1.0E-3
	2.01	1.89	1.50	.90	.71	.43
512	2.1E-8	3.9E-7	1.7E-6	1.2E-5	8.3E-5	7.5E-4
	2.25	1.94	1.52	.99	.73	.53
1024	4.4E-9	1.0E-7	5.9E-7	5.9E-6	5.0E-5	5.2E-4

The corresponding results for $\beta = 1/2$ and $\beta = 3/4$ are displayed in Tables 2 and 3. These results are consistent with (3.17a) and also indicate that the estimate (3.17a) is not sharp unless $\varepsilon = h^2$. The results in Tables 4 and 5 are for the case $\beta = 1/4$ and $\beta = 3/4$ when the original El-Mistikawy Werle scheme ($P(x)$ not modified near the turning point) is used to solve (5.3). Note that when $\varepsilon \geqslant h^2$ the rates for the modified and original El-Mistikawy Werle schemes are similar while the magnitude of the errors is actually in general smaller for the original method. However, when $\varepsilon = h^3$ the results suggest that the rate of convergence of the original scheme deteriorates.

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1. L. R. ABRAHAMSSON, "A priori estimates for solutions of singular perturbations with a turning point," *Stud. Appl. Math.*, v. 56, 1977, pp. 51–69.

2. L. R. ABRAHAMSSON, *Difference Approximations for Singular Perturbations With a Turning Point*, Dept. of Comput. Sci., Uppsala Univ., Rep. 58, 1975.

3. M. ABRAMOWITZ & I. A. STEGUN, *Handbook of Mathematical Functions*, National Bureau of Standards, Washington, D. C., 1964.

4. A. E. BERGER, J. M. SOLOMON & M. CIMENT, "An analysis of a uniformly accurate difference method for a singular perturbation problem," *Math. Comp.*, v. 37, 1981, pp. 79–94.

5. E. CODDINGTON & N. LEVINSON, *Theory of Ordinary Differential Equations*, McGraw-Hill, New York, 1955.

6. T. M. EL - MISTIKAWY & M. J. WERLE, "Numerical method for boundary layers with blowing—The exponential box scheme," *AIAA J.*, v. 16, 1978, pp. 749–751.

7. K. V. EMELJANOV, "A difference scheme for the equation $\varepsilon u'' + xa(x)u' - b(x)u = f(x)$," *Proc. Inst. Math. Mech.*, Ural Science Center Acad. USSR, v. 21, 1976, pp. 5–18. (In Russian.)

8. P. A. FARRELL, *A Uniformly Convergent Difference Scheme for Turning Point Problems*, Proc. Internat. Conf. on Boundary and Interior Layers, Computational and Asymptotic Methods, June 3–6, 1980, Trinity College, Dublin, Ireland (J. J. H. Miller, ed.), Boole Press, Dublin, 1980, pp. 270–274.

9. P. A. FARRELL, *Uniformly Convergent Difference Schemes for Singularly Perturbed Turning and Non-Turning Point Problems*, Doctoral Thesis, Trinity College, Dublin, Ireland, 1982.

10. A. F. HEGARTY, J. J. H. MILLER & E. O'RIORDAN, *Uniform Second Order Difference Schemes for Singular Perturbation Problems*, Proc. Internat. Conf. on Boundary and Interior Layers, Computational and Asymptotic Methods, June 3–6, 1980, Trinity College, Dublin, Ireland (J. J. H. Miller, ed.), Boole Press, Dublin, 1980, pp. 301–305.

11. R. B. KELLOGG, "Difference approximation for a singular perturbation problem with turning points," *Analytical and Numerical Approaches to Asymptotic Problems in Analysis* (O. Axelsson, L. S. Frank and A. van der Sluis, eds.), North-Holland, Amsterdam, 1981, pp. 133–139.

12. R. B. KELLOGG & A. TSAN, "Analysis of some difference approximations for a singular perturbation problem without turning points," *Math. Comp.*, v. 32, 1978, pp. 1025–1039.

13. B. KREISS & H. O. KREISS, "Numerical methods for singular perturbation problems," *SIAM J. Numer. Anal.*, v. 18, 1981, pp. 262–276.

14. W. L. MIRANKER & J. P. MORREEUW, "Semianalytic numerical studies of turning point problems arising in stiff boundary value problems," *Math. Comp.*, v. 28, 1974, pp. 1017–1034.

15. C. E. PEARSON, "On a differential equation of boundary layer type," *J. Math. Phys.*, v. 49, 1968, pp. 134–154.

16. M. H. PROTTER & H. F. WEINBERGER, *Maximum Principles in Differential Equations*, Prentice-Hall, Englewood Cliffs, N. J., 1967.

17. S. A. PRUESS, "Solving linear boundary value problems by approximating the coefficients," *Math. Comp.*, v. 27, 1973, pp. 551–561.

18. M. E. ROSE, "Weak-element approximations to elliptic differential equations," *Numer. Math.*, v. 24, 1975, pp. 185–204.

19. I. STAKGOLD, *Green's Functions and Boundary Value Problems*, Wiley, New York, 1979.

20. E. T. WHITTAKER & G. N. WATSON, *A Course of Modern Analysis*, MacMillan, New York, 1947.

21. P. A. FARRELL, *Sufficient Conditions for the Uniform Convergence of Difference Schemes for Singularly Perturbed Turning and Non-Turning Point Problems*, Proc. Second Internat. Conf. on Boundary and Interior Layers—Computational and Asymptotic Methods, June 16–18, 1982, Dublin, Ireland (J. J. H. Miller, ed.), Boole Press, Dublin, 1982, pp. 230–235.

22. R. E. O'MALLEY, JR., "On boundary value problems for a singularly perturbed differential equation with a turning point", *SIAM J. Math. Anal.*, v. 1, 1970, pp. 479–490.