Computing an Arithmetic Constant Related to the Ring of Gaussian Integers

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Abstract. We compute the analogue for $\mathbf{Z}[i]$ of Euler's constant, that is $\delta = \lim_{n \rightarrow +\infty} \delta_n$, where $\delta_n = (\sum_{2 \leq k \leq n} 1/\pi r_k^2) - \log n$. For this purpose we give an estimate for

$$r_k = \min \{ r \geq 0; \text{there exists } z \in \mathbf{C} \text{ such that } \text{card}(\mathbf{Z}[i] \cap \overline{D}(z, r)) \geq k \},$$

and we compute a great number of values of r_k .

0. Results and Notations. Let $\mathbf{Z}[i] = \{a + ib \in \mathbf{C}; a, b \in \mathbf{Z}\}$ be the ring of Gaussian integers. For each integer $k \geq 2$ set

$$r_k = \min \{ r > 0; \text{there exists } z \in \mathbf{C} \text{ such that } \text{card}(\mathbf{Z}[i] \cap \overline{D}(z, r)) \geq k \},$$

where $\overline{D}(z, r) = \{w \in \mathbf{C}; |w - z| \leq r\}$ is the closed disk with center z and radius r .

Clearly, this minimum is attained and a closed disk with radius r_k containing at least k integer (i.e., Gaussian integer) points will be called a minimal disk D_k and its boundary Γ_k will be called a minimal circle. It is not known whether all the minimal disks with index k are deduced from one another by an isometry of $\mathbf{R}^2 \simeq \mathbf{C}$ which stabilizes the lattice $\mathbf{Z}[i]$. Nevertheless, Sections 1 and 2 deal with some properties of these disks. In particular, we prove in Section 2 an estimate of r_k from which we deduce the existence of $\delta = \lim_{n \rightarrow +\infty} \delta_n$, where

$$\delta_n = \left(\sum_{2 \leq k \leq n} \frac{1}{\pi r_k^2} \right) - \log n.$$

This constant δ has been introduced by D. W. Masser (cf. [7] and [5]) in the study of entire functions which map $\mathbf{Z}[i]$ into $\mathbf{Z}[i]$. It is a bidimensional analogue of Euler's constant; it appears in a formula of the type of Stirling's formula and is related to the ring of Gaussian integers (see [7, Lemme 2] or [5, Proposition 3]) which leads to the conjecture that $\delta = 1.822825 \dots$ (cf. remark following Theorem 1 of [5]).

As no formula for r_k is known, it has been necessary, in order to evaluate δ , to compute a great number of values of r_k with only the use of the above definition of r_k .

For this purpose, we have used a personal computer with a 6502 microprocessor (8 bits, $1\mu s$) and 8 Kbytes memory. We have first developed various models in BASIC language which have enabled us to establish a table of the minimal disks D_k for $k \leq 300$ (see Appendix 3), to compute fairly good approximations of δ_{1000} (1.80849), and then of δ_{1400} (1.81105: compare with the final result below!) in order to define

Received December 13, 1982.

1980 *Mathematics Subject Classification*. Primary 10E05, 10-04, 40-04.

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0025-5718/85 \$1.00 + \$.25 per page

the optimal specifications of the final program. (All appendices are in the supplements section at the end of this issue.)

Indeed, as the BASIC interpreter used floating-point rounded values with little known errors, it was impossible to guarantee more than 4 decimal digits for the final result. Then, as all computed values were rational numbers (Propositions 1 and 2) with easily bounded sizes, we could develop a program in 6502 code working only with positive fixed-point values and multiprecision routines. With this method the accuracy can be as good as needed and the error computation is rigorous.

After approximately 175 hours of computation we obtained the following results

$$1.811\,090\,114 < \delta_{1400} < 1.811\,090\,234$$

and

$$21.055\,212\,94 < r_{1400} < 21.055\,214\,02$$

which, considering the estimates of Section 2 give

$$1.811\,447\,299 < \delta < 1.897\,327\,117$$

and the constant λ_0 of [7] therefore satisfies

$$0.170\,7339 < \lambda_0 < 0.186\,0446.$$

In particular, we have $\lambda_0 > 1/6$ (conjecture of [7]), but even if we notice that the lower bound for δ is better than its upper bound, we cannot yet assert that the above conjecture about the value of δ (which would lead to $\lambda_0 = 1/2e = 0.183\,939\dots$) is reasonable. Indeed, the estimates of Section 2 allow us to see that, to obtain δ with an accuracy of 10^{-n} (for $n \geq 4$), it is necessary to compute δ_k for k near 50×10^{3n} . So it seems impossible, even with big recent computers, to obtain the first 4 digits of δ , unless the algorithm of calculation of r_k or the estimate of r_k are improved, or another way of computing δ is found.

1. Some Properties of Minimal Disks. In this section, two simple geometrical properties of minimal disks are given. With the notations of Section 0 we have

PROPOSITION 1. *Let Γ_k , with $k \geq 3$, be a minimal circle. If Γ_k does not pass through (at least) 3 integer points, then Γ_k has one diameter with integer extremities. This diameter is parallel to the real or imaginary axis and its length is an odd integer each of whose prime factors is congruent to 3 modulo 4.*

Such a minimal circle will be called exceptional. One can verify that, for $3 \leq k \leq 1500$, there is no exceptional disk. For this purpose it is sufficient to compute the number k of integer points belonging to the closed disk of center $1/2$ and of diameter $2r$, where $2r$ is an odd integer whose prime factors are all congruent to 3 (mod 4) and satisfying $2r < 44$ (according to Proposition 3). If one has $r < \sqrt{(k-1)/\pi}$ one compares r with $r'_k = \min_{l \geq k} r'_l$, where

$$r'_k = \min \{ r > 0; \text{there exists } z \in \mathbb{C} \text{ such that } \text{card}(\overline{D}(z, r) \cap \mathbb{Z}[i]) = k \\ \text{and } \text{card}(\partial D(z, r) \cap \mathbb{Z}[i]) \geq 3 \}$$

is given by the BASIC program of Appendix 1 or the final program. This work is given in full detail in Appendix 2.

However, there exist minimal disks Γ_k with $k \geq 3$, whose one diameter is the segment parallel to the real axis connecting 2 integer points, and with an odd length (for instance Γ_{22} , with radius $r_{22} = 5/2$). But in each known case ($k \leq 1500$), Γ_k passes through at least 3 (therefore 6 by symmetry) integer points. Thus, it is natural to state the following:

CONJECTURE. *For $k \geq 3$, any minimal circle Γ_k passes through at least 3 integer points.*

Proof of Proposition 1. That a minimal circle passes through at least 2 integer points is obvious.

Let Γ be a circle passing through exactly 2 integer points a and b not on the same diameter. The radius r of Γ satisfies $2r > |a - b|$ and if ω denotes the center of Γ one has

$$\max\{|\omega - \zeta|; \zeta \in \mathbf{Z}[i] \cap \bar{D}(\omega, r), \zeta \neq a, \zeta \neq b\} = r - \varepsilon < r.$$

Let ω' be a point inside the triangle ωab lying on the perpendicular from ω to $[ab]$, and such that $|\omega - \omega'| \leq \varepsilon/2$. The triangle inequality shows that the closed disk with center ω' and radius $|\omega' - a| < r$ contains the same integer points as $\bar{D}(\omega, r)$, so Γ is not minimal.

Let us now study the case where Γ has a diameter with integer extremities, for instance a and b . (Note that the circle Γ is not assumed to be minimal.) If $b - a \notin \mathbf{R} \cup i\mathbf{R}$, then the center $\omega = (a + b)/2$ of Γ belongs to $\frac{1}{2}\mathbf{Z}[i]$, thus, by symmetry with respect to the line $\omega + i\mathbf{R}$ or $\omega + \mathbf{R}$, the circle Γ passes through at least 4 integer points. If $b - a \in 2\mathbf{Z} \cup 2i\mathbf{Z}$, then $\omega \in \mathbf{Z}[i]$ and it is easily seen (using a rotation of center ω and angle $\pi/2$) that Γ passes through at least 4 integer points.

In the only remaining case, we can suppose that Γ has center $1/2$ and radius $m + 1/2$ with m an integer ≥ 1 . Then, the only integer points of Γ are $-m$ and $m + 1$ if and only if $(2m + 1)^2$ is not a sum of 2 squares, i.e., (see [6, Theorem 278]) if the prime factors of $2m + 1$ are all congruent to 3 (mod 4). Q.E.D.

It is also possible to get more information on the position of the integer points of a minimal circle.

PROPOSITION. 2. *If a minimal circle passes through (at least) 3 integer points, then it is circumscribed to some triangle with integer vertices and angles $\leq \pi/2$.*

Proof. Let D be a closed disk with boundary Γ passing through at least 3 integer points and set $E = \Gamma \cap \mathbf{Z}[i]$. Suppose that each triplet of points in E forms a triangle with one angle $> \pi/2$. Then the convex hull of E does not contain the center ω of Γ , for otherwise, by Carathéodory's theorem (cf. [1, Vol. 3, 11.1.8.6]), the center ω would be contained in the convex hull of 3 points of E , which would thus form a triangle with angles $\leq \pi/2$.

Let $[a, b]$ be the chord of Γ with extremities in E and maximal length: Every point in E is on the other side of $[a, b]$ with respect to ω . Then, it is clear that any disk translated from D by a vector orthogonal to $[a, b]$ directed from ω to $[a, b]$, and sufficiently small, will contain all the integer points of D in its interior; thus Γ is not minimal. Q.E.D.

2. Estimate of r_k and Existence of δ . The values of r_k which we computed show that the following upper bound is rather good.

PROPOSITION 3. *For each integer $k \geq 2$ one has $r_k < \sqrt{(k-1)/\pi}$.*

Proof. By a classical result ([8, no. 452, p. 151] or [5, Proposition 1]), there exists a closed disk of radius $((k-1)/\pi)^{1/2}$ containing at least k integer points. We thus have $r_k^2 \leq (k-1)/\pi$. But it follows from Propositions 1 and 2 that $r_k^2 \in \mathbf{Q}$ and, as π is irrational, the inequality is strict. Q.E.D.

The following lower bound for r_k is a refinement of classical calculations (see, for instance, [4]).

PROPOSITION 4. *Let k be the number of integer points lying in a closed disk of radius $r \geq 2$. Then*

$$\begin{aligned} k < \pi r^2 + 2\left(\sqrt{2} + \frac{4}{\sqrt{5}} - 2\right)r + 27 - \frac{8\sqrt{2}}{3\sqrt{5}} - \frac{10\sqrt{5}}{3} + \frac{1}{r}\left(\frac{2\sqrt{2} + 20\sqrt{5}}{3}\right) \\ & - \frac{1}{r^2}\left(\left(\frac{6}{5}\left(\frac{7}{8}\right)^{5/2} - 1\right)\left(\frac{2\sqrt{2}}{\sqrt{5}} - 1\right) + \frac{5}{4}\left(6\left(\frac{11}{16}\right)^{5/2} - 5\right)(\sqrt{5} - 2)\right) \\ & + \frac{1}{r^3}\left(\frac{1}{\sqrt{2}}\left(\frac{6}{5}\left(\frac{7}{8}\right)^{5/2} - 1\right) + \frac{5\sqrt{5}}{2}\left(6\left(\frac{11}{16}\right)^{5/2} - 5\right)\right). \end{aligned}$$

For $r \geq 21$, one can replace on the right-hand side $(7/8)^{5/2}$ by $(881/882)^{5/2}$ and $(11/16)^{5/2}$ by $(1759/1764)^{5/2}$.

Remark. This obviously gives a lower bound for r_k . If the above result is written in the form $k < \pi r^2 + 2ar + f(r)$, it is easily seen that f is a decreasing function of r and we can use this inequality in the following manner: For $k \geq k_0$ the radius r_k is greater than the positive root of the trinomial $\pi X^2 + 2aX + f(\rho) - k$, where ρ satisfies $0 < \rho \leq r_{k_0}$. This lower bound is much less precise than the upper bound of Proposition 3 (as a matter of fact it is efficient for any disk, either minimal or not). Nevertheless, if k is not too big ($k \leq 1.364 \cdot 10^7$) this lower bound is better than the best known results taken from the work of H. Chaix [2] on convex compact sets.

In order to obtain Proposition 4 the following calculation will be used:

LEMMA. *The area T_1 of the convex curvilinear right-angled and isosceles triangle with sides 1 and hypotenuse of radius of curvature r is*

$$T_1 = \frac{1}{2} - \frac{r}{\sqrt{2}}\left(1 - \frac{1}{2r^2}\right)^{1/2} + \frac{r^2}{2}\text{Arcsin}\frac{\sqrt{2}(1 - 1/(2r^2))^{1/2}}{r}.$$

The area T_2 of the convex curvilinear right-angled triangle with sides 1 and 2 and hypotenuse of radius of curvature r is

$$T_2 = 1 - \frac{r\sqrt{5}}{2}\left(1 - \frac{5}{4r^2}\right)^{1/2} + \frac{r^2}{2}\text{Arcsin}\frac{\sqrt{5}(1 - 5/(4r^2))^{1/2}}{r}.$$

Proof. This is a straightforward computation of integrals.

Proof of Proposition 4. The closed disk D with center $x + iy$ and radius r is the disjoint union of $\{x + iy\}$ and of the 4 quarter disks

$$Q_1 = \{z = \xi + i\eta \in D; \xi > x, \eta \geq y\}, \quad Q_2 = iQ_1, \quad Q_3 = -Q_1 \quad \text{and} \quad Q_4 = -iQ_1.$$

FIGURE 1

FIGURE 2

For each integer point belonging to Q_1 (resp. Q_2 , resp. Q_3 , resp. Q_4), construct the square with unit sides parallel to the real and imaginary axes with vertex equal to the above-mentioned integer point, and lying down left (resp. down right, resp. up right, resp. up left) as seen in Figure 1.

Let A_j be the area of the union of the squares generated by the integer points of Q_j and let k be the number of integer points belonging to D . One has $k = \varepsilon + \sum_{1 \leq j \leq 4} A_j$, where $\varepsilon = 1$ if $x + iy \in \mathbb{Z}[i]$ and $\varepsilon = 0$ otherwise.

The area A_j is less than the sum of the area of Q_j and the area of the union of the parts of the squares lying outside the radii bounding Q_j minus the area of the union of curvilinear triangles whose vertices are the integer points nearest to the quarter circle bounding Q_j , whose rectilinear sides are some sides of squares generated by the integer points of Q_j , and whose hypotenuses are circular arcs of radius r . These triangles are indeed contained in D since its radius is r , and they have been hatched in Figure 2 relating to Q_1 .

If we study separately the case when $x + iy \in \mathbb{Z}[i]$ and its contrary, we get $k < \pi r^2 + 4r + 1 - T$, where T is the area of the union of the above triangles.

In order to bound T from below, notice that the point of the first quadrant lying on the circle with center 0 and radius r such that the tangent to the circle at this point has gradient -2 has abscissa $r \cos(\text{Arctg } \frac{1}{2}) = 2r/\sqrt{5}$. Thus if the points $x + iy$ and $x + 1 + iy$ of the first quadrant belong to the closed disk $\overline{D}(0, r)$ and if $x \geq 2r/\sqrt{5}$, then the point $x + i(y + 2)$ also belongs to the disk. Similarly, if the points $x + iy$ and $x + 1 + iy$ of the first quadrant belong to the disk $\overline{D}(0, r)$ and if $x \geq r/\sqrt{2}$, then the point $x + i(y + 1)$ also belongs to the disk.

FIGURE 3

Consider again the disk of center $x + iy$ and radius r , and more precisely its first quarter Q_1 (for the other quarters make a similar calculation). Any segment with integer extremities and length 1 contained in the interval $[x + 2r/\sqrt{5}, x + r]$ generates a triangle with area T_2 (notation of the above lemma) contained in D . Similarly, any segment with integer extremities and length 1 contained in the interval $[x + r/\sqrt{2}, x + 2r/\sqrt{5}]$ generates a triangle with area T_1 contained in D . These triangles are hatched in Figure 3 and the area of their union is greater than $(r(1 - 2/\sqrt{5}) - 2)T_2 + (r(2/\sqrt{5} - 1/\sqrt{2}) - 2)T_1$.

The same calculation can be applied to the second half-quarter, exchanging x and y , but by Figure 3 it is easily seen that a few triangles in the neighborhood of the cuts $x + r/\sqrt{2}$ and $x + 2r/\sqrt{5}$ have been left aside. As one bounds k from above by an increasing function of r , one can suppose without loss of generality that D is minimal and thus that r^2 , x and y are rational numbers.

If $x + 2r/\sqrt{5} \in \mathbf{Z}$, then $x + r/\sqrt{2} \notin \mathbf{Z}$, so that a triangle of area T_1 and a triangle of area T_2 can be added on either side of $x + 2r/\sqrt{5}$. Similarly, if $x + r/\sqrt{2} \in \mathbf{Z}$, a triangle of area T_1 can be added in the neighborhood of $x + r/\sqrt{2}$ and another triangle of area T_1 near $x + 2r/\sqrt{5}$. If $x + 2r/\sqrt{5} \notin \mathbf{Z}$ and $x + r/\sqrt{2} \notin \mathbf{Z}$, then a triangle of area T_1 can be added in the neighborhood of $x + 2r/\sqrt{5}$. Moreover, in the neighborhood of $x + r/\sqrt{2}$, a triangle with area at least T_1 can be added, but one must not add more than $T_1/2$ because it can have a vertex of ordinate greater than $y + r/\sqrt{2}$, and therefore it may have been taken into account in the computation related to the second half-quarter. This allows us to add $8 \times \frac{3}{2}T_1$ to the lower bound for T , which becomes

$$T \geq 8(r(1 - 2/\sqrt{5}) - 2)T_2 + 8(r(2/\sqrt{5} - 1/\sqrt{2}) - 1/2)T_1.$$

Then finite expansions for T_1 and T_2 deduced from the lemma give the announced inequalities. Q.E.D.

If r is large enough, this upper bound for k can be improved by dividing into 3, 4, ... parts each half- Q_j eliciting triangles with areas $T_3 \approx 3/2$, $T_4 \approx 2, \dots$, but, for r near 20, the present cutting gives the best result.

Using the estimate for r_k given by Propositions 3 and 4 we get the following

THEOREM. *The sequence $\delta_n = (\sum_{2 \leq k \leq n} 1/\pi r_k^2) - \log n$ is increasing and has a finite limit δ as n tends to infinity.*

Proof. From $r_{n+1}^2 < n/\pi$ (Proposition 3) we deduce, for $n \geq 2$,

$$\delta_{n+1} - \delta_n > \frac{1}{n} + \log n - \log(n+1) > 0,$$

so δ_n is an increasing sequence. Proposition 4 and Euler-Maclaurin's formula (see, for instance, [3, Chapter IX, Section 7]) give an upper bound for δ_n . Further details are left to the reader. Q.E.D.

Actually, it is easy to obtain an estimate for $\delta - \delta_n$. Indeed, by Proposition 3, one has

$$\delta > \delta_n + \lim_{N \rightarrow +\infty} \left(\log \frac{n}{N+1} + \sum_{n \leq k \leq N} \frac{1}{k} \right)$$

and Euler-Maclaurin's formula enables one to estimate this limit. Similarly, one gets an upper bound for δ using the lower bound for r_k given by Proposition 4 for $1401 \leq k \leq 1.364 \times 10^7$, and then the lower bound $k < \pi r^2 + 30.842\,747\,23 \times r^{2/3}$ due to H. Chaix [2] for $k > 1.364 \times 10^7$. By this process, we get the estimate of δ given in Section 0 from the computed value of δ_{1400} and we can see that $\delta - \delta_n = O(n^{-2/3})$.

3. The Computation of δ_N .

(a) *General Principles of the Programs.* Our purpose is to compute r_n for $n \leq N$ (see the definitions of r'_n and r''_n following Proposition 1 in Section 1). We first compute r'_n and to this end we consider the circumcircles of triangles with integer vertices and radii $\leq \sqrt{(N-1)/\pi}$ (in (b) we will explain how to restrict the number of triangles to be considered): We compute the number of integer points contained in the associated disk.

For a given number of points $n < N$, the minimal radius of the studied disks containing exactly n integer points is memorized, and for $n = N$ one memorizes the smallest radius giving at least N integer points. When all the triangles have been studied, for a given n , there are 3 possibilities:

(α) Memory r'_n is empty (or strictly speaking contains 0).

(β) Memory r'_n contains a nonzero value. Then, either

(β_1) memories r'_{n+t} ($t \geq 1$) contain 0 or values $\geq r'_n$, or

(β_2) there exists a memory r'_{n+t} ($t \geq 1$) containing a nonzero value $< r'_n$.

Indeed, it is possible that none of the tested disks contains exactly n integer points (case (α) arising for $n = 3$, for example); it is also possible that each disk containing exactly n integer points has a radius greater than the radius of some disks containing $n + t$ integer points (case (β_2)). For instance, the circumcircle of the triangle of vertices $-1, 1$ and $3i$ induces a disk containing $n = 10$ integer points and with radius $5/3$, whereas the disk centered at $(1+i)/2$ and with radius $\sqrt{5/2} < 5/3$ contains 12 integer points).

It is thus necessary to alter the table of r'_n in order to get the increasing sequence r''_n . For this purpose, we let n run from $N - 1$ to 2, and we store in the memory of index n the value r''_{n+1} contained in the memory of index $n + 1$ in case the memory of index n contains 0 or a value greater than r''_{n+1} . Then it is sufficient to verify that there is no exceptional disk (cf. Section 1 and Appendix 2) and to compute δ_N .

(b) *Selection of the Circles to be Considered.* Our purpose is to reduce the time of computation by avoiding, as far as possible, repeating the computations for triangles deduced from one another by the isometries of $\mathbf{R}^2$ which stabilize the lattice of Gaussian integers.

The circle considered here passes through the origin 0 and the 2 points b and $c \in \mathbf{Z}[i]$ which can be picked up from different quadrants (the method used to construct the table in Appendix 3), or from the same quadrant. (This is the case of the model in Appendix 2 and of the final program which are described below).

It is easily verified that, as the triangle $0bc$ has angles $\leq \pi/2$, one can suppose b and c to be in the first quadrant. Moreover, $r_N < \sqrt{(N-1)/\pi}$; thus, one can assume that $|b| \leq |c| \leq L = 2\sqrt{N/\pi}$, and, by symmetry with respect to the first bisector of the axes, one can suppose that $\arg b \in [\pi/4, \pi/2]$.

The range of $b = x_b + iy_b$ is thus defined by

$$\{(x_b, y_b) \in \mathbf{Z}^2; 0 \leq x_b \leq L/\sqrt{2}, \max(1, x_b) \leq y_b \leq \sqrt{L^2 - x_b^2}\}.$$

For fixed b , one has $|c| \geq |b|$ and, since the angles of the triangle $0bc$ are $< \pi/2$, point c is on the same side as 0 with respect to the perpendicular from b to $[0, b]$ whose equation is

$$x_b X + y_b Y = |b|^2,$$

and which intersects the real axis at the point $|b|^2/x_b$.

So, for fixed b , the range of $c = x_c + iy_c$ is defined by

$$\left\{ (x_c, y_c) \in \mathbf{Z}^2; 0 \leq x_c \leq \min(L, |b|^2/x_b), \min\left(\sqrt{L^2 - x_c^2}, \frac{|b|^2 - x_b x_c}{y_b}\right) \geq y_c, \right. \\ \left. y_c \geq \begin{cases} 0 & \text{if } x_c > |b| \\ \sqrt{|b|^2 - x_c^2} & \text{otherwise} \end{cases} \right\},$$

and $b \neq c$ (or equivalently $x_b y_c - x_c y_b \neq 0$).

It is easily seen that the circumcircle of the triangle $0bc$ has center $x + iy$ and radius r with $x = N(x)/D$, $y = N(y)/D$ and $r^2 = N(r^2)/D^2$, where $N(x) = y_c |b|^2 - y_b |c|^2$, $N(y) = x_b |c|^2 - x_c |b|^2$, $N(r^2) = |b|^2 |c|^2 |b - c|^2$ and $D = 2(x_b y_c - x_c y_b)$.

One verifies without trouble that r^2 is a rational number with a denominator not greater than $D^2/2$.

(c) *Counting the Integer Points in a Disk.*

(α) *Method used in the BASIC models.* In the closed disk of center $x + iy$ and radius r integer points belong to the straight lines parallel to the imaginary axis and with integer abscissa $k \in [x - r, x + r]$. Therefore they amount to

$$\sum_{\substack{x-r \leq k \leq x+r \\ k \in \mathbf{Z}}} \left(\left[y + \sqrt{r^2 - (x - k)^2} \right] - \left[y - \sqrt{r^2 - (x - k)^2} \right] \right),$$

where $[\cdot]$ denotes the integral part.

With respect to BASIC rounded values, a parameter ε (EP in Appendix 1) has been introduced in order not to leave out any integer point in this counting, by actually considering the disk of center $x + iy$ and radius $\sqrt{r^2 + \varepsilon}$. The computation is valid if ε is sufficiently small, and one chooses its values noticing that, from the denominators of x , y and r^2 , if the point $\alpha \in \mathbf{Z}[i]$ does not belong to the circle of center $x + iy$ and radius r , then one has $||\alpha - (x + iy)| - r| \geq (2L^2(L + 1))^{-1}$.

Nevertheless, the BASIC computation is not strictly justified (except for N sufficiently small, in practice $N < 500$). This is the reason why the final program and all needed arithmetical multiprecision routines have been developed directly in 6502 machine code.

(β) *Method used in the final program.* One computes the number of points in $D\mathbf{Z}[i] - D(x + iy)$ contained in the disk of center 0 and radius Dr (which is $\text{card}(\mathbf{Z}[i] \cap \overline{D}(x + iy, r))$).

Using an Euclidean division routine (64 bits dividend, 32 bits divisor) we compute the representative X_1 (resp. Y_1) of $N(x)$ (resp. $N(y)$) modulo D belonging to $[0, D[$, and the desired number of points is

$$\sum_X \left(\left\lfloor \frac{Y_1 + \sqrt{N(r^2) - X^2}}{D} \right\rfloor + \left\lfloor \frac{D - Y_1 + \sqrt{N(r^2) - X^2}}{D} \right\rfloor \right),$$

where the sum is taken over all X in

$$\begin{aligned} & \{ X \in \mathbf{Z}; X \equiv X_1 \pmod{D}, X_1 \leq X \leq N(r^2) \} \\ & \cup \{ X \in \mathbf{Z}; X \equiv D - X_1 \pmod{D}, D - X_1 \leq X \leq N(r^2) \}. \end{aligned}$$

The routine used for the square root gives the integral part of the exact square root, thus we obtain an accurate result.

(d) *Values of r_k^{-2} Memorized.*

(α) *BASIC models.* The computation of $r_k^{-2} = D^2/N(r^2)$ has been achieved with a 32 bit floating-point routine, but one does not know the sign of the error.

(β) *Final program.* It actually computes the integral part of $2^{32}r_k^{-2} = 2^{32}D^2/N(r^2)$ using an Euclidean division routine (64 bit dividend, 32 bit divisor and quotient). Thus, we get an approximation of r_k^{-2} from below with an accuracy of 2^{-32} .

(e) *Summation of r_k^{-2} .*

(α) *BASIC models.* In view of the size of the memory the BASIC model of Appendix 1 has been used to compute $\sum_{2 \leq k \leq 1000} r_k^{-2}$. A second similar program was used to compute and to add up the r_k^{-2} for $1001 \leq k \leq 1400$.

(β) *Final program.* For the same reasons mentioned above, the final program actually computes $\Delta_{N_1, N_2} = \sum_{N_1 \leq k \leq N_2} 2^{32}r_k^{-2}$ (with $N_1 \geq 6$ so that $2^{32}r_k^{-2} < 2^{32}$). The work has been achieved in 3 steps giving

$$\Delta_{6,500} \approx 69\,500\,353\,773, \quad \Delta_{501,1000} \approx 9\,453\,770\,216 \quad \text{and}$$

$$\Delta_{1001,1400} \approx 4\,574\,912\,626.$$

As $\Delta_{2,5} = 9 \cdot 2^{32}$ one easily gets an approximation of δ_{1400} with an accuracy near $1395 \pi^{-1} 2^{-32}$.

Acknowledgment. The authors are grateful to Isabelle Plaud-Proietti for her help in translating the original manuscript into English.

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1. M. BERGER, *Géométrie*, CEDIC, Paris; Nathan Information, Paris, 1978.
2. H. CHAIX, "Sur les points entiers voisins d'une courbe plane convexe-application," *Journées de Théorie Analytique et Élémentaire des Nombres*, Reims, 1981 and private communication.
3. J. DIEUDONNÉ, *Calcul Infinitésimal*, Hermann, Paris, 1968.
4. A. O. GEL'FOND, "Sur les propriétés arithmétiques des fonctions entières," *Tôhoku Math. J.*, v. 30, 1929, pp. 280–285.
5. F. GRAMAIN, "Sur le théorème de Fukasawa-Gel'fond-Gruman-Masser," *Séminaire Delange-Pisot-Poitou (Théorie des Nombres)*, 1980-1981, Birkhäuser, Basel, 1982.
6. G. H. HARDY & E. M. WRIGHT, *An Introduction to the Theory of Numbers*, Clarendon Press, Oxford, 1971.
7. D. W. MASSER, "Sur les fonctions entières à valeurs entières," *C. R. Acad. Sci. Paris Sér. A-B*, v. 291, 1980, pp. 1–4.
8. G. PÓLYA & G. SZEGÖ, *Problems and Theorems in Analysis*, Vol. II, Springer-Verlag, Berlin and New York, 1976.

Supplement to Computing an Arithmetic Constant Related to the Ring of Gaussian Integers

By F. Gramain and M. Weber

Appendix 1 : BASIC program for §1000

```

10 NP = 0 : YC = 0 : NX = 0 : R2 = 0 : DN = 0 : C = 0 : F2 = 0 : W2 = 0 :
  EP = .000001 : K = 0 : X = 0 : Y = 0 : W1 = 0
20 S = 0 : G = 0 : IR = 0 : L2 = 0 : L = 0 : U = 1 : B2 = 0 : B = 0 : RM = 0 :
  M1 = 0 : HI = 0 : M2 = 0 : LX = 0 : V2 = 0
30 XB = 0 : C2 = 0 : D = 2 : YX = 0 : N = 1000 : YB = 0 : BX = 0 : L1 = 0 :
  Z = 0 : P = 1.0001
40 DIM RR(N) : RR(D) = 4 : L = DSQR(N/W) : L2 = L*L : L1 = L2/4
100 FOR XB = Z TO L/SQR(D) : B2 = XB*XB : LX = XB : IF XB = Z THEN LX = U
110 FOR YB = LX TO SQR(L2-B2) : B = B2+YB*YB : PRINT TI$ ; XB, YB
120 FOR XC = Z TO L : BX = YB*XC : C2 = XC*XC : YX = YB*XC : M1 = Z :
  M2 = (B-C2)/P : IF M2 > Z THEN M1 = SQR(M2)
130 RM = Z : HI = P*(B-XB*XC)/YB : IF HI < Z THEN NEXT YB : NEXT : GO TO 260
140 RM = SQR(L2-C2) : IF RM > HI THEN RM = HI
150 FOR YC = INT(RM) TO M1 STEP -U : C = C2+YC*YC
160 DN = D*(YX-XB*YC) : IF DN = Z THEN NEXT : NEXT : GO TO 260
170 NX = YB*C-YC*B : NY = BX-XB*C : S = NX*NX+NY*NY : G = DN/DN : R2 = S/G
180 IF R2 > L1 THEN NEXT : NEXT : NEXT : NEXT : GO TO 260
190 Y = NY/DN : X = NX/DN : NP = Z : F1 = SQR(R2+EP) : W1 = W-F1
200 IF W1 < INT(W1) THEN W1 = INT(W1+U)
210 FOR K = W1 TO X+F1 : F2 = X-K : F2 = SQR(R2+EP-F2*F2) : W2 = Y-F2
220 NP = NP+INT(Y+F2)-INT(W2) : NEXT
230 IF NP > N THEN NP = N
240 IR = G/S : IF RR(NP) > IR THEN NEXT : NEXT : NEXT : NEXT : GO TO 260
250 RR(NP) = IR : NEXT : NEXT : NEXT : NEXT
260 OPEN 1 : FOR K = N TO U STEP -U : PRINT = 1, RR(K) : NEXT : CLOSE 1 : END

```

This first program computes the r_k^{-2} and stores their values in the table $RR(K)$, then in the file # 1. The latter is exploited by the program below which computes the r_k^{-2} (the conjecture in §.1 is that $r_k'' = r_k$ for $k \geq 3$) and then computes δ_n for $n = SM$.

```

10 K = 0 : N = 0 : PT = 0 : TT = 0 : L = 0 : MX = 0 : SM = 0 : ES$ = ""
20 INPUT "TAB LENGTH" : L : DIM T(L) : OPEN 1 : FOR K = L TO 2 STEP -1 :
  INPUT # 1, T(K) : NEXT : CLOSE 1
30 PT = 0 : K = L
40 IF T(K) = 0 THEN K = K-1 : GO TO 40
50 MX = K : PRINT "MAX. TAB = " ; MX : FOR K = MX TO 2 STEP -1 : IF T(K) < PT
  THEN T(K) = PT : NEXT : GO TO 70
60 PT = T(K) : NEXT
70 INPUT "TOTAL FROM 2 TO" : SM : IF SM > MX THEN 70
80 TT = 0 : FOR K = SM TO 2 STEP -1 : TT = TT+T(K) : NEXT
90 PRINT SM : TT/W : TT/W - LOG(SM) : STOP : GO TO 50

```

Appendix 2 : Prospecting exceptional disks.

The following table gives for integers of the form $2r = 2m+1$ ($1 \leq m \leq 21$) having only prime factors congruent to 3 (mod 4) the number k of integer points belonging to the closed disk of center $1/2$ and radius r ; it shows when $r \leq \sqrt{(k-1)/\pi}$ and gives r_k^* in this case. One checks that $r > r_k^*$ therefore, for $k \leq 1500$, there exists no exceptional disk.

$2r$	k	$r \leq \sqrt{\frac{k-1}{\pi}}$	r_k^*
3	8	N O	
7	40	Y E S	3.424 13...
$9=3^2$	62	N O	
11	96	N O	
19	280	N O	
$21=3 \cdot 7$	346	N O	
23	412	N O	
$27=3^3$	576	Y E S	13.45362...
31	756	Y E S	15.44445...
$39=3 \cdot 11$	850	N O	
43	1448	N O	

$$= r_{40}^*$$

$$= r_{577}^*$$

$$= r_{756}^*$$

Appendix 3 : Table of r_k ($2 \leq k \leq 300$)

For each value of k ($2 \leq k \leq 300$) the coordinates of the center of some minimal circle Γ_k passing through the origin and the integer points b and c are $x = N(x)/D$ and $y = N(y)/D$, and its squared radius is $r_k^2 = N(r^2)/D^2$. The last column gives an approximation to r_k .

If the line of index k_0 is empty there exists a smallest $k_1 > k_0$ such that the line of index k_1 is nonempty and then the minimal disks of indices $k_0, k_0+1, \dots, k_1$ are identical : For instance, $r_{18} = r_{19} = r_{20} = r_{21} = \sqrt{5}$. Notice that not all the centers of the minimal disks are on the symmetry axes of the lattice $\mathcal{T}[1]$ (see for instance D_{47}).

k	b	c	$N(x)$	$N(y)$	D	$N(r^2)$	r_k
2	1		0	1	2	1	0.5
3							
4	1		1	1	2	2	0.7071 06781
5	-1+ i	1+ i	0	1	1	1	1
6	2i		1	2	2	5	1.1180 33989
7	-1+ 2i	1+ 2i	0	5	4	25	1.25
8							
9	2i		1	1	1	2	1.4142 13562
10							
11	-2+ 2i	1+ i	-1	3	2	10	1.5811 38830
12	-2+ 3i	1+ i	-7	17	10	338	1.8384 77631
13	-2+ 3i	1+ 3i	-3	11	6	130	1.9602 92375
14							
15	4i		1	4	2	17	2.0615 52813
16	-2+ 3i	2+ 3i	0	13	6	169	2.1666 66667
17							
18							
19							
20	-3+ 3i	1+ i	-1	2	1	5	2.2360 67977
21	-4+ 2i	1+ 2i	-3	4	2	25	2.5
22							
23	-3+ 3i	2+ 2i	-1	5	2	26	2.5495 09757
24							
25	5i		2	5	2	29	2.6925 82404
26	-1+ 5i	3+ 4i	21	103	38	11 050	2.7662 88953
27			-2	11	4	125	2.7950 84972
28	-3+ 4i	2+ 4i					
29							
30							
31	-4+ 4i	1+ i	-3	5	2	34	2.9154 75947
32	-4+ 4i	2+ 3i	-11	29	10	962	3.1016 12484
33							
34							
35							
36	-4+ 4i	2+ 2i	-1	3	1	10	3.1522 77560
37	6i		-1	6	2	45	3.3544 01985
38	-5+ 4i	4+ i	-1	17	6	416	3.3747 42789
39	-1+ 4i	4+ 5i	61	159	58	39 442	3.4241 39567
40			-10	33	10	1 169	3.4481 87930
41	-4+ 5i	2+ 5i					
42							
43							
44	-6+ 3i	1+ 2i	-5	5	2	50	3.5355 33906
45	-5+ 5i	1+ i	-2	3	1	13	3.6855 51275
46	-4+ 6i	2+ 2i	-7	17	5	338	3.6769 55262
47	-2+ 5i	5+ 4i	89	227	66	59 450	3.6942 98598
48	-4+ 5i	3+ 5i	-5	37	10	1 394	3.7336 30941
49							
50							
51	-5+ 5i	2+ 2i	-3	7	2	58	3.8078 86553
52							
53							
54							
55	-6+ 4i	2+ 3i	-4	7	2	65	4.0511 28874
56	-5+ 5i	3+ 3i	-1	4	1	17	4.1231 05626
57	-2+ 8i	1+ 8i	-4	33	8	1 105	4.1551 92353
58	-4+ 7i	2+ 7i	-14	57	14	3 445	4.1524 37511
59							
60							

K	b	c	N(x)	N(y)	D	N(r ²)	r _k	K	b	c	N(x)	N(y)	D	N(r ²)	r _k	
61	6i	5i	3	85	1	18	4.2425 40687	121	-7+ 7i	5+ 5i	-1	3	10	37	5.0827 62530	
62	-6+ 5i	2+ 2i	-41	85	22	8 996	4.2896 18450	122	-11+ 3i	1+ 2i	-49	67	10	3 770	5.1400 32573	
63	-7+ 5i	1+ 4i	-211	133	66	81 770	4.3326 44573	123	-8+ 9i	3+ 4i	-335	635	118	529 250	5.1552 17835	
64	-5+ 7i	2+ 6i	-41	87	22	9 230	4.3716 78196	124	-8+ 9i	3+ 4i	-54	97	18	12 325	5.1676 67589	
65	-7+ 5i	1+ 5i	-15	16	5	481	4.3863 42440	125	-9+ 7i	3+ 4i	-115	205	38	55 250	5.1856 10145	
66								126	-9+ 7i	3+ 7i	-21	38	7	1 885	5.2023 69527	
67								127	-8+ 8i	4+ 7i	-47	129	22	18 850	5.2406 95555	
68								128	11i	6	6	11	2	157	5.2649 82045	
69	-6+ 6i	2+ 2i	-2	4	1	20	4.4721 35955	129	-11+ 6i	4+ 10i	-5	31	5	986	5.2801 27388	
70	-8+ 4i	1+ 2i	-7	6	2	85	4.6897 72229	130	-11+ 6i	1+ 2i	-71	53	14	7 850	5.3285 87552	
71	-3+ 6i	6+ 5i	47	151	34	25 010	4.6513 36313	131	-11+ 6i	1+ 6i	-60	47	12	5 809	5.3513 95671	
72	-6+ 6i	1+ 7i	-35	55	14	4 250	4.6565 75147	132	-2+ 11i	7+ 10i	389	1173	194	1 527 250	5.3702 03948	
73	-4+ 8i	4+ 4i	-2	14	3	200	4.7140 45208	133	-7+ 10i	3+ 10i	-40	121	20	16 241	5.3720 05100	
74	-4+ 8i	5	5	8	2	89	4.7169 90566	134								
75								135								
76	-6+ 6i	3+ 3i	-3	9	2	90	4.7434 16490	136	-9+ 9i	1+ 1	-4	5	1	41	5.4831 24237	
77	-7+ 6i	2+ 6i	-15	43	6	830	4.8409 61805	137								
78	-7+ 6i	4+ 3i	-102	421	86	179 430	4.8521 26379	138	-12+ 4i	1+ 3i	-11	289	7	2 370	5.5192 02485	
79	-3+ 9i	6+ 7i	182	289	62	93 323	4.8530 84931	139	-12+ 4i	1+ 3i	-457	289	82	292 370	5.5940 54791	
80	-8+ 9i	1+ 6i	-17	57	14	4 810	4.9538 68824	140	-12+ 4i	1+ 3i	-13	171	26	29 410	5.5959 01542	
81	-7+ 7i	3+ 7i	-21	67	14	4 930	5.0152 82766	141	-7+ 10i	5+ 8i	-151	684	106	430 657	5.6881 99325	
82	-6+ 7i	3+ 7i	-21	67	14	4 930	5.0152 82766	142	-10+ 4i	2+ 2i	-25	39	7	2 146	5.6178 48470	
83	-1+ 10i	1+ 10i	0	101	20	10 201	5.05	143								
84	-5+ 8i	4+ 6i	-59	306	62	98 345	5.0580 65544	144								
85	-8+ 3i	1+ 1	-67	89	22	12 410	5.0635 44524	145								
86								146								
87								147								
88								148	-8+ 8i	5+ 5i	-3	13	2	178	5.6708 32030	
89	-6+ 6i	4+ 4i	-1	5	1	26	5.0990 19514	149	-12+ 6i	1+ 2i	-11	8	2	185	5.8807 35255	
90	-8+ 6i	2+ 3i	-111	152	36	35 425	5.2282 02840	150	-4+ 13i	4+ 11i	-127	644	96	430 865	5.8375 32056	
91	-7+ 6i	3+ 3i	-49	127	26	18 530	5.2355 74715	151	-9+ 9i	4+ 5i	-49	113	18	15 170	5.8425 86328	
92	-3+ 7i	7+ 6i	247	661	134	497 930	5.2659 81732	152	-6+ 11i	6+ 9i	-21	274	40	75 517	5.8700 89155	
93	-7+ 7i	3+ 6i	-13	29	6	1 010	5.2967 49528	153	-11+ 7i	1+ 1	-13	16	3	425	5.8718 42710	
94	-5+ 9i	4+ 4i	-17	73	14	5 618	5.3538 08486	154	-3+ 12i	7+ 11i	119	527	78	291 890	5.9265 18613	
95								155	-7+ 12i	4+ 11i	-33	149	22	23 290	5.9368 46164	
96								156	-7+ 12i	4+ 8i	-146	333	52	132 205	5.9523 09750	
97	-7+ 7i	3+ 3i	-2	5	1	29	5.3851 64807	157	-12+ 7i	7+ 7i	-77	61	14	3 830	5.9617 40638	
98	-8+ 5i	3+ 5i	-25	49	10	3 026	5.3895 09016	158	-3+ 13i	4+ 10i	197	163	242	2 890	5.9688 56971	
99	-8+ 7i	2+ 7i	-42	65	14	5 385	5.3277 33261	159	-10+ 4i	2+ 4i	-2	9	1	5	7 0710	5.9781 12362
100	-10+ 4i	1+ 4i	-18	13	4	453	5.3505 06628	160	-12+ 6i	2+ 4i	-5	5	1	50	7.0710 67812	
101								161	-12+ 6i	2+ 4i	-5	5	1	50	7.0710 67812	
102								162	-13+ 5i	1+ 5i	-30	19	5	1 261	7.1021 12362	
103								163	-1+ 10i	1+ 1	-9	11	2	202	7.1063 35200	
104	-8+ 6i	3+ 4i	-5	10	2	125	5.5901 69944	164								
105								165	-10+ 8i	4+ 5i	-6	13	2	205	7.1589 10530	
106								166	-5+ 11i	1+ 14i	-28	199	28	40 385	7.1771 49825	
107								167	-5+ 11i	6+ 9i	281	1893	266	3 662 410	7.1945 20327	
108								168	-10+ 10i	2+ 2i	-4	6	1	52	7.2111 02551	
109								169	-6+ 13i	3+ 12i	-157	511	74	285 770	7.2239 81043	
110								170	-10+ 10i	3+ 4i	-55	85	14	10 250	7.2315 91693	
111								171	-8+ 11i	5+ 9i	-499	1773	254	3 392 530	7.2515 04685	
112	-10+ 5i	1+ 2i	-9	7	2	130	5.7008 77125	172	-11+ 4i	2+ 2i	-121	181	30	47 402	7.2573 33457	
113	-10+ 6i	1+ 2i	-121	93	26	23 290	5.8696 39062	173								
114	-8+ 7i	1+ 1	-29	37	8	2 210	5.8763 29637	174								
115	-3+ 8i	8+ 7i	393	923	170	1 006 565	5.9010 81947	175								
116	-7+ 8i	4+ 8i	-6	23	4	42 050	5.9424 32162	176								
117	-8+ 9i	1+ 1	-127	161	34	42 050	6.0312 04897	177	-9+ 9i	5+ 5i	-2	7	1	53	7.2801 09869	
118								178	-5+ 14i	5+ 9i	-101	327	46	117 130	7.4400 56609	
119								179	-10+ 9i	4+ 4i	-109	261	38	80 002	7.4453 22316	
120	-8+ 8i	3+ 3i	-5	11	2	146	6.0415 22987	180	-4+ 10i	10+ 9i	383	942	136	1 034 053	7.4770 87565	

k	b	c	N(x)	N(y)	D	N(r ²)	r _k	k	b	c	N(x)	N(y)	D	N(r ²)	r _k
181	-12+9;	1+8;	-81	67	14	11 050	7.5084 98586	241	-12+12;	2+2;	-5	7	1	74	8.6823 25267
182	-1+14;	7+13;	491	1597	222	2 791 490	7.5260 14725	242	-8+13;	7+13;	-13	225	26	50 794	8.6682 78565
183	-12+8;	1+1	-24	29	5	1 417	7.5286 12090	243	-10+11;	7+8;	-525	2677	314	7 441 954	8.6878 88576
184	15;	2+15;	2	15	2	229	7.5663 72975	244	-3+17;	2+17;	-17	295	34	87 314	8.6908 65426
185	-14+5;	1+5;	-65	39	10	5 746	7.5802 37483	245	-9+11;	8+7;	-171	2633	302	6 961 930	8.7369 10427
187								246	-11+10;	6+6;	-101	353	42	134 810	8.7420 13560
188	13;	8	8	13	2	233	7.6521 68760	247	-12+12;	3+3;	-9	15	2	306	8.7464 27840
189	-8+13;	4+5;	-156	315	46	124 189	7.6609 95500	248	-7+12;	5+6;	-477	1351	166	2 125 850	8.7833 10193
190	-2+11;	8+11;	1181	1617	282	4 656 250	7.6846 92533	249	-10+13;	4+6;	-479	1351	166	2 125 850	8.7858 61256
191	-14+12;	1+5;	-39	29	36	85 850	7.7105 71254	250	-12+13;	5+11;	-459	1592	18	2 745 145	8.8130 28603
192	-16+14;	7+12;	-58	1120	146	1 276 965	7.7408 52117	251	-13+5;	3+3;	-41	68	9	6 365	8.8226 76013
193	-4+16;	7+12;	3	4	4	4	7.7659 12333	252							
194	-7+12;	6+12;	-2	28	5	1 513	7.7791 60136	253							
195	-11+11;	2+8;	-27	28	5	1 513	7.7791 60136	254							
196	-13+8;	2+7;	-1207	1155	214	2 790 874	7.8064 99533	255	-11+11;	6+6;	-5	17	2	314	8.8500 22575
197	-10+10;	5+9;	-37	103	14	11 978	7.8174 32114	256	-3+11;	14+9;	-63	109	14	15 850	8.9926 27564
198	-9+12;	5+9;	-251	693	94	543 250	7.8410 10112	257	-11+14;	2+14;	-63	109	14	15 850	8.9926 27564
199	-2+14;	8+13;	331	1033	138	1 176 650	7.8603 98486	258							
200	-13+8;	2+8;	-44	45	8	3 961	7.8670 59489	259							
201								260							
202								261							
203								262	-16+8;	1+2;	-15	10	2	325	9.0138 78190
204								263	-14+9;	4+7;	-677	1009	134	1 476 410	9.0677 32306
205								264	-2+18;	1+18;	-9	163	18	26 650	9.0693 48733
206								265	-13+12;	4+7;	-1411	2097	278	6 388 330	9.0917 79097
207								266	-5+17;	4+17;	-17	309	34	95 770	9.1019 78947
208	-14+7;	1+2;	-13	9	2	250	7.9056 94150	267	-6+17;	7+12;	-619	3433	382	12 168 650	9.1318 30401
209								268	-17+6;	1+6;	-96	53	12	12 025	9.1382 13417
210								269							
211								270	16;	9	9	16	2	337	9.1787 78875
212								271							
213	-15+5;	1+3;	-7	4	1	65	8.0622 57748	272							
214	-15+6;	1+3;	-241	137	34	76 850	8.1534 80162	273							
215	-3+16;	5+16;	-132	259	22	68 105	8.1532 92091	274							
216	-12+5;	3+3;	-139	241	34	77 402	8.1627 10274	275							
217								276							
218								277							
219								278	-16+8;	2+4;	-7	6	1	85	9.2195 44457
220	-10+10;	6+6;	-2	8	1	68	8.2462 11251	279	-11+15;	4+6;	-108	163	21	38 233	9.3110 76476
221								280	-12+14;	4+11;	-911	1502	188	3 085 925	9.3280 13412
222	-14+8;	2+8;	-1451	2103	306	6 528 010	8.3103 95070	281	-14+11;	14+10;	-36	43	6	3 145	9.3467 15595
223	-12+11;	3+10;	-1451	2103	306	6 528 010	8.3103 95070	282	-4+11;	6+14;	-14	64	7	4 292	9.3590 51060
224	-2+13;	11+12;	1369	2433	334	7 793 650	8.3584 16063	283	-10+14;	11+12;	-463	2071	226	4 503 410	9.3899 28920
225	-4+15;	8+13;	181	1430	172	2 077 661	8.3802 87053	284	-14+4;	1+1	-233	271	38	127 730	9.4050 87913
226	-2+15;	7+6;	-277	2179	262	4 824 770	8.3837 24844	285	-8+17;	4+15;	-599	1670	188	3 147 701	9.4371 07165
227	-9+11;	9+11;	371	3057	366	9 482 890	8.4137 43557	286	-13+13;	4+5;	-89	145	18	28 946	9.4519 57794
228	-6+13;	7+11;	-11	92	11	8 585	8.4232 06835	287	-11+14;	6+12;	-214	647	72	464 405	9.4648 98194
229	-9+11;	7+11;	-709	2913	354	8 968 250	8.4690 42449	288	-14+5;	3+3;	-191	305	38	129 506	9.4702 47916
230	-9+13;	5+13;	-26	107	13	12 125	8.4702 75208	289							
231	-15+6;	2+5;	-13	11	2	290	8.5146 53185	290							
232	-15+6;	2+5;	-13	11	2	290	8.5146 53185	291							
233	-11+11;	5+5;	-3	8	1	73	8.5440 03745	292	-12+12;	6+6;	-3	9	1	90	9.4868 32981
234	-11+11;	5+5;	-3	8	1	73	8.5440 03745	293							
235	-15+8;	1+8;	-112	79	16	18 785	8.5586 21385	294							
236	-2+15;	10+13;	529	1414	176	2 279 237	8.5661 48856	295							
237	-10+13;	4+13;	-78	209	26	49 765	8.5779 21653	296							
238								297							
239								298							
240								299							
								300	-18+6;	1+3;	-17	9	2	370	9.6176 92830