On the Differential-Difference Properties of the Extended Jacobi Polynomials

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Abstract. We discuss differential-difference properties of the extended Jacobi polynomials

$$P_n(x) = {}_{p+2}F_q(-n, n + \lambda, a_p; b_q; x) \quad (n = 0, 1, \dots).$$

The point of departure is a corrected and reformulated version of a differential-difference equation satisfied by the polynomials $P_n(x)$, which was derived by Wimp (*Math. Comp.*, v. 29, 1975, pp. 577-581).

1. Introduction. Here we are concerned with the properties of the extended Jacobi polynomials [3, Vol. 1, Section 7.4],

$$(1) \quad \begin{aligned} P_n(x) &= {}_{p+2}F_q \left(\begin{matrix} -n, n + \lambda, a_p \\ b_q \end{matrix} \middle| x \right) \\ &= \sum_{k=0}^n \frac{(-n)_k (n + \lambda)_k (a_p)_k}{k! (b_q)_k} x^k \quad (n = 0, 1, \dots), \end{aligned}$$

where

$$(\alpha)_k = \Gamma(\alpha + k)/\Gamma(\alpha)$$

is Pochhammer's symbol and the above contracted notation will be used throughout the paper:

$$f(a_p) = \prod_{i=1}^p f(a_i), \quad f(b_q) = \prod_{j=1}^q f(b_j),$$

f being a given function. We assume that no a_i equals any b_j ($i = 1, 2, \dots, p$; $j = 1, 2, \dots, q$) and set

$$(2) \quad b_j = 1 \quad \text{for } j = q + 1.$$

Let

$$(3) \quad \sigma = \max\{p + 1, q\}.$$

Wimp ([5]; see also [3, Vol. 2, Section 12.2]) has proved the following.

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THEOREM 1.1. *Let λ, a_i, b_j ($i = 1, 2, \dots, p$; $j = 1, 2, \dots, q + 1$) be complex constants such that none of the quantities $\lambda, \lambda + 1 - b_j$ ($j = 1, 2, \dots, q$) are negative integers or zero. Then the polynomials $P_n(x)$, see (1), satisfy the difference equation*

$$(4) \quad \sum_{m=0}^{\sigma+1} [C_m(n; \sigma + 1) + xD_m(n; \sigma + 1)] P_{n-m}(x) = 0 \quad (n \geq \sigma + 1),$$

where

$$(5) \quad C_m(n; t) = \frac{(n - m + 1)_m (2n + \lambda - 2m)_{2m} (n - m - 1 + b_{q+1})}{m! (n + \lambda - m)_m (2n + \lambda - m - t)_m (n - 1 + b_{q+1})} \\ \times {}_{q+3}F_{q+2} \left(\begin{matrix} -m, 2n + \lambda - m - t, n - m + b_{q+1} \\ 2n + \lambda - 2m + 1, n - m - 1 + b_{q+1} \end{matrix} \middle| 1 \right),$$

$$(6) \quad D_m(n; t) = \frac{(n - m + 1)_m (2n + \lambda - 2m)_{2m} (n - m + a_p)}{\Gamma(m) (n + \lambda - m)_m (2n + \lambda - m - t + 1)_{m-1} (n - 1 + b_{q+1})} \\ \times {}_{p+2}F_{p+1} \left(\begin{matrix} 1 - m, 2n + \lambda - m - t + 1, n - m + 1 + a_p \\ 2n + \lambda - 2m + 1, n - m + a_p \end{matrix} \middle| 1 \right).$$

Moreover, these polynomials do not satisfy a nontrivial equation of the form (4) of lower order than $\sigma + 1$.

The next theorem supplements Theorem 1.1 in a certain particular case.

THEOREM 1.2 [2]. *If $p + 1 = q$ and $x = 1$ in (1) then the sequence $P_n(1)$ satisfies the recurrence relation*

$$(7) \quad \sum_{m=0}^{\sigma} [C_m(n; \sigma) + D_m(n; \sigma)] P_{n-m}(1) = 0 \quad (n \geq \sigma).$$

The above result contains as a special case the recurrence relation given by Bailey [1] for

$$(8) \quad {}_3F_2 \left(\begin{matrix} -n, n + \lambda, a_1 \\ b_1, b_2 \end{matrix} \middle| 1 \right).$$

In this paper, our attention is focussed on another result of Wimp which is contained in the following theorem.

THEOREM 1.3 ([6]; see also [4, Section 5.13]). *Let $\lambda \neq 1, 2, \dots$ and let the assumptions of Theorem 1.1 be satisfied. Then the polynomials $P_n(x)$ satisfy the differential-difference equation*

$$(9) \quad x(\delta x - \varepsilon) \frac{d}{dx} P_n(x) = \sum_{m=0}^{\sigma} [A_m(n) + xB_m(n)] P_{n-m}(x),$$

where

$$(10) \quad \delta = \begin{cases} 1 & (p+1 \geq q), \\ 0 & (p+1 < q), \end{cases} \quad \varepsilon = \begin{cases} 0 & (p+1 > q), \\ 1 & (p+1 \leq q), \end{cases}$$

and

$$(11) \quad A_m(n) = \begin{cases} \alpha_m(n) \left[\frac{1}{m!} \sum_{k=0}^m \frac{(-m)_k (n-k-1+b_{q+1})}{k!(2n+\lambda-k-\sigma)_{\sigma+1-m}} - \varepsilon \right] & (m > 0), \\ \omega(n) - \varepsilon \alpha_0(n) & (m = 0), \end{cases}$$

$$(12) \quad B_m(n) = \begin{cases} \alpha_m(n) \left[\delta - \frac{1}{\Gamma(m)} \sum_{k=0}^{m-1} \frac{(1-m)_k (n-k-1+a_p)}{k!(2n+\lambda-k-\sigma)_{\sigma-m}} \right] & (m > 0), \\ \delta \alpha_0(n) & (m = 0), \end{cases}$$

and

$$(13) \quad \alpha_m(n) = \begin{cases} (-1)^m \frac{(n-m+1)_m (2n+\lambda-2m)}{(n+\lambda-m)_m} & (m > 0), \\ n & (m = 0), \end{cases}$$

$$(14) \quad \omega(n) = \frac{n-1+b_{q+1}}{(2n+\lambda-\sigma)_\sigma}.$$

Moreover, no equation of the type (9) of lower order $\sigma' < \sigma$ exists.

Note that the formulae (5) and (6) of the paper [6], defining the coefficients A_m and B_m , respectively, are not correct as can be seen by considering the case of $p+1=q=2$ and $x=1$ and observing that the resulting second-order (pure) difference equation for quantities (8) disagrees with that given by Bailey [1]. An inspection of Wimp's proof of the equation (9) (see [6, esp. the last paragraph of Section II]) reveals how the formulae should be corrected.

Wimp's theorem was reproduced in the book [4] (see Theorem 2 in Section 5.13.2). Unfortunately, in the Russian edition of that book (Mir Publ., Moscow, 1980) the list of errors was increased by six other misprints.

2. Alternative Forms for A_m and B_m . In Theorem 2.1, below, we give some alternative forms for the coefficients A_m and B_m , see (1.11) and (1.12), respectively. We need the following lemma.

LEMMA 2.1. *Let m, r, s be integers ≥ 0 . Let none of the complex constants γ, c_i ($i = 1, 2, \dots, r$) be integers. Then the identity*

$$(1) \quad \sum_{k=0}^m \frac{(-m)_k (c_r - k)}{k! (\gamma - s - k)_{s+1-m}} = (-1)^m \frac{(\gamma - 2m + 1)_{2m-1} (c_r - m)}{(\gamma - s - m)_{s+m}} \times_{r+2} F_{r+1} \left(\begin{matrix} -m, \gamma - s - m, 1 - m + c_r \\ \gamma + 1 - 2m, c_r - m \end{matrix} \middle| 1 \right)$$

holds.

Proof. Making use of some properties of Pochhammer's symbol [3, Vol. 1, Section 2.1], we obtain

$$\begin{aligned}
 & (-1)^m \frac{(\gamma - s - m)_{s+m}}{(\gamma - 2m + 1)_{2m-1} (c_r - m)} \sum_{k=0}^m \frac{(-m)_k (c_r - k)}{k! (\gamma - s - k)_{s+1-m}} \\
 &= (-1)^m \sum_{k=0}^m \frac{(-m)_k (\gamma - s - m)_{m-k} (c_r - k)}{k! (\gamma - 2m + 1)_{m-k} (c_r - m)} \\
 &= (-1)^m \sum_{k=0}^m \frac{(-m)_{m-k} (\gamma - s - m)_k (c_r - m + k)}{(m - k)! (\gamma - 2m + 1)_k (c_r - m)} \\
 &= \sum_{k=0}^m \frac{(-m)_k (\gamma - s - m)_k (c_r - m + 1)_k}{k! (\gamma - 2m + 1)_k (c_r - m)_k} \\
 &= {}_{r+2}F_{r+1} \left(\begin{matrix} -m, \gamma - s - m, c_r - m + 1 \\ \gamma - 2m + 1, c_r - m \end{matrix} \middle| 1 \right). \quad \square
 \end{aligned}$$

THEOREM 2.1. *The equations (1.11) and (1.12) can be rewritten in the form*

$$\begin{aligned}
 (2) \quad & A_m(n) = \omega(n) C_m(n; \sigma) - \varepsilon \alpha_m(n), \\
 (3) \quad & B_m(n) = \omega(n) D_m(n; \sigma) + \delta \alpha_m(n) \quad (m = 0, 1, \dots, \sigma),
 \end{aligned}$$

respectively. Here the notation is that of (1.3), (1.5), (1.6), (1.10), (1.13) and (1.14).

Proof. Putting $r = q + 1$, $\gamma = 2n + \lambda$, $s = \sigma$, $c_i = n - 1 + b_i$ ($i = 1, 2, \dots, q + 1$) in (1), we obtain

$$\begin{aligned}
 & \sum_{k=0}^m \frac{(-m)_k (n - k - 1 - b_{q+1})}{k! (2n + \lambda - \sigma - k)_{\sigma+1-m}} \\
 &= (-1)^m \frac{(2n + \lambda - 2m + 1)_{2m-1} (n - m - 1 + b_{q+1})}{(2n + \lambda - \sigma - m)_{\sigma+m}} \\
 & \quad \times {}_{q+2}F_{q+1} \left(\begin{matrix} -m, 2n + \lambda - m - \sigma, n - m + b_{q+1} \\ 2n + \lambda - 2m + 1, n - m - 1 + b_{q+1} \end{matrix} \middle| 1 \right) \\
 &= \frac{m! \omega(n)}{\alpha_n(n)} C_m(n; \sigma).
 \end{aligned}$$

Now, it readily follows that the first part of (1.11) may be written in the form (2). Obviously, the second part of (1.11) can be written as

$$A_0(n) = \omega(n) C_0(n; \sigma) - \varepsilon \alpha_0(n)$$

because $C_0(n; \sigma) = 1$. Proceeding in a similar fashion, one arrives at (3). $\square$

Note that if $p + 1 = q$ and $x = 1$ then the equation (1.9) takes the form (1.7) as, in view of (2) and (3), we have

$$A_m(n) + B_m(n) = \omega(n) [C_m(n; \sigma) + D_m(n; \sigma)].$$

3. Further Differential-Difference Equations. With the aid of Theorems 1.3 and 2.1 we derive further differential-difference equations satisfied by the polynomials $P_n(x)$. We require one more lemma.

LEMMA 3.1. *We have*

- (1) $C_m(n; \sigma) + \theta(n)C_{m-1}(n-1; \sigma) = C_m(n; \sigma+1) \quad (m = 1, 2, \dots, \sigma+1),$
- (2) $C_{\sigma+1}(n; \sigma) = \varepsilon \alpha_{\sigma+1}(n)/\omega(n),$
- (3) $D_m(n; \sigma) + \theta(n)D_{m-1}(n-1; \sigma) = D_m(n; \sigma+1) \quad (m = 1, 2, \dots, \sigma+1),$
- (4) $D_{\sigma+1}(n; \sigma) = -\delta \alpha_{\sigma+1}(n)/\omega(n),$

where

$$(5) \quad \theta(n) = \frac{n\omega(n-1)}{(n+\lambda-1)\omega(n)},$$

and the notation used is that of (1.3), (1.5), (1.6), (1.10), (1.13) and (1.14).

Proof. Equations (1) and (3) can be checked by a straightforward calculation, using the definitions (1.5) and (1.6), respectively, and (5).

We prove the formula (2). We have

$$C_{\sigma+1}(n; \sigma) = \frac{(n-\sigma)_{\sigma+1}(2n+\lambda-2\sigma-2)_{2\sigma+2}(n-\sigma-2+b_{q+1})}{(\sigma+1)!(n+\lambda-\sigma-1)_{\sigma+1}(2n+\lambda-2\sigma-1)_{\sigma+1}(n-1+b_{q+1})} f,$$

where

$$f = {}_{q+2}F_{q+1} \left(\begin{matrix} -\sigma-1, n-\sigma-1+b_{q+1} \\ n-\sigma-2+b_{q+1} \end{matrix} \middle| 1 \right)$$

(see (1.5)). Now, observe that

$$\begin{aligned} f &= \frac{(-1)^{\sigma+1}(\sigma+1)!}{(n-\sigma-2+b_{q+1})} \quad (\sigma = q \geq p+1), \\ &= 0 \quad (\sigma = p+1 > q) \end{aligned}$$

(cf. [3, Vol. 1, Section 2.9]). Thus, for arbitrary p and q ,

$$\begin{aligned} C_{\sigma+1}(n; \sigma) &= \varepsilon(-1)^{\sigma+1} \frac{(n-\sigma)_{\sigma+1}(2n+\lambda-\sigma)_{\sigma}(2n+\lambda-2-\sigma)}{(n+\lambda-\sigma-1)_{\sigma+1}(n-1+b_{q+1})} \\ &= \varepsilon \alpha_{\sigma+1}(n)/\omega(n). \end{aligned}$$

Equation (4) can be proved in a similar way. $\square$

THEOREM 3.1. *Under the assumptions of Theorem 1.3, the polynomials $P_n(x)$ satisfy the equations*

$$(6) \quad x \left[\frac{dP_n(x)}{dx} + \frac{n}{n+\lambda-1} \frac{dP_{n-1}(x)}{dx} \right] = n[P_n(x) - P_{n-1}(x)],$$

$$(7) \quad x \frac{dP_n(x)}{dx} = \sum_{k=1}^n (-1)^k \frac{(-n)_k(2n+\lambda-2k)}{(1-\lambda-n)_k} P_{n-k}(x) + nP_n(x).$$

Proof. First observe that Eq. (1.9) can, by virtue of Theorem 2.1, be written in the form

$$(8) \quad x(\delta x - \varepsilon) \frac{dP_n(x)}{dx} = (\delta x - \varepsilon) \sum_{m=0}^{\sigma} \alpha_m(n) P_{n-m}(x) + \omega(n) \sum_{m=0}^{\sigma} [C_m(n; \sigma) + xD_m(n; \sigma)] P_{n-m}(x).$$

Now, replace in the above equation n by $n - 1$, multiply the resulting equation by $n/(n + \lambda - 1)$ and add to (8). The result is

$$(9) \quad x(\delta x - \varepsilon) \left[\frac{dP_n(x)}{dx} + \frac{n}{n + \lambda - 1} \frac{dP_{n-1}(x)}{dx} \right] = (\delta x - \varepsilon) \left[\sum_{m=0}^{\sigma} \alpha_m(n) P_{n-m}(x) + \frac{n}{n + \lambda - 1} \sum_{m=1}^{\sigma+1} \alpha_{m-1}(n-1) P_{n-m}(x) \right] + \omega(n) \left\{ \sum_{m=0}^{\sigma} [C_m(n; \sigma) + xD_m(n; \sigma)] P_{n-m}(x) + \theta(n) \sum_{m=1}^{\sigma+1} [C_{m-1}(n-1; \sigma) + xD_{m-1}(n-1; \sigma)] P_{n-m}(x) \right\}.$$

Using Lemma 3.1 and considering that $C_0(n; t) = 1$, $D_0(n; t) = 0$, we write the right-hand side of (9) in the form

$$(10) \quad (\delta x - \varepsilon) \sum_{m=0}^{\sigma+1} \beta_m(n) P_{n-m}(x) + \omega(n) \sum_{m=0}^{\sigma+1} [C_m(n; \sigma + 1) + xD_m(n; \sigma + 1)] P_{n-m}(x),$$

in which

$$\beta_m(n) = \begin{cases} \alpha_m(n) + \frac{n}{n + \lambda - 1} \alpha_{m-1}(n-1) & (m > 0), \\ \alpha_0(n) & (m = 0), \end{cases}$$

or, in view of (1.13),

$$\beta_m(n) = \begin{cases} n & (m = 0), \\ -n & (m = 1), \\ 0 & (m > 1). \end{cases}$$

By virtue of Theorem 1.1, the second sum of (10) is zero, therefore the right-hand side of (9) reduces to

$$(\delta x - \varepsilon) n [P_n(x) - P_{n-1}(x)],$$

and Eq. (6) follows.

Formula (6) is a first-order difference equation with respect to $U_n(x) = xP'_n(x)$. Using the well-known formula for the general solution of such an equation and remembering that $U_0(x) = 0$ (see (1.1)), we get (7). $\square$

Of course, Eqs. (6) and (7) provide a generalization of the classical differential-difference formulae for the Jacobi polynomials, see [7, p. 262].

Introducing the notation

$$Q_n(x) = (-1)^n (\lambda)_n P_n(x) / n!$$

yields somewhat simpler forms of (6) and (7):

$$\begin{aligned} x[Q'_n(x) - Q'_{n-1}(x)] &= nQ_n(x) + (n + \lambda - 1)Q_{n-1}(x), \\ xQ'_n(x) &= \sum_{k=0}^{n-1} (2k + \lambda)Q_k(x) + nQ_n(x). \end{aligned}$$

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1. W. N. BAILEY, "Contiguous hypergeometric functions of the type ${}_3F_2(1)$," *Proc. Glasgow Math. Assoc.*, v. 2, 1954, pp. 62-65.
2. S. LEWANOWICZ, *Recurrence Relations for Hypergeometric Functions of Unit Argument*, Report N-127, Inst. Computer Sci., Univ. of Wrocław, 1983. Also *Math. Comp.*, v. 45, 1985. (To appear.)
3. Y. L. LUKE, *The Special Functions and Their Approximations*, Academic Press, New York, 1969.
4. Y. L. LUKE, *Mathematical Functions and Their Approximations*, Academic Press, New York, 1975.
5. JET WIMP, "Recursion formulae for hypergeometric functions," *Math. Comp.*, v. 22, 1968, pp. 363-373.
6. JET WIMP, "Differential-difference properties of hypergeometric polynomials," *Math. Comp.*, v. 29, 1975, pp. 577-581.
7. E. D. RAINVILLE, *Special Functions*, Macmillan, New York, 1960.